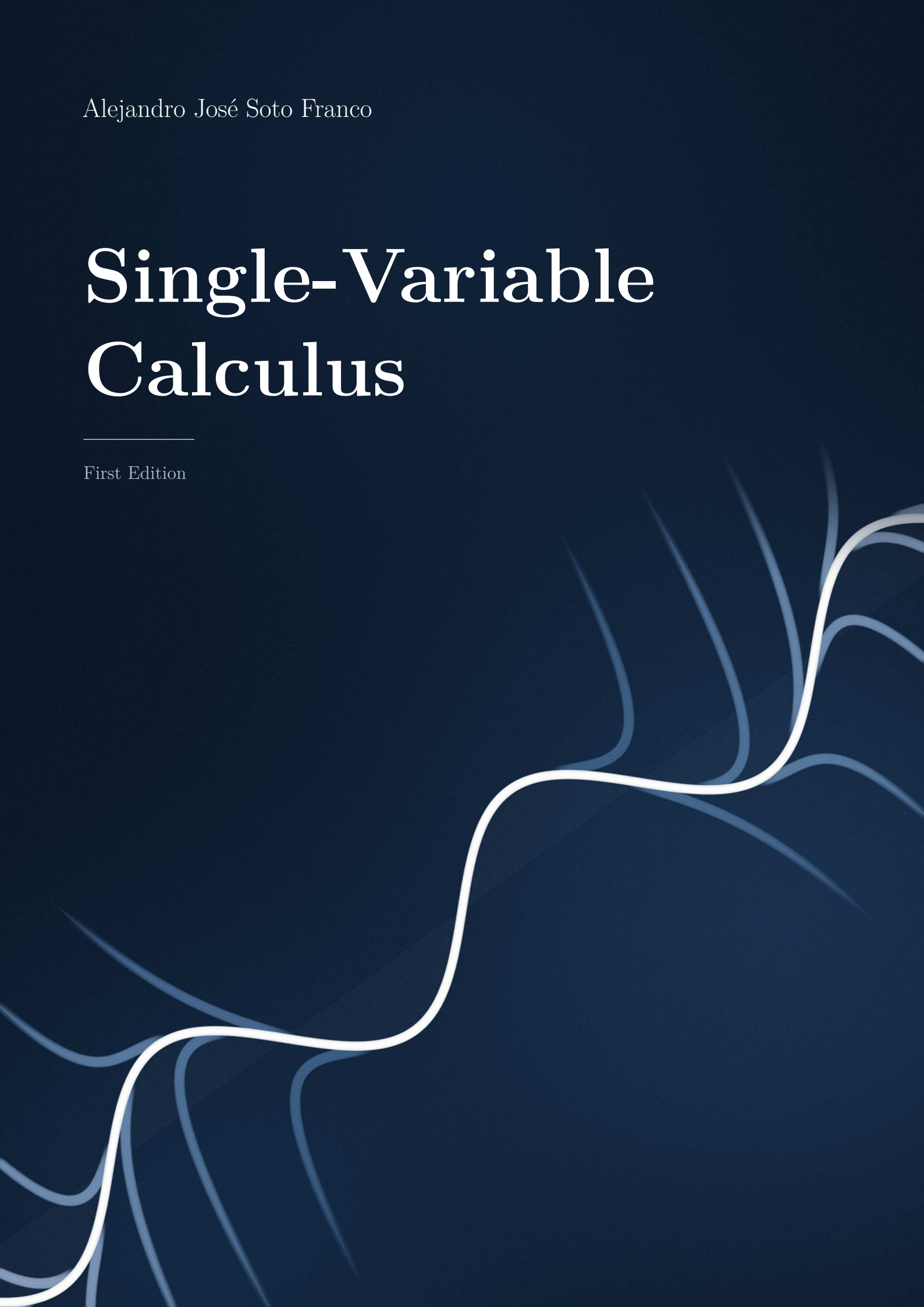
Alejandro José Soto Franco

Single-Variable Calculus

First Edition



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First Edition

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Preface

This book is a course in the calculus of one real variable. It is written for a reader who has met algebra and trigonometry in school and is ready to see why the methods of calculus work.

The book has four parts followed by four appendices. Part I sets out logic and proof, the real numbers, vectors and linear maps, normed spaces, the elementary functions and the limit. A norm measures the size of a vector or a function. The absolute value is the first norm on the line and the Euclidean length is one norm among several. Chapter 5 states limits and continuity in normed spaces. Part II defines the derivative and the 1-forms, states the derivative of a map between normed spaces, and treats differentiation as a linear operator measured in the sup and C^1 norms, with the Mean Value Theorem and Taylor's theorem. Part III defines the integral, proves the Fundamental Theorem and develops the techniques of integration. Part IV studies sequences, series and power series. Its first chapter defines complete normed spaces and reads uniform convergence as convergence in the sup norm. Its second chapter applies that convergence to power series.

I have organised the calculus in this book around the construction and use of the differential operator. The differential operator d applied to the coordinate function x produces $\mathrm{d}x$, the linear functional that returns the displacement it is given. Multiplying $\mathrm{d}x$ by a function f gives the 1-form $f(x)\,\mathrm{d}x$. Integrals integrate 1-forms. Section 6.2 defines these objects.

Thomas' Calculus: Early Transcendentals by Thomas, Weir and Hass [2] supplied the original choice of topics for the integration and series chapters. The development, the proofs, the worked examples and the problems are my own.

The problems are the core of the book. Every section ends with a set of them marked with one, two or three stars by difficulty. Every one has a hint, an answer and a complete solution in Appendices B to D, linked in both directions. Every computational answer has been checked by computer algebra. Appendix A contains the proofs too long to set in the chapters that use them.

This is the first edition.[1]

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14 October 2026
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How to Use This Book

Each section opens with an overview of its objectives, its prerequisites and its key results. A box of notation follows when the section introduces new symbols. Definitions, theorems and their proofs come next, then worked examples, a study guide and the problems. Results, equations and figures are numbered within their section: the number 3.1.2 names the second numbered item in the first section of the third chapter.

Study method

Study a section by reproducing its proofs and worked examples on paper before reading on. Then attempt its problems with the appendices closed. A hint is the first thing to consult when stuck and a solution the last. Every hint, answer and solution is in the appendices, away from the page in front of you.

Standard for a proof

A proof is a piece of writing and is marked as one. The proofs in this book are written to the standard below. It applies equally to your own work.

Standard for a proof

1. **Logical form.** Identify the form of the claim before the first line: implication, biconditional, universal, existential or unique existence. The form fixes the shape of the proof (Section 1.3).
2. **Hypotheses in full.** Write each hypothesis out in terms of its definition. "f is continuous at a" becomes the ε–δ statement of Definition 5.3.1.
3. **Variables introduced once.** Introduce every variable with its range at the moment it enters: "Let $\varepsilon > 0$." A variable chosen at an $\exists$ may depend only on the variables introduced before it (Section 1.2).
4. **Justified steps.** Justify every step by naming the definition, the theorem, the identity or the rule that produced it: "by Theorem 8.3.5", "by the chain rule". A step without a reason is an assertion. Cite only results proved in this book or in Appendix A, and none whose proof depends on the claim.
5. **Exhaustive cases.** A proof by cases names the cases and shows that together they cover every possibility.
6. **Conclusion.** The last step reaches the claim and names it.

Standard for a computation

A computation proves that its answer is correct and meets the same standard. Integration adds requirements of its own.

Standard for an integral

1. **Domain conditions.** Write down where the integrand is defined and where each step is valid *before* the step that needs it. The integral $\int \ln x \, dx$ is worked for $x > 0$ and a denominator $\sqrt{1 - x^2}$ appears only on $-1 < x < 1$.
2. **Constant of integration.** Write $+C$ the moment an indefinite integral becomes an antiderivative and keep it to the last line. A single constant serves the whole answer on one interval (Theorem 8.1.3).
3. **Substitutions.** Declare the new variable, its differential and the transformed limits before use. Write $u = g(x)$ and $du = g'(x)\,dx$ on their own line. For a definite integral, convert the limits to $g(a)$ and $g(b)$ there and then (Theorem 8.5.2) or convert back to x at the end. Never leave an integral with u inside and x-limits outside.
4. **Trigonometric substitutions.** Before the first line of algebra, state $x = a \sin\theta$, the range of θ, and the reference triangle that converts the answer back. The range licenses $\sqrt{a^2 - x^2} = a\cos\theta$ without an absolute value.
5. **Checking.** An antiderivative differentiates back to the integrand in one line. A definite integral has a sign and a size you can check against the picture.

Algebraic simplification first

Before integrating, simplify the integrand algebraically. Expand a product, split a fraction, factor a denominator, complete a square, or apply a log law first. Many hard-looking integrals become routine in the right algebraic shape. Section 4.3 treats the reshaping and Section 9.1 builds its strategy on it.

Difficulty of the problems

Each problem has one, two or three stars. A one-star problem applies a definition, a theorem or a worked example of its section directly. A two-star problem combines several results of the section or asks for a short proof. A three-star problem needs a longer argument or an idea that the worked examples do not show.

Appendices and indexes

Appendix A contains the proofs too long to set in the chapters that use them. Each is cited at the result it proves. Appendices B, C and D contain the hints, the answers and the complete solutions of every problem, grouped by section. Problems and entries are linked in both directions. The back matter has the notes, an index of notation that lists each symbol with its meaning in order of first appearance, an index of the problems by topic with their stars, and the bibliography.

Part I

Foundations of Analysis

Chapter 1

Logic, Sets and Proof

Calculus is written in statements of a precise logical shape. The definition of a limit (Definition 5.2.1) contains three quantifiers and an implication. Every theorem of the subject is an implication whose hypotheses must be checked before its conclusion may be used. To say what it means for a limit to fail you must be able to negate such a statement. To decide whether a theorem applies you must be able to read its hypotheses exactly. To check a claim of your own you must be able to write a proof.

This chapter sets out that language and those methods. Section 1.1 treats statements and the connectives that join them. Section 1.2 adds the quantifiers "for every" and "there exists" and the mechanical rules for negating them. Section 1.3 describes the standard methods of proof, and Section 1.4 the method of induction on which every statement about all natural numbers rests. Section 1.5 fixes the vocabulary of sets and functions (domain, codomain, image, injective, surjective, inverse, composition) that the rest of the book uses without further comment.

1.1 Statements and Connectives

Overview

Objectives: Decide whether a sentence is a statement; compute the truth value of a compound statement with a truth table; tell an implication from its converse and recognise its contrapositive as equivalent to it; explain why an implication with a false hypothesis is true; test two statements for logical equivalence.
Prerequisites: Arithmetic of integers and real numbers.
Key results: Truth tables of the connectives (Table 1.1.1); an implication is equivalent to its contrapositive and its negation is $P \wedge \neg Q$ (Theorem 1.1.10); De Morgan's laws for statements (Theorem 1.1.11).

Notation

$\neg P$	not P (negation)
$P \wedge Q$	P and Q (conjunction)
$P \vee Q$	P or Q or both (disjunction)
$P \implies Q$	if P then Q (implication)
$P \iff Q$	P if and only if Q (biconditional)
$\equiv$	is logically equivalent to
T, F	the truth values true and false

A theorem of calculus states that a function differentiable at a point is continuous there. Its converse, that a function continuous at a point is differentiable there, is false: $|x|$ is continuous at 0 with no derivative at 0. A student who confuses the two will apply the theorem in the wrong direction and reach false conclusions. Ordinary speech uses "if", "or" and "not" loosely enough to hide such confusions. Mathematics fixes the meaning of each word by a rule.

Definition 1.1.1 (Statement)**.** A *statement* (or *proposition*) is a declarative sentence that is either true or false and not both. Its *truth value* is T if it is true and F if it is false.

Example 1.1.2. "7 is a prime number" is a true statement and "$2 + 2 = 5$" is a false one. The sentence "every even integer greater than 2 is the sum of two primes" (Goldbach's conjecture) is also a statement. It is either true or false although nobody yet knows which.

Three sentences fail the definition. "Compute $3 + 4$" is a command and "Is 91 prime?" is a question. "$x > 2$" is neither true nor false until a value of x is given. A sentence with a free variable of this kind is an *open sentence*. Section 1.2 shows two ways of turning it into a statement.

1.1.3 Negation conjunction and disjunction

Statements combine into compound statements through *connectives*. The truth value of a compound statement depends only on the truth values of its parts. A *truth table* lists that dependence row by row.

Definition 1.1.4 (Negation, conjunction, disjunction)**.** Let P and Q be statements.

- The *negation* $\neg P$ is true when P is false and false when P is true.
- The *conjunction* $P \wedge Q$ is true when both P and Q are true and false otherwise.
- The *disjunction* $P \vee Q$ is true when at least one of P and Q is true and false when both are false.

The "or" of mathematics is inclusive: $P \vee Q$ is true when both parts are true. "2 is even or 2 is prime" is true even though both parts are true. Where an exclusive meaning is intended a mathematician writes "either … or … but not both".

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Table 1.1.1: Truth tables of negation, conjunction, disjunction, implication and the biconditional. Each row is one assignment of truth values to P and Q.

1.1.5 Implication

Definition 1.1.6 (Implication)**.** The *implication* $P \implies Q$ is false when P is true and Q is false and true in the other three cases (Table 1.1.1). The statement P is its *hypothesis* and Q its *conclusion*. Each of the following phrases means $P \implies Q$: "if P then Q", "P implies Q", "Q if P", "P only if Q", "Q whenever P", "P is sufficient for Q" and "Q is necessary for P".

Example 1.1.7. For a real number x the implication "if $x > 2$ then $x^2 > 4$" is true at every x. At $x = 3$ both parts are true. At $x = 0$ and at $x = -5$ the hypothesis is false and the implication

is true by Definition 1.1.6, whatever the conclusion. The implication "if $x^2 > 4$ then $x > 2$" is false at $x = -3$, where its hypothesis $9 > 4$ is true and its conclusion $-3 > 2$ is false.

The rows of Table 1.1.1 in which the hypothesis is false surprise most readers. An implication is a promise about what happens *when* its hypothesis is true and makes no claim about the other case. A teacher who says "if you score above 90 you will receive an A" has broken the promise only if a student scores above 90 and receives no A. A student scoring 80 cannot convict the teacher of a false statement, whatever grade that student receives.

The convention is also forced by mathematics. Everyone agrees that "for every real x, if $x > 2$ then $x^2 > 4$" is a true statement. A statement about every real x is true only when it is true at each x. At $x = 0$ the hypothesis $0 > 2$ is false. So the instance at $x = 0$ counts as true. An implication whose hypothesis is false is said to be *vacuously true*.

Definition 1.1.8 (Converse, contrapositive, biconditional)**.** The *converse* of $P \implies Q$ is $Q \implies P$ and its *contrapositive* is $\neg Q \implies \neg P$. The *biconditional* $P \iff Q$, read "P if and only if Q", is true when P and Q have the same truth value and false when they differ.

Definition 1.1.9 (Logical equivalence)**.** Two compound statements built from the same parts are *logically equivalent*, written $\equiv$, when they have the same truth value in every row of their truth table. A compound statement that is true in every row is a *tautology*.

Two equivalent statements say the same thing in different words. A proof of one is a proof of the other. A mathematician replaces a statement by an equivalent one whenever the second is easier to prove.

Theorem 1.1.10 (Implication and its contrapositive)**.** *For statements P and Q,*

1. $P \implies Q \equiv \neg Q \implies \neg P$;
2. $P \implies Q \equiv \neg P \vee Q$;
3. $\neg(P \implies Q) \equiv P \wedge \neg Q$;
4. $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$.

Proof. Each part is checked by comparing columns of one truth table:

P	Q	$P \implies Q$	$\neg Q \implies \neg P$	$\neg P \vee Q$	$P \wedge \neg Q$	$Q \implies P$	$P \iff Q$
T	T	T	T	T	F	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	F
F	F	T	T	T	F	T	T

Agreement of the third, fourth and fifth columns in every row proves the first two parts. The sixth column is the negation of the third row by row, as the third part states. For the fourth part the conjunction of the third and seventh columns is T exactly in the first and last rows, the same rows in which the eighth column is T. ∎

The first part of Theorem 1.1.10 justifies proof by contrapositive in Section 1.3. The third part says what it takes to refute an implication: one case in which the hypothesis is true and the conclusion false. The converse is absent from the theorem. It is not equivalent to the implication: in the second row of the table $P \implies Q$ is false while $Q \implies P$ is true.

Theorem 1.1.11 (De Morgan's laws for statements)**.** *For statements P and Q,*

$$\neg\neg P \equiv P, \qquad \neg(P \wedge Q) \equiv \neg P \vee \neg Q, \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q. \tag{1.1.1}$$

Proof. Negating twice exchanges T and F twice and returns the original value. For the second law, $\neg(P \wedge Q)$ is false exactly when $P \wedge Q$ is true, that is, when both P and Q are true. The disjunction $\neg P \vee \neg Q$ is false exactly when both $\neg P$ and $\neg Q$ are false, that is, again when both P and Q are true. Two statements false in exactly the same rows are true in exactly the same rows. For the third law, apply the second to $\neg P$ and $\neg Q$ to get $\neg(\neg P \wedge \neg Q) \equiv \neg\neg P \vee \neg\neg Q \equiv P \vee Q$ and negate both sides. ∎

Common misconception

"An implication and its converse say the same thing." They do not. The implication "if n is divisible by 4 then n is even" is true for every integer n. Its converse fails at $n = 2$. The statement equivalent to an implication is its contrapositive: "if n is odd then n is not divisible by 4".

Caution

In a *definition* the word "if" means "if and only if". "An integer is even if it is twice an integer" defines the even integers to be exactly the integers of that form. In a *theorem* "if" means only "if" and the converse needs a proof of its own.

Worked Example 1.1.1

Show that $(P \implies Q) \wedge (Q \implies P)$ is true exactly when P and Q have the same truth value. Then use a truth table to show that $(P \wedge Q) \implies P$ is a tautology.

The conjunction $(P \implies Q) \wedge (Q \implies P)$ is false when either implication is false. By Definition 1.1.6 the first is false only in the row $P = \mathrm{T}$, $Q = \mathrm{F}$ and the second only in the row $P = \mathrm{F}$, $Q = \mathrm{T}$. The conjunction is therefore true in exactly the two rows in which P and Q agree. These are the rows in which $P \iff Q$ is true, as the fourth part of Theorem 1.1.10 states.

For the second statement,

P	Q	$P \wedge Q$	$(P \wedge Q) \implies P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

In the first row the hypothesis and the conclusion are both true. In the other three rows the hypothesis $P \wedge Q$ is false and the implication is vacuously true. Every row is T and the statement is a tautology.

Worked Example 1.1.2

For an integer n consider the implication "if n^2 is even then n is even". Write down its converse, its contrapositive and its negation, and decide which of them is true for every integer n.

With P the statement "n^2 is even" and Q the statement "n is even":

- converse $Q \implies P$: if n is even then n^2 is even;
- contrapositive $\neg Q \implies \neg P$: if n is odd then n^2 is odd;

- negation $P \wedge \neg Q$ (Theorem 1.1.10): n^2 is even and n is odd.

The converse is true: $n = 2k$ gives $n^2 = 2(2k^2)$. The contrapositive is true: $n = 2k + 1$ gives $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. The original implication is equivalent to its contrapositive and is therefore true for every n. Its negation is false for every n. The implication and its converse together give the biconditional "n^2 is even if and only if n is even", proved in full as Theorem 1.3.7.

Study guide

Learn Table 1.1.1 until you can reproduce it without thought. The row to watch is the implication with a false hypothesis: it is true. When you meet an implication, write down its contrapositive and its converse. The contrapositive says the same thing and the converse says something else. To refute $P \implies Q$ you need exactly one case where P is true and Q is false. A truth table settles any question of logical equivalence between statements built from finitely many parts. The two laws to have ready are $\neg(P \implies Q) \equiv P \wedge \neg Q$ and De Morgan's laws.

Problems

1.1.1 Decide the truth value of each statement.

(a)If $3 < 2$ then 5 is even.

(b)If $3 > 2$ then 5 is even.

(c)If 5 is even then $3 > 2$.

(d)$3 > 2$ if and only if 5 is odd.

⋆ Hint · Answer · Solution

1.1.2 For a real number x consider "if $x > 3$ then $x^2 > 9$". Write its converse, its contrapositive and its negation. Which of the four is true for every real x?

⋆ Hint · Answer · Solution

1.1.3 Use a truth table to show that $((P \implies Q) \implies P) \equiv P$.

⋆⋆ Hint · Answer · Solution

1.1.4 Show that $P \implies (Q \implies R) \equiv (P \wedge Q) \implies R$. Then show that $(P \implies Q) \implies R$ and $P \implies (Q \implies R)$ are *not* equivalent. ⋆⋆ Hint · Answer · Solution

1.1.5 Rewrite each sentence as an implication about an integer n and decide whether it is true for every n.

(a)Divisibility of n by 4 is sufficient for n to be even.

(b)Divisibility of n by 4 is necessary for n to be even.

(c)n is even only if n^2 is even.

⋆⋆ Hint · Answer · Solution

1.1.6 Express $P \vee Q$ and $P \wedge Q$ using only the connectives $\neg$ and $\implies$, and verify each expression. ⋆⋆⋆ Hint · Answer · Solution

1.2 Quantifiers

Overview

Objectives: Turn an open sentence into a statement with a quantifier; decide the truth of a quantified statement and refute a universal one by a counterexample; read nested quantifiers in order and explain why "for every ε there is a δ" differs from "there is a δ for every ε"; negate a nested quantified statement mechanically; state boundedness, continuity and the limit of a sequence in quantifiers.
Prerequisites: Connectives, implication and De Morgan's laws (Section 1.1).
Key results: Negation of quantifiers (Theorem 1.2.4); $\exists y\,\forall x$ implies $\forall x\,\exists y$ and not conversely (Theorem 1.2.6); the negation procedure for nested statements.

Notation

$\forall x \in S$	for every x in the set S
$\exists x \in S$	there exists at least one x in S
$\exists!\, x \in S$	there exists exactly one x in S
$\mathbb{N},\ \mathbb{Z}$	natural numbers $1, 2, 3, \ldots$; all integers
$\mathbb{Q},\ \mathbb{R}$	rational numbers; real numbers

The open sentence $x > 2$ of Example 1.1.2 has no truth value. It acquires one in two ways: by a value for x, or by a claim about how many values of x make it true. "Every real x satisfies $x > 2$" is a false statement and "some real x satisfies $x > 2$" is a true one. Every definition of analysis is built in the second way. A set of real numbers is bounded above when *some* number exceeds *every* element. A limit exists when for *every* tolerance there is *some* response. Reading these definitions correctly, and negating them correctly, is most of the work of Chapter 5.

Definition 1.2.1 (Quantifiers)**.** Let $P(x)$ be an open sentence whose variable x ranges over a set S.

- The *universal* statement $\forall x \in S,\ P(x)$, read "for every x in S, $P(x)$", is true when $P(x)$ is true for each element x of S and false otherwise.
- The *existential* statement $\exists x \in S,\ P(x)$, read "there exists x in S such that $P(x)$", is true when $P(x)$ is true for at least one element x of S and false otherwise.

The variable x is *bound* by the quantifier. Renaming it changes nothing: $\forall x \in S,\ P(x)$ and $\forall t \in S,\ P(t)$ are the same statement.

Example 1.2.2. The statement $\forall x \in \mathbb{R},\ x^2 \geq 0$ is true. The statement $\forall x \in \mathbb{R},\ x^2 > 0$ is false because the single element $x = 0$ gives the false inequality $0 > 0$. An element that makes a universal statement fail is a *counterexample*. The statement $\exists x \in \mathbb{R},\ x^2 = 4$ is true with $x = 2$. The statement $\exists n \in \mathbb{N},\ n + 1 = 0$ is false because $n + 1 \geq 2$ for every natural number n.

The set over which a variable ranges is part of the statement. The statement $\exists x \in \mathbb{R},\ x^2 = 2$ is true by the completeness of $\mathbb{R}$ (Section 2.4). The statement $\exists x \in \mathbb{Q},\ x^2 = 2$ is false by Theorem 2.1.3.

Over the empty set every universal statement is vacuously true and every existential statement false. "Every element of $\emptyset$ is negative" has no counterexample and "some element of $\emptyset$ is negative" has no example.

Quantifiers are often restricted by a condition, as in "for every $\varepsilon > 0$". A restricted universal statement is an implication in disguise and a restricted existential statement is a conjunction:

$$\forall x \in S\ (x > 0 \implies P(x)) \qquad \text{and} \qquad \exists x \in S\ (x > 0 \wedge P(x)). \tag{1.2.1}$$

The first is written $\forall x > 0,\ P(x)$ and the second $\exists x > 0,\ P(x)$ when S is understood.

1.2.3 Negation of quantifiers

Theorem 1.2.4 (Negating a quantifier)**.** *For an open sentence $P(x)$ on a set S,*

$$\neg(\forall x \in S,\ P(x)) \equiv \exists x \in S,\ \neg P(x) \qquad \textit{and} \qquad \neg(\exists x \in S,\ P(x)) \equiv \forall x \in S,\ \neg P(x). \tag{1.2.2}$$

Proof. The statement $\forall x \in S,\ P(x)$ is false exactly when $P(x)$ fails for at least one $x \in S$, that is, exactly when $\exists x \in S,\ \neg P(x)$ is true. The statement $\exists x \in S,\ P(x)$ is false exactly when no $x \in S$ makes $P(x)$ true, that is, when $\neg P(x)$ is true for every $x \in S$. ∎

Negation passes through a quantifier and flips it. A restricted quantifier flips while keeping its restriction, as in (1.2.1): by (1.2.2) and the third part of Theorem 1.1.10,

$$\neg(\forall \varepsilon > 0,\ P(\varepsilon)) \equiv \exists \varepsilon\ \neg(\varepsilon > 0 \implies P(\varepsilon)) \equiv \exists \varepsilon > 0,\ \neg P(\varepsilon). \tag{1.2.3}$$

The negation of "for every $\varepsilon > 0$" is "for some $\varepsilon > 0$", never "for every $\varepsilon \leq 0$".

Negation procedure

To negate a statement built from quantifiers and connectives, write it out in symbols and work from left to right.

1. Change each $\forall$ to $\exists$ and each $\exists$ to $\forall$. Keep every restriction ($\varepsilon > 0$, $x \in S$, $n > N$) as it is.
2. Negate the statement that remains after the last quantifier, using $\neg(P \implies Q) \equiv P \wedge \neg Q$ and De Morgan's laws (Theorems 1.1.10 and 1.1.11).
3. Translate back into words and check the result by asking what an example of the negation would look like.

1.2.5 Order of quantifiers

A statement with two quantifiers of the same kind does not depend on their order: $\forall x\, \forall y$ and $\forall y\, \forall x$ say the same thing, as do $\exists x\, \exists y$ and $\exists y\, \exists x$. Quantifiers of different kinds do not commute. Consider the two statements

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ y > x \qquad \text{and} \qquad \exists y \in \mathbb{R}\ \forall x \in \mathbb{R},\ y > x. \tag{1.2.4}$$

The first is true: given x, the number $y = x + 1$ exceeds it. The choice of y depends on x. The second claims a single real number y larger than every real number including y itself and is false. In the first statement y is chosen after x is known. In the second y is chosen first and must then work for every x.

Theorem 1.2.6 (Order of quantifiers)**.** *Let $P(x, y)$ be an open sentence with x ranging over S and y over T. Then*

$$(\exists y \in T\ \forall x \in S,\ P(x, y)) \implies (\forall x \in S\ \exists y \in T,\ P(x, y)), \tag{1.2.5}$$

and the converse implication is false in general.

Proof. Suppose $y_0 \in T$ satisfies $P(x, y_0)$ for every $x \in S$. Given any $x \in S$, the element $y = y_0$ satisfies $P(x, y)$. The right-hand statement is therefore true. The pair of statements (1.2.4), with $P(x, y)$ the sentence $y > x$ on $S = T = \mathbb{R}$, has a true right-hand side and a false left-hand side and refutes the converse. ∎

1.2.7 Quantifiers in calculus

Three definitions met later in the book show the pattern.

Boundedness. A set S of real numbers is *bounded above* (Definition 2.4.1) when

$$\exists M \in \mathbb{R}\ \forall x \in S,\ x \leq M. \tag{1.2.6}$$

By the negation procedure applied to (1.2.6), S is *not* bounded above when $\forall M \in \mathbb{R}\ \exists x \in S,\ x > M$: every proposed bound is beaten by some element. Swapping the quantifiers gives $\forall x \in S\ \exists M,\ x \leq M$. This swapped statement is true for every set (take $M = x$) and therefore says nothing about S.

Limit of a sequence. A sequence $a_1, a_2, a_3, \ldots$ converges to L (Definition 5.1.3) when

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n > N \implies |a_n - L| < \varepsilon). \tag{1.2.7}$$

Informally: whatever tolerance ε is named, from some index on every term lies within ε of L. The index N is chosen after ε and may depend on it. The negation procedure applied to (1.2.7) gives the meaning of "a_n does not converge to L":

$$\exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \in \mathbb{N}\ (n > N \wedge |a_n - L| \geq \varepsilon). \tag{1.2.8}$$

Some tolerance is violated by terms arbitrarily far along the sequence.

Continuity. A function f is continuous at a (Definition 5.3.1) when

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ (|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon). \tag{1.2.9}$$

The negation of (1.2.9) is

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ (|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon). \tag{1.2.10}$$

Exchanging the first two quantifiers of (1.2.9) produces the far stronger statement studied in the next example.

Example 1.2.8. Suppose that f satisfies the swapped statement

$$\exists \delta > 0\ \forall \varepsilon > 0\ \forall x\ (|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon). \tag{1.2.11}$$

Fix that δ and a point x with $|x - a| < \delta$. The nonnegative number $|f(x) - f(a)|$ is then smaller than every $\varepsilon > 0$. If it were positive, the choice $\varepsilon = |f(x) - f(a)|$ would give a number smaller than itself. Hence $f(x) = f(a)$ and f is constant on the interval $(a - \delta, a + \delta)$. The function $f(x) = 2x$ is continuous at every point and constant on no interval. For $f(x) = 2x$ the definition is true and the swapped statement false. In "for every ε there is a δ" the δ may shrink as ε shrinks. In "there is a δ for every ε" one δ must serve every ε at once.

Common misconception

"The order of the quantifiers is a matter of style." It is part of the meaning. "Every person has a mother" is $\forall p\,\exists m$ and is true. "Some woman is the mother of every person" is $\exists m\,\forall p$ and is false. Read a nested statement from left to right. Each $\exists$ may depend on every variable introduced before it and on none introduced after it.

Worked Example 1.2.1

A function f is *bounded* on a set A when $\exists M \in \mathbb{R}\ \forall x \in A,\ |f(x)| \leq M$. Write the negation and use it to prove that $f(x) = 1/x$ is not bounded on the open interval $(0, 1)$.
By the negation procedure, f is unbounded on A when

$$\forall M \in \mathbb{R}\ \exists x \in A,\ |f(x)| > M.$$

To prove a statement of the form $\forall M\,\exists x$, take an arbitrary M and exhibit an x that may depend on M. Given M, put $x = 1/(|M| + 2)$. Then $0 < x \leq \frac{1}{2}$ and $x \in (0, 1)$, and

$$|f(x)| = |M| + 2 > |M| \geq M.$$

Every M is beaten and $1/x$ is unbounded on $(0, 1)$. The term 2 in the denominator keeps x inside $(0, 1)$ even when M is negative or zero.

Worked Example 1.2.2

Prove that the sequence $a_n = (-1)^n$ converges to no real number L.
Fix a real number L. By the negation of convergence it suffices to find an $\varepsilon > 0$ such that for every N some $n > N$ has $|a_n - L| \geq \varepsilon$. Take $\varepsilon = 1$ and let N be given. The two consecutive indices $n = 2N + 1$ and $n = 2N + 2$ both exceed N and give $a_n = -1$ and $a_n = 1$. By the triangle inequality (Lemma 2.3.8),

$$2 = |1 - (-1)| \leq |1 - L| + |L - (-1)|.$$

Since the two terms on the right sum to at least 2, one of them is at least 1. For that index $n > N$ we have $|a_n - L| \geq 1$. Since L was arbitrary, the sequence converges to no real number.

Study guide

Before reading a quantified definition, write it in symbols. Read it from left to right as a game: the player who chooses at a $\forall$ is an opponent and the player who chooses at an $\exists$ is you. You choose with knowledge of every earlier move. To prove $\forall\varepsilon\,\exists\delta$, start "Let $\varepsilon > 0$" and construct δ from ε. To refute it, find one ε that defeats every δ. Negate by flipping each quantifier, keeping each restriction, and negating the final clause with $\neg(P \implies Q) \equiv P \wedge \neg Q$. Never swap $\forall$ and $\exists$ without a proof that the swap is allowed.

Problems

1.2.1 Decide the truth value of each statement and justify your answer.

(a) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ x + y = 0.$

(b) $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R},\ x + y = 0.$

(c)$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ xy = 1.$

(d)$\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z},\ xy = 0.$

⋆ Hint · Answer · Solution

1.2.2 Negate each statement mechanically and decide which of the statement and its negation is true.

(a)$\forall x \in \mathbb{R}\ \exists n \in \mathbb{N},\ n > x.$

(b)$\exists x \in \mathbb{R}\ \forall y \in \mathbb{R},\ y^2 \geq x.$

⋆ Hint · Answer · Solution

1.2.3 A function f is *strictly increasing* on an interval I when $\forall x, y \in I\ (x < y \implies f(x) < f(y))$. Write the negation and show that $f(x) = x^2$ is not strictly increasing on $\mathbb{R}$.

⋆ Hint · Answer · Solution

1.2.4 For the sequence $a_n = 1/n$ and the limit $L = 0$, find the least N such that $n > N$ implies $|a_n - 0| < \varepsilon$, first for $\varepsilon = \frac{1}{100}$ and then for $\varepsilon = 0.003$.

⋆ Hint · Answer · Solution

1.2.5 Prove that the sequence $a_n = 1/n$ does not converge to $\frac{1}{2}$.

⋆⋆ Hint · Answer · Solution

1.2.6 The statement $\exists!\, x \in S,\ P(x)$ means $\exists x \in S\ (P(x) \wedge \forall y \in S\ (P(y) \implies y = x))$. Write its negation in words and symbols, and decide the truth of $\exists!\, x \in \mathbb{R},\ x^2 = 4$ and of $\exists!\, x \in [0, \infty),\ x^2 = 4$.

⋆⋆⋆ Hint · Answer · Solution

1.3 Methods of Proof

Overview

Objectives: Write a direct proof, a proof by contrapositive and a proof by contradiction; refute a universal statement by a counterexample; split a proof into cases; prove an "if and only if" in two directions; prove existence and uniqueness separately; apply these methods to parity, divisibility and inequalities.
Prerequisites: Implication, contrapositive and negation (Section 1.1); quantifiers (Section 1.2).
Key results: Every integer is even or odd and not both (Lemma 1.3.3); n^2 is even if and only if n is even (Theorem 1.3.7); divisibility is transitive and passes to sums (Theorem 1.3.5); $x + 1/x \geq 2$ for $x > 0$ with equality only at $x = 1$ (Theorem 1.3.17).

Notation

$a \mid b$ a divides b

A proof is a finite chain of statements. Each of them is an assumption, a definition, a previously proved result, or a consequence of earlier statements in the chain by a rule of logic. Its last statement is the claim. No list of checked cases, however long, proves a statement about infinitely many objects (Example 1.3.11). Only a proof establishes such a statement. This section describes the standard shapes a proof takes. The examples use the integers because the facts needed about them are few and familiar. They are the arithmetic and the order of $\mathbb{Z}$ learnt in school. One consequence of the order is cited by number below: no integer lies strictly between 0 and 1. Equivalently,

$$m \in \mathbb{Z} \text{ and } m \neq 0 \implies |m| \geq 1. \tag{1.3.1}$$

Definition 1.3.1 (Parity and divisibility)**.** An integer n is *even* if $n = 2k$ for some integer k and *odd* if $n = 2k+1$ for some integer k. For integers a and b, a *divides* b, written $a \mid b$, if $b = ak$ for some integer k.

Example 1.3.2. The integer $-6 = 2 \cdot (-3)$ is even and $7 = 2 \cdot 3 + 1$ is odd. Zero is even because $0 = 2 \cdot 0$. The relation $3 \mid 12$ holds with $k = 4$, and $3 \mid 0$ with $k = 0$. The relation $3 \mid 7$ fails because $7/3$ is not an integer. The relation $0 \mid 5$ fails because $0 \cdot k = 0 \neq 5$ for every integer k.

Lemma 1.3.3 (Every integer is even or odd)**.** *Every integer is even or odd and no integer is both.*

Proof. Suppose some integer were both, with $2k = 2l + 1$. Then $2(k - l) = 1$. If the integer $k - l$ is 0 then $2(k - l) = 0$. Otherwise $|k - l| \geq 1$ by (1.3.1) and $|2(k - l)| \geq 2$. Neither equals 1. This is a contradiction.

For existence we first treat $n \geq 0$ by induction from the base 0 (Theorem 1.4.2 and Corollary 1.4.3), proved in Section 1.4 without any use of this lemma. The integer $0 = 2 \cdot 0$ is even. If $n = 2k$ then $n + 1 = 2k + 1$ is odd. If $n = 2k + 1$ then $n + 1 = 2(k + 1)$ is even. Every $n \geq 0$ is therefore even or odd. For $n < 0$ apply this to $-n > 0$: $-n = 2k$ gives $n = 2(-k)$ and $-n = 2k + 1$ gives $n = 2(-k - 1) + 1$. ∎

1.3.4 Direct proof

To prove $P \implies Q$ directly, assume P and deduce Q. To prove $\forall x \in S,\ P(x)$, let x be an arbitrary element of S, assume nothing about it beyond membership of S and prove $P(x)$.

Theorem 1.3.5 (Divisibility rules)**.** *Let a, b, c be integers.*

1. *If $a \mid b$ and $b \mid c$, then $a \mid c$.*
2. *If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for all integers x and y.*

Proof. (1) By Definition 1.3.1 there are integers k and l with $b = ak$ and $c = bl$. Then $c = (ak)l = a(kl)$ with kl an integer. Hence $a \mid c$.

(2) Write $b = ak$ and $c = al$ with integers k and l. Then $bx + cy = akx + aly = a(kx + ly)$ with $kx + ly$ an integer. ∎

A direct proof begins by writing out the definitions in the hypothesis ($b = ak$ for some k) and ends by repacking the conclusion into the form a definition requires ($c = a \cdot$ integer). Most short proofs consist of nothing else.

1.3.6 Proof by contrapositive

By Theorem 1.1.10 a proof of $\neg Q \implies \neg P$ is a proof of $P \implies Q$. This route helps when $\neg Q$ gives something to compute with and P does not.

Theorem 1.3.7 (Parity of squares)**.** *For every integer n, n^2 is even if and only if n is even.*

Proof. A biconditional is the conjunction of two implications (Theorem 1.1.10). We prove each in turn.

($\Leftarrow$) If $n = 2k$ then $n^2 = 4k^2 = 2(2k^2)$ is even.

($\Rightarrow$) We prove the contrapositive: if n is not even then n^2 is not even. By Lemma 1.3.3 an integer that is not even is odd. Write $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd and, by Lemma 1.3.3 again, not even. ∎

A direct attempt at ($\Rightarrow$) starts from $n^2 = 2m$ and needs to extract information about n from a statement about n^2 with no obvious way to do so. The contrapositive starts from a formula for n and computes n^2.

1.3.8 Proof by contradiction

To prove P by contradiction, assume $\neg P$ and deduce a statement known to be false, typically of the form $R \wedge \neg R$. Since a true assumption cannot lead by valid steps to a false conclusion, $\neg P$ is false and P is true. For an implication $P \implies Q$ the assumption is its negation $P \wedge \neg Q$ (Theorem 1.1.10).

Example 1.3.9. There is no smallest positive real number. Suppose there were one, $s > 0$. Then $s/2$ is a real number with $0 < s/2 < s$. This contradicts the choice of s. Contradiction suits statements of nonexistence because their negation, an existence statement, provides an object to work with. Irrationality of $\sqrt{2}$ (Theorem 2.1.3) is proved in the same way.

1.3.10 Counterexample

A universal statement $\forall x \in S,\ P(x)$ is false as soon as a single $x \in S$ makes $P(x)$ false (Theorem 1.2.4). To refute a universal claim, exhibit one counterexample and verify it. No further argument is needed.

Example 1.3.11. The polynomial $p(n) = n^2 + n + 41$ takes prime values at $n = 0, 1, 2, \ldots, 39$. Each case can be checked directly. The claim "$p(n)$ is prime for every natural number n" is nevertheless false. At $n = 40$,

$$p(40) = 40^2 + 40 + 41 = 40 \cdot 41 + 41 = 41^2 = 1681.$$

Forty confirming cases do not prove a universal statement. One counterexample refutes it.

1.3.12 Proof by cases

When the hypothesis splits into finitely many cases that together cover every possibility, prove the conclusion in each case. The cases need not be exclusive. They must be exhaustive.

Example 1.3.13. For every integer n, $n^2 + n$ is even. By Lemma 1.3.3 every integer is even or odd. If $n = 2k$ then $n^2 + n = 2(2k^2 + k)$. If $n = 2k + 1$ then $n^2 + n = (2k + 1)(2k + 2) = 2(2k + 1)(k + 1)$. In both cases $n^2 + n$ is even.

1.3.14 Existence and uniqueness

A statement $\exists!\, x \in S,\ P(x)$ is two claims, proved separately. *Existence*: some $x \in S$ satisfies $P(x)$, usually shown by producing it. *Uniqueness*: if x and y in S both satisfy P, then $x = y$.

Example 1.3.15. For real numbers $a \neq 0$ and b, the equation $ax + b = 0$ has exactly one real solution. Existence: $x = -b/a$ gives $a(-b/a) + b = 0$. Uniqueness: if $ax + b = 0$ and $ay + b = 0$, subtracting gives $a(x - y) = 0$. Dividing by $a \neq 0$ gives $x = y$. An existence proof need not produce the object. The Intermediate Value Theorem (Theorem 5.4.2) proves that $x^5 + x - 1 = 0$ has a real root without giving a formula for it.

1.3.16 Inequalities

Proofs of inequalities rest on one fact: the square of a real number is nonnegative and is zero only for the number 0.

Theorem 1.3.17 (Sum of a number and its reciprocal)**.** *For every real $x > 0$, $x + \dfrac{1}{x} \geq 2$, with equality if and only if $x = 1$.*

Proof. For $x > 0$,

$$x + \frac{1}{x} - 2 = \frac{x^2 - 2x + 1}{x} = \frac{(x - 1)^2}{x}.$$

The numerator is a square and hence at least 0. The denominator is positive. The quotient is therefore at least 0 as claimed. The quotient is 0 if and only if $(x - 1)^2 = 0$, that is, if and only if $x = 1$. ∎

The proof was found by working backwards from the claim to the square $(x - 1)^2$ and is written forwards from that square to the claim. A chain that starts from the conclusion and reaches a true statement proves nothing unless every step can be reversed.

Caution

Starting from the claim $x + 1/x \geq 2$ and multiplying by x to reach $(x - 1)^2 \geq 0$ is a sound way to *discover* the proof because each step is reversible when $x > 0$. It is not a proof as it stands. From the false statement $-1 = 1$, squaring gives the true statement $1 = 1$. A true conclusion never certifies the statement it came from.

Common misconception

"A proof by contradiction may begin by assuming the conclusion." It begins by assuming the *negation* of the conclusion. Assuming the conclusion and then deriving something true establishes nothing.

Worked Example 1.3.1

Prove that for every integer n, n is odd if and only if $n^2 + 2n$ is odd.
($\Rightarrow$) Direct. Let $n = 2k + 1$. Then $n^2 + 2n = n(n + 2) = (2k + 1)(2k + 3) = 4k^2 + 8k + 3 = 2(2k^2 + 4k + 1) + 1$. The result is odd.
($\Leftarrow$) Contrapositive: if n is not odd then $n^2 + 2n$ is not odd. By Lemma 1.3.3, n is even. Write $n = 2k$. Then $n^2 + 2n = 4k^2 + 4k = 2(2k^2 + 2k)$ is even and, by Lemma 1.3.3, not odd.

Worked Example 1.3.2

Prove that the equation $x^2 - y^2 = 2$ has no solution in integers x and y.
We argue by contradiction. Suppose integers x and y satisfy $x^2 - y^2 = 2$, that is, $(x - y)(x + y) = 2$. The two factors differ by $2y$ and therefore have the same parity. If both are even, say $x - y = 2k$ and $x + y = 2l$, their product $4kl$ is divisible by 4 and $4 \nmid 2$. If both are odd their product is odd and 2 is even. Both cases contradict $(x - y)(x + y) = 2$. No such integers exist. The claim that the factors have the same parity uses $(x + y) - (x - y) = 2y$: if one factor were even and the other odd, their difference $2y$ would be odd by Lemma 1.3.3.

Study guide

Before writing, identify the logical form of the claim: implication, biconditional, universal, existential or unique existence. Choose the method by the form. Start every proof by writing the hypotheses in terms of definitions. Know the target expression before you begin. Try a direct proof first. Switch to the contrapositive when the negated conclusion is the easier thing to compute with. Use contradiction for statements of nonexistence or irrationality. A universal statement is refuted by one checked counterexample. A biconditional needs both directions. Unique existence needs both existence and uniqueness.

Problems

1.3.1 Prove that the product of two odd integers is odd and that the sum of two odd integers is even. ⋆ Hint · Answer · Solution

1.3.2 Prove that if n^2 is odd then n is odd. ⋆ Hint · Answer · Solution

1.3.3 Give a counterexample to each statement.

(a) For every real x, if $x^2 > 1$ then $x > 1$.

(b) Every odd integer greater than 1 is prime.

(c) For all positive reals a and b, $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$.

⋆ Hint · Answer · Solution

1.3.4 Prove that $a^2 + b^2 \geq 2ab$ for all real a and b, with equality if and only if $a = b$. Deduce that $\dfrac{a + b}{2} \geq \sqrt{ab}$ for all $a, b \geq 0$. ⋆⋆ Hint · Answer · Solution

1.3.5 Prove by cases that $3n^2 + n + 14$ is even for every integer n.

⋆⋆ Hint · Answer · Solution

1.3.6 Prove that $x = 2$ is the unique real number with $x^3 = 8$.

⋆⋆ Hint · Answer · Solution

1.4 Mathematical Induction

Overview

Objectives: State the principle of induction and derive it from the well-ordering property; prove formulas for sums and inequalities such as $2^n > n$ by induction; start an induction at any integer; use strong induction when a step needs more than the previous case; locate the error in a faulty induction; divide integers with remainder and prove Bézout's identity and Euclid's lemma from well-ordering.
Prerequisites: Methods of proof, in particular proof by contradiction (Section 1.3).
Key results: Principle of mathematical induction (Theorem 1.4.2); induction from any base (Corollary 1.4.3); strong induction (Theorem 1.4.8 and Corollary 1.4.9); the sum formulas of Theorem 1.4.6; every integer $n \geq 2$ is a product of primes and there are infinitely many primes (Theorem 1.4.10); division with remainder (Theorem 1.4.12); Bézout's identity (Theorem 1.4.14); Euclid's lemma (Theorem 1.4.15) and lowest terms (Corollaries 1.4.16 and 1.4.17).

Notation

$P(n)$	a statement depending on the integer n
$\sum_{k=1}^{n} a_k$	the sum $a_1 + a_2 + \cdots + a_n$
$n!$	$1 \cdot 2 \cdots n$ for $n \in \mathbb{N}$, and $0! = 1$
$\gcd(a, b)$	the greatest common divisor of a and b

Many statements in calculus are claims about every natural number n: a formula for $1 + 2 + \cdots + n$, the inequality $2^n > n$, the nth derivative of $x \mapsto e^{2x}$. No finite amount of checking proves such a claim (Example 1.3.11). Induction proves it in two finite steps by a property of the natural numbers themselves.

Assumption 1.4.1 (Well-ordering of $\mathbb{N}$)**.** Every nonempty set of natural numbers has a least element.

We take well-ordering as a basic property of $\mathbb{N} = \{1, 2, 3, ...\}$. It fails for other familiar sets. The set $\mathbb{Z}$ has no least element, and the set of positive rationals has none either, by the argument of Example 1.3.9.

Theorem 1.4.2 (Principle of mathematical induction)**.** *For each $n \in \mathbb{N}$ let $P(n)$ be a statement. Suppose that*

1. (base case) $P(1)$ is true, and

2. (inductive step) for every $k \in \mathbb{N}$, $P(k) \implies P(k+1)$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Proof. Suppose for a contradiction that the set $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$ is nonempty. By Assumption 1.4.1 it has a least element m. By the base case $m \neq 1$. Hence $m \geq 2$ and $k = m-1$ is a natural number. Because $k < m$ and m is the least element of S, the number k is not in S and $P(k)$ is true. The inductive step then makes $P(k+1) = P(m)$ true, contradicting $m \in S$. Hence S is empty. ∎

In the inductive step, $P(k)$ is the *inductive hypothesis*. The step proves an implication: it assumes $P(k)$ for an arbitrary k and deduces $P(k+1)$. It makes no assertion that $P(k)$ is true. The base case then starts a chain $P(1) \implies P(2) \implies P(3) \implies \cdots$ that reaches every natural number.

Corollary 1.4.3 (Induction from any base)**.** *Let n_0 be an integer and let $P(n)$ be a statement for each integer $n \geq n_0$. If $P(n_0)$ is true and $P(k) \implies P(k+1)$ for every integer $k \geq n_0$, then $P(n)$ is true for every integer $n \geq n_0$.*

Proof. For $m \in \mathbb{N}$ let $Q(m)$ be the statement $P(m + n_0 - 1)$. Then $Q(1) = P(n_0)$ is true. The implication $Q(m) \implies Q(m+1)$ is the hypothesis $P(k) \implies P(k+1)$ with $k = m+n_0-1 \geq n_0$. By Theorem 1.4.2, $Q(m)$ is true for every $m \in \mathbb{N}$. That statement is $P(n)$ for every $n \geq n_0$. ∎

Example 1.4.4. For every $n \in \mathbb{N}$, $2^n > n$. The base case is $2^1 = 2 > 1$. For the inductive step suppose $2^k > k$ for some $k \in \mathbb{N}$. Then

$$2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1,$$

the last step because $k \geq 1$. Hence $2^{k+1} > k + 1$. By Theorem 1.4.2 the inequality holds for every n. A closely related inequality, Bernoulli's inequality $(1 + x)^n \geq 1 + nx$ for real $x \geq -1$, is proved by the same method in Theorem 2.3.11.

1.4.5 Sum formulas

Write $\sum_{k=1}^{n} a_k$ for $a_1 + a_2 + \cdots + a_n$. Sigma notation is developed in Section 4.6. Here we need only the recursion $\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}$ through which an inductive step reaches a sum.

Theorem 1.4.6 (Power sums)**.** *For every $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{1.4.1}$$

Proof. We prove the first formula. The other two are Problem 1.4.1 and Problem 1.4.2. Let $P(n)$ be the statement $\sum_{k=1}^{n} k = n(n+1)/2$. The base case $P(1)$ is the true equation $1 = 1 \cdot 2/2$. Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2},$$

the second equality by the inductive hypothesis. The result is $P(k+1)$, and Theorem 1.4.2 completes the proof. ∎

These are the formulas Eqs. (4.6.7) to (4.6.9) used to evaluate Riemann sums. Induction verifies a formula without finding it. The first formula can also be found by writing the sum forwards and backwards and adding, as Gauss is said to have done.

1.4.7 Strong induction

Some inductive steps need more than the single previous case. To show that an integer n is a product of primes, one factors $n = ab$ with a and b smaller than n and possibly far from $n - 1$.

Theorem 1.4.8 (Strong induction)**.** *For each $n \in \mathbb{N}$ let $P(n)$ be a statement. Suppose that $P(1)$ is true and that for every $k \in \mathbb{N}$, if $P(1), P(2), \ldots, P(k)$ are all true then $P(k+1)$ is true. Then $P(n)$ is true for every $n \in \mathbb{N}$.*

Proof. Let $Q(n)$ be the statement "$P(1), P(2), \dots, P(n)$ are all true". Then $Q(1) = P(1)$ is true. If $Q(k)$ is true then the hypothesis gives $P(k+1)$, and together with $Q(k)$ this is $Q(k+1)$. By Theorem 1.4.2, $Q(n)$ is true for every n. In particular $P(n)$ is true. ∎

Corollary 1.4.9 (Strong induction from any base)**.** *Let n_0 be an integer and let $P(n)$ be a statement for each integer $n \geq n_0$. Suppose that $P(n_0)$ is true and that for every integer $k \geq n_0$, if $P(n_0), \dots, P(k)$ are all true then $P(k+1)$ is true. Then $P(n)$ is true for every integer $n \geq n_0$.*

Proof. For $m \in \mathbb{N}$ let $Q(m)$ be the statement $P(m + n_0 - 1)$. Then $Q(1) = P(n_0)$ is true. If $Q(1), \dots, Q(m)$ are all true then $P(n_0), \dots, P(k)$ are all true with $k = m + n_0 - 1 \geq n_0$, and the hypothesis gives $P(k+1) = Q(m+1)$. By Theorem 1.4.8, $Q(m)$ is true for every $m \in \mathbb{N}$. That statement is $P(n)$ for every $n \geq n_0$. ∎

Strong induction is no stronger than ordinary induction because the proof of Theorem 1.4.8 derives it from Theorem 1.4.2. Its name refers to the stronger inductive hypothesis it allows.

An integer $p \geq 2$ is *prime* if its only positive divisors are 1 and p. An integer $n \geq 2$ that is not prime is *composite*: it has a divisor a with $1 < a < n$ and a factorisation $n = ab$ with $1 < b < n$ as well.

Theorem 1.4.10 (Prime factorisation and infinitude of primes)**.** *Every integer $n \geq 2$ is a product of one or more primes. There are infinitely many primes.*

Proof. Let $P(n)$ be the statement "n is a product of primes", for $n \geq 2$. The base case $P(2)$ is true because 2 is prime. Let $k \geq 2$ and suppose $P(2), \dots, P(k)$ are true. If $k+1$ is prime then $P(k+1)$ is true. If $k+1$ is composite, write $k+1 = ab$ with $2 \leq a, b \leq k$. By the inductive hypothesis a and b are products of primes. Their product $k+1$ is therefore a product of primes. By strong induction from the base 2 (Corollary 1.4.9), every $n \geq 2$ is a product of primes.

Suppose for a contradiction that there are only finitely many primes $p_1, p_2, \dots, p_m$. Let $N = p_1 p_2 \cdots p_m + 1$. Since $N \geq 2$, the first part gives a prime p dividing N with $p = p_i$ for some i. The prime p_i also divides $p_1 p_2 \cdots p_m$. By Theorem 1.3.5 it divides the difference $N - p_1 p_2 \cdots p_m = 1$. No prime divides 1. This is a contradiction. ∎

1.4.11 Division with remainder and greatest common divisors

Reducing a fraction to lowest terms and the proof of the rational root theorem (Theorem 4.2.11) both divide out common factors. They rest on three facts about the integers proved here from well-ordering: division with remainder, Bézout's identity and Euclid's lemma.

Theorem 1.4.12 (Division with remainder)**.** *Let a be an integer and b a natural number. There are unique integers q and r with*

$$a = qb + r, \qquad 0 \leq r < b. \tag{1.4.2}$$

Proof. Let S be the set of natural numbers of the form $a - qb + 1$ with q an integer and $a - qb \geq 0$. The choice $q = -|a|$ gives $a - qb = a + |a|b \geq a + |a| \geq 0$ because $b \geq 1$. Hence S is nonempty and has a least element $m = a - qb + 1$ by Assumption 1.4.1. Put $r = m - 1 = a - qb \geq 0$. If $r \geq b$ then $a - (q+1)b = r - b \geq 0$ and $m - b$ is an element of S smaller than m. This is a contradiction. Hence $0 \leq r < b$.

For uniqueness suppose $qb + r = q'b + r'$ with $0 \leq r, r' < b$. Then $r - r' = (q' - q)b$ and $|r - r'| < b$. If $q' \neq q$ then $|q' - q| \geq 1$ by (1.3.1) and $|r - r'| \geq b$. This is a contradiction. Hence $q = q'$ and $r = r'$. ∎

The integer r is the *remainder* on dividing a by b. For example $-7 = (-3) \cdot 3 + 2$ has remainder 2 on division by 3. Every integer is therefore of exactly one of the forms $3k$, $3k+1$ and $3k+2$. With $b = 2$ the theorem recovers Lemma 1.3.3.

Definition 1.4.13 (Greatest common divisor)**.** Let a and b be integers, not both 0. A *common divisor* of a and b is an integer that divides both. The *greatest common divisor* $\gcd(a, b)$ is the largest common divisor. The integers a and b are *coprime* if their only positive common divisor is 1, that is, if $\gcd(a, b) = 1$. A quotient r/s of integers with $s \in \mathbb{N}$ is *in lowest terms* when r and s are coprime.

A common divisor c of a and b with $a \neq 0$ satisfies $|c| \leq |a|$ because $a = ck$ with $k \neq 0$ gives $|a| = |c||k| \geq |c|$ by (1.3.1). The common divisors are therefore bounded above. Theorem 1.4.14 shows that a largest one exists. For example $\gcd(12, 18) = 6$. The integers 8 and 15 are coprime. As a non-example, $\gcd(0, 0)$ is undefined because every integer divides 0.

Theorem 1.4.14 (Bézout's identity)**.** *Let a and b be integers that are not both 0 and let d be the least element of the set*

$$S = \{ax + by : x, y \in \mathbb{Z},\ ax + by > 0\}. \tag{1.4.3}$$

Then d divides a and b. Every common divisor of a and b divides d. Hence $d = \gcd(a, b)$ is of the form $ax + by$ with integers x and y. In particular if a and b are coprime then $ax + by = 1$ for some integers x and y.

Proof. The set S of (1.4.3) consists of natural numbers and contains $a \cdot a + b \cdot b > 0$. By Assumption 1.4.1 it has a least element $d = ax_0 + by_0$. Divide a by d by (1.4.2): $a = qd + r$ with $0 \leq r < d$. Then

$$r = a - q(ax_0 + by_0) = a(1 - qx_0) + b(-qy_0) \tag{1.4.4}$$

is of the form $ax + by$. If $r > 0$ it would be an element of S smaller than d. Hence $r = 0$ and $d \mid a$. The same argument with the roles of a and b exchanged gives $d \mid b$. A common divisor c of a and b divides $ax_0 + by_0 = d$ by Theorem 1.3.5. Since $d \neq 0$, the argument of the remark after Definition 1.4.13, applied to $c \mid d$, gives $|c| \leq d$. The common divisor d is therefore the largest one. ∎

Theorem 1.4.15 (Euclid's lemma)**.** *Let r, a and b be integers with r and a coprime. If $r \mid ab$ then $r \mid b$.*

Proof. By Theorem 1.4.14 there are integers x and y with $rx + ay = 1$. Multiplying by b gives $b = r(bx) + (ab)y$. The integer r divides $r(bx)$ and, by hypothesis, ab. By Theorem 1.3.5 it divides b. ∎

The hypothesis that r and a are coprime cannot be dropped: $6 \mid 4 \cdot 3$ and $6 \nmid 3$ with $\gcd(6, 4) = 2$.

Corollary 1.4.16 (Powers of coprime integers)**.** *If r and s are coprime integers, then r and s^n are coprime for every $n \in \mathbb{N}$.*

Proof. By induction on n. The case $n = 1$ is the hypothesis. Suppose r and s^k are coprime and let d be a positive common divisor of r and $s^{k+1} = s \cdot s^k$. A positive common divisor of d and s divides r by Theorem 1.3.5 and is therefore 1. Hence d and s are coprime. Euclid's lemma

(Theorem 1.4.15) applied to $d \mid s \cdot s^k$ gives $d \mid s^k$. Then d is a positive common divisor of r and s^k and $d = 1$ by the inductive hypothesis. ∎

Corollary 1.4.17 (Existence of lowest terms)**.** *For integers m and n with $n \in \mathbb{N}$ there are coprime integers r and s with $s \in \mathbb{N}$ and $ms = rn$. In the notation of Definition 2.1.1, every rational number can be written in lowest terms.*

Proof. Since $n \neq 0$, the integer $d = \gcd(m, n) \geq 1$ exists. Write $m = dr$ and $n = ds$. Then $s \in \mathbb{N}$ and $ms = drs = rn$. If c is a positive common divisor of r and s, then cd divides m and n. Hence $cd \leq d$ and $c = 1$. ∎

1.4.18 Common errors

Both parts of an induction are needed. Either can fail on its own.

Example 1.4.19. Let $P(n)$ be the statement $n = n + 1$. If $P(k)$ is true, adding 1 to both sides gives $P(k + 1)$, the equation $k + 1 = k + 2$. The inductive step is valid. The base case $1 = 2$ is false, as is $P(n)$ for every n. A valid step with no base proves nothing.

A step can also be valid only from some point on. In Worked Example 1.4.2 the step from k to $k + 1$ uses $k^2 > 2k + 1$. That inequality is false for $k = 1$ and $k = 2$. An induction with base 1 would break at its first link even though the step is valid for every $k \geq 3$. Test the step at the smallest value of k it is meant to cover.

Common misconception

"The inductive step assumes what is to be proved." It assumes $P(k)$ for one value k and proves $P(k + 1)$ for the next. The conclusion $P(n)$ for every n is never assumed. An argument that assumes $P(n)$ for every n is circular, as is any argument that uses $P(k+1)$ to prove $P(k + 1)$. Write the inductive hypothesis out in full with the letter k and mark the line where you use it.

Worked Example 1.4.1

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every $n \in \mathbb{N}$.
Let $P(n)$ be the statement $\sum_{j=1}^{n}(2j - 1) = n^2$. Base case: for $n = 1$ the sum is $1 = 1^2$. Inductive step: suppose $P(k)$ is true for some $k \in \mathbb{N}$. The next term is $2(k+1)-1 = 2k+1$, and

$$\sum_{j=1}^{k+1}(2j - 1) = \sum_{j=1}^{k}(2j - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2,$$

using $P(k)$ at the second equality. The result is $P(k + 1)$. By Theorem 1.4.2 the formula holds for every n.

Worked Example 1.4.2

Prove that $2^n > n^2$ for every integer $n \geq 5$. Check that it fails at $n = 2, 3, 4$.
At $n = 2$, $2^2 = 4 = 2^2$; at $n = 3$, $8 < 9$; at $n = 4$, $16 = 16$. The inequality is strict and fails at all three. We use Corollary 1.4.3 with $n_0 = 5$. Base case: $2^5 = 32 > 25 = 5^2$. Inductive step: suppose $2^k > k^2$ for some integer $k \geq 5$. Then

$$2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2.$$

Since $k \geq 5$ we have $k^2 \geq 5k = 2k + 3k > 2k + 1$. Therefore $2^{k+1} > k^2 + 2k + 1 = (k+1)^2$. The inductive step needed $k \geq 3$ to give $k^2 > 2k + 1$. The base case needed $n_0 = 5$. Both parts of the induction set the range of n.

Study guide

Every induction proof has the same frame: state $P(n)$ explicitly, check the base case, write the inductive hypothesis with the letter k and derive $P(k+1)$ while marking where the hypothesis is used. For a sum, split off the last term. For an inequality, write the $(k+1)$st quantity in terms of the kth, apply the hypothesis and prove the small inequality that remains. That inequality usually needs k to be large enough and fixes the base. Use strong induction when $P(k+1)$ depends on cases other than $P(k)$. A faulty induction usually lacks a base case or has a step that fails for one small value of k. Test the step at the smallest k.

Problems

1.4.1 Prove that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for every $n \in \mathbb{N}$.

⋆ Hint · Answer · Solution

1.4.2 Prove that $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every $n \in \mathbb{N}$. ⋆⋆ Hint · Answer · Solution

1.4.3 Prove that $3 \mid (4^n - 1)$ for every $n \in \mathbb{N}$. ⋆⋆ Hint · Answer · Solution

1.4.4 Prove that for every real $r \neq 1$ and every integer $n \geq 0$, $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$. Evaluate $1 + 3 + 3^2 + \cdots + 3^5$. ⋆⋆ Hint · Answer · Solution

1.4.5 Find the error in the following argument. "Claim: for every $n \in \mathbb{N}$, $n^2 + n$ is odd. Suppose $k^2 + k$ is odd. Then $(k+1)^2 + (k+1) = (k^2 + k) + 2(k+1)$ is an odd number plus an even number and is odd. By induction the claim is true."

⋆⋆ Hint · Answer · Solution

1.4.6 Prove that every integer $n \geq 12$ can be written as $n = 4a + 5b$ with integers $a, b \geq 0$.

⋆⋆⋆ Hint · Answer · Solution

1.4.7 Find $\gcd(84, 120)$ and integers x and y with $84x + 120y = \gcd(84, 120)$.

⋆ Hint · Answer · Solution

1.4.8 Let p be a prime and a, b integers with $p \mid ab$. Prove that $p \mid a$ or $p \mid b$.

⋆⋆ Hint · Answer · Solution

1.4.9 Let a and b be coprime integers and let c be an integer with $a \mid c$ and $b \mid c$. Prove that $ab \mid c$. Show by an example that the hypothesis that a and b are coprime is needed. ⋆⋆⋆ Hint · Answer · Solution

1.5 Sets and Functions

Overview

Objectives: Describe a set by listing or by a defining property; prove that two sets are equal by double inclusion; form unions, intersections, complements and Cartesian products and apply De Morgan's laws; identify the domain, codomain and image of a function; compute images and preimages; decide whether a function is injective, surjective or bijective; compose functions; find an inverse, if necessary after restricting the domain.
Prerequisites: Quantifiers (Section 1.2); methods of proof (Section 1.3).
Key results: De Morgan's laws for sets (Theorem 1.5.6); the definitions of function, injective, surjective, bijective, composition and inverse (Definitions 1.5.8, 1.5.13 to 1.5.15, 1.5.20 and 1.5.24); a function has an inverse if and only if it is bijective (Theorem 1.5.26); the horizontal-line test (Theorem 1.5.17).

Notation

$x \in A$, $x \notin A$	x is (is not) an element of A
$\{x \in A : P(x)\}$	the elements of A that satisfy P
$\emptyset$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B,\ A \cap B,\ A \setminus B$	union, intersection, difference
$A^c,\ A \times B$	complement, Cartesian product
$f : A \to B$	a function from A to B
$f(S),\ f^{-1}(T)$	image of S, preimage of T
$g \circ f$	composition, $x \mapsto g(f(x))$
f^{-1}	inverse function of a bijection f
$f\|_S$	restriction of f to S

A function such as $f(x) = x^2$ is often treated as a formula alone. The formula does not settle whether f has an inverse. On all of $\mathbb{R}$ it has none because $f(2) = f(-2)$. On $[0, \infty)$ it has the inverse $\sqrt{x}$. The functions arcsin and ln of Chapter 4 exist only because domains are chosen with care. This section makes the domain and the target of a function part of its definition and builds the vocabulary of sets needed to do so.

1.5.1 Sets

A *set* is a collection of objects, its *elements*. A set is determined by its elements: two sets are equal when they have the same elements. A set is described by listing its elements, as in $\{1, 2, 3\}$, or by a defining property. The set $\{x \in \mathbb{R} : x^2 < 4\}$, read "the set of real x such that $x^2 < 4$", is the interval $(-2, 2)$ of Definition 2.1.8. Order and repetition in a list are irrelevant: $\{1, 2, 3\} = \{3, 1, 2, 2\}$. The *empty set* $\emptyset$ has no elements. The standard sets of numbers are $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Definition 1.5.2 (Subset)**.** A set A is a *subset* of a set B, written $A \subseteq B$, if every element of A is an element of B: $\forall x\,(x \in A \implies x \in B)$.

Example 1.5.3. The relations $\{1,3\} \subseteq \{1,2,3\}$, $\mathbb{N} \subseteq \mathbb{Z}$ and $[0,1] \subseteq [0,2)$ are all true. For every set A, $A \subseteq A$ and $\emptyset \subseteq A$, the second vacuously because $\emptyset$ has no element that could fail to lie in A. The relation $\{1,4\} \subseteq \{1,2,3\}$ is false because 4 is an element of the first set and not of the second. The number 1 is an element of $\{1,2\}$ and is not a subset of it: $1 \in \{1,2\}$ and $\{1\} \subseteq \{1,2\}$.

Two sets A and B are equal exactly when $A \subseteq B$ and $B \subseteq A$. A proof of a set identity therefore has two parts, one for each inclusion. Each part takes an arbitrary element of one side and shows that it lies in the other.

Definition 1.5.4 (Set operations)**.** Let A and B be subsets of a set U.

- The *union* is $A \cup B = \{x : x \in A \vee x \in B\}$.
- The *intersection* is $A \cap B = \{x : x \in A \wedge x \in B\}$. The sets are *disjoint* if $A \cap B = \emptyset$.
- The *difference* is $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
- The *complement* of A in U is $A^c = U \setminus A$.
- The *Cartesian product* is the set of ordered pairs $A \times B = \{(a,b) : a \in A,\ b \in B\}$, with $(a,b) = (a',b')$ exactly when $a = a'$ and $b = b'$.

Example 1.5.5. In $U = \mathbb{R}$ take $A = [0,2]$ and $B = (1,3)$. Then $A \cup B = [0,3)$, $A \cap B = (1,2]$, $A \setminus B = [0,1]$ and $A^c = (-\infty, 0) \cup (2, \infty)$. The plane is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and $[0,1] \times [0,1]$ is the unit square. The intervals $[0,1)$ and $[1,2]$ are disjoint. The intervals $[0,1]$ and $[1,2]$ share the point 1.

The connectives of Section 1.1 and the set operations correspond: $\cup$ to $\vee$, $\cap$ to $\wedge$ and complement to $\neg$. De Morgan's laws for statements become De Morgan's laws for sets.

Theorem 1.5.6 (De Morgan's laws)**.** *For subsets A and B of U,*

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c. \tag{1.5.1}$$

Proof. For $x \in U$ the following are equivalent by Definition 1.5.4 and Theorem 1.1.11:

$$x \in (A \cup B)^c \iff \neg(x \in A \vee x \in B) \iff (x \notin A) \wedge (x \notin B) \iff x \in A^c \cap B^c.$$

Read from left to right the chain gives $(A \cup B)^c \subseteq A^c \cap B^c$. Read from right to left it gives the reverse inclusion. The second law follows in the same way from the second law of (1.1.1). ∎

1.5.7 Functions

Definition 1.5.8 (Function)**.** Let A and B be sets. A *function* f from A to B, written $f : A \to B$, assigns to each element $x \in A$ exactly one element $f(x) \in B$, the *value* of f at x. The set A is the *domain* of f and B its *codomain*. The *image* (or *range*) of f is the set of values

$$f(A) = \{f(x) : x \in A\} \subseteq B. \tag{1.5.2}$$

The *graph* of f is $\{(x, f(x)) : x \in A\} \subseteq A \times B$. Two functions are equal when they have the same domain, the same codomain and the same value at every point of the domain.

A function is often given by a formula such as $f(x) = x^2$ or $x \mapsto x^2$, read "x maps to x^2". When no domain is stated the domain is taken to be the largest set of real numbers at which the formula is defined and the codomain is $\mathbb{R}$.

Example 1.5.9. The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, has domain and codomain $\mathbb{R}$ and image $[0, \infty)$: every value is a square and hence nonnegative. Conversely every $y \geq 0$ equals $f(\sqrt{y})$ with $\sqrt{y}$ the nonnegative square root of Theorem 2.4.8. Its image is a proper subset of its codomain. The formula $g(x) = 1/(x-1)$ has largest domain $\mathbb{R} \setminus \{1\}$.

The rule "assign to $x \geq 0$ a number y with $y^2 = x$" does not define a function from $[0, \infty)$ to $\mathbb{R}$: it gives two values ± 2 at $x = 4$ where a function assigns exactly one. Choosing the nonnegative value gives the function $\sqrt{x}$. A curve in the plane is the graph of a function of x exactly when every vertical line meets it at most once. The circle $x^2 + y^2 = 1$ fails this test at $x = 0$.

Definition 1.5.10 (Image and preimage)**.** Let $f\colon A \to B$. For $S \subseteq A$ the *image* of S is $f(S) = \{f(x) : x \in S\}$. For $T \subseteq B$ the *preimage* of T is $f^{-1}(T) = \{x \in A : f(x) \in T\}$.

The preimage $f^{-1}(T)$ is defined for every function, including a function with no inverse in the sense of Definition 1.5.24. It is a set of points of the domain and may be empty.

Example 1.5.11. For $f(x) = x^2$ on $\mathbb{R}$ the image $f([-1, 2])$ is $[0, 4]$ because x^2 takes every value in $[0, 1]$ on $[-1, 1]$ and every value in $[1, 4]$ on $[1, 2]$. The preimages are $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$, $f^{-1}(\{0\}) = \{0\}$ and $f^{-1}(\{-1\}) = \emptyset$.

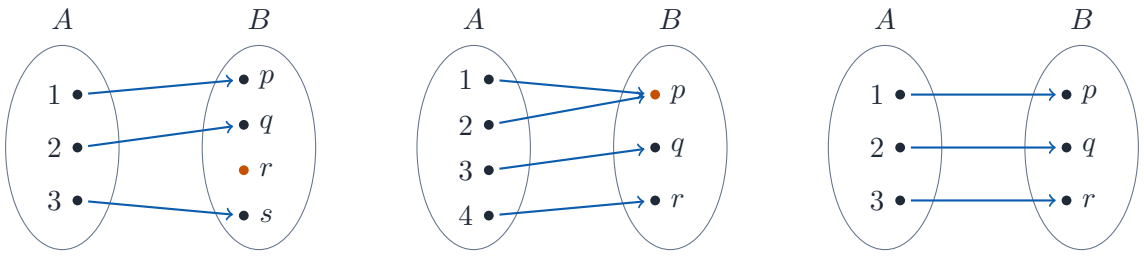
1.5.12 Injective surjective and bijective functions

Definition 1.5.13 (Injective)**.** A function $f\colon A \to B$ is *injective* (or *one-to-one*) if distinct points have distinct values: $\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \implies x_1 = x_2)$.

Definition 1.5.14 (Surjective)**.** A function $f\colon A \to B$ is *surjective* (or *onto*) if every element of the codomain is a value: $\forall y \in B\ \exists x \in A,\ f(x) = y$. Equivalently, $f(A) = B$.

Definition 1.5.15 (Bijective)**.** A function is *bijective* (a *bijection*) if it is both injective and surjective. Equivalently, for every $y \in B$ there is exactly one $x \in A$ with $f(x) = y$.

Figure 1.5.1 shows the three properties for functions between finite sets. An arrow runs from each element of the domain to its value. Injectivity says that no element of the codomain receives two arrows. Surjectivity says that every element of the codomain receives at least one.



(a) Injective and not surjective (b) Surjective and not injective (c) Bijective

Figure 1.5.1: Functions between finite sets drawn as arrow diagrams. In panel (a) the element r (orange) receives no arrow. In panel (b) the element p (orange) receives two arrows. In panel (c) every element of B receives exactly one.

Example 1.5.16. One formula has different properties on different domains and codomains.

- $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is neither injective nor surjective: $f(-1) = f(1)$ and -1 is not a value.
- $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = x^2$, is injective: if $x_1^2 = x_2^2$ with $x_1, x_2 \geq 0$, then $(x_1 - x_2)(x_1 + x_2) = 0$. Either $x_1 = x_2$ or $x_1 + x_2 = 0$. The second case forces $x_1 = x_2 = 0$. It is not surjective.

- $f\colon [0,\infty) \to [0,\infty)$, $f(x) = x^2$, is bijective.
- $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3x - 7$, is bijective: for each y the equation $3x - 7 = y$ has the single solution $x = (y + 7)/3$.

For functions between sets of real numbers, injectivity and surjectivity can be read off the graph.

Theorem 1.5.17 (Horizontal-line test)**.** *Let $f\colon A \to B$ with $A, B \subseteq \mathbb{R}$. Then f is injective if and only if every horizontal line $y = c$ meets the graph of f at most once, and f is surjective if and only if every horizontal line $y = c$ with $c \in B$ meets the graph at least once.*

Proof. The line $y = c$ meets the graph at the points (x, c) with $x \in A$ and $f(x) = c$, one point for each element of $f^{-1}(\{c\})$. The function is injective exactly when each $f^{-1}(\{c\})$ has at most one element, and surjective exactly when $f^{-1}(\{c\})$ is nonempty for every $c \in B$. ∎

A function f on a set $A \subseteq \mathbb{R}$ is *strictly increasing* if $x_1 < x_2$ implies $f(x_1) < f(x_2)$. It is *strictly decreasing* if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Proposition 1.5.18 (Strict monotonicity implies injectivity)**.** *A strictly increasing or strictly decreasing function is injective.*

Proof. Let $x_1 \neq x_2$ be labelled so that $x_1 < x_2$. Strict monotonicity gives $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. In either case $f(x_1) \neq f(x_2)$. This is the contrapositive of Definition 1.5.13. ∎

1.5.19 Composition

Definition 1.5.20 (Composition)**.** Let $f\colon A \to B$ and $g\colon B \to C$. The *composition* $g \circ f\colon A \to C$ is the function

$$(g \circ f)(x) = g(f(x)), \qquad x \in A. \tag{1.5.3}$$

More generally $g \circ f$ is defined on $\{x \in A : f(x) \in \text{domain of } g\}$ whenever that set is nonempty.

Example 1.5.21. With $f(x) = x + 1$ and $g(x) = x^2$ on $\mathbb{R}$, $(g \circ f)(x) = (x + 1)^2$ and $(f \circ g)(x) = x^2 + 1$. At $x = 1$ these are 4 and 2. Composition is therefore not commutative. The function $h(x) = \sqrt{1 - x^2}$ is the composition $s \circ q$ of $q(x) = 1 - x^2$ and $s(u) = \sqrt{u}$. Its domain is $\{x : 1 - x^2 \geq 0\} = [-1, 1]$, the set of x at which the inner value lies in the domain $[0, \infty)$ of s. Recognising a function as a composition is the first step of the chain rule (Theorem 6.3.4).

Composition is associative: for $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ send x to $h(g(f(x)))$. The brackets may therefore be omitted.

Theorem 1.5.22 (Composition preserves injectivity and surjectivity)**.** *Let $f\colon A \to B$ and $g\colon B \to C$. If f and g are injective then $g \circ f$ is injective. If f and g are surjective then $g \circ f$ is surjective. Consequently a composition of bijections is a bijection.*

Proof. Suppose f and g are injective and $g(f(x_1)) = g(f(x_2))$. Injectivity of g gives $f(x_1) = f(x_2)$. Injectivity of f then gives $x_1 = x_2$.

Suppose f and g are surjective and let $z \in C$. Surjectivity of g gives $y \in B$ with $g(y) = z$. Surjectivity of f gives $x \in A$ with $f(x) = y$. Then $(g \circ f)(x) = z$. ∎

1.5.23 Inverse functions

Definition 1.5.24 (Inverse function)**.** Let $f\colon A \to B$. A function $g\colon B \to A$ is an *inverse* of f if

$$g(f(x)) = x \text{ for every } x \in A \qquad \text{and} \qquad f(g(y)) = y \text{ for every } y \in B. \tag{1.5.4}$$

The inverse, when it exists, is unique (Theorem 1.5.26) and is written f^{-1}.

Example 1.5.25. The bijection $g(x) = 3x - 7$ of Example 1.5.16 has inverse $g^{-1}(y) = (y+7)/3$ because $g^{-1}(g(x)) = (3x - 7 + 7)/3 = x$ and $g(g^{-1}(y)) = 3 \cdot \frac{y+7}{3} - 7 = y$. The function x^2 on $\mathbb{R}$ has no inverse. An inverse g would satisfy $g(4) = g(f(2)) = 2$ and $g(4) = g(f(-2)) = -2$. No function takes two values at 4.

Theorem 1.5.26 (Bijections are invertible)**.** *A function $f\colon A \to B$ has an inverse if and only if it is bijective. The inverse is then unique and is itself a bijection, with $(f^{-1})^{-1} = f$.*

Proof. ($\Rightarrow$) Let g be an inverse. If $f(x_1) = f(x_2)$ then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$. Hence f is injective. For $y \in B$ the point $x = g(y)$ satisfies $f(x) = y$. Hence f is surjective.

($\Leftarrow$) Let f be bijective. For each $y \in B$ there is exactly one $x \in A$ with $f(x) = y$ (Definition 1.5.15). Define $g(y)$ to be that x. Then $f(g(y)) = y$ by construction. For $x \in A$, the point $g(f(x))$ is the unique element sent to $f(x)$. The point x is such an element. Hence $g(f(x)) = x$.

Uniqueness: if g and h are both inverses, then for $y \in B$, $g(y) = g(f(h(y))) = h(y)$, the first equality because $f(h(y)) = y$ and the second because $g \circ f$ is the identity of A. The two conditions of Definition 1.5.24 are symmetric in f and g. Therefore f is an inverse of f^{-1}. By the first part f^{-1} is bijective, and by uniqueness $(f^{-1})^{-1} = f$. ∎

To find an inverse in practice, solve the equation $y = f(x)$ for x. If for every y in the codomain the equation has exactly one solution in the domain, f is bijective and the solution is $f^{-1}(y)$. Since (a, b) lies on the graph of f exactly when (b, a) lies on the graph of f^{-1}, the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

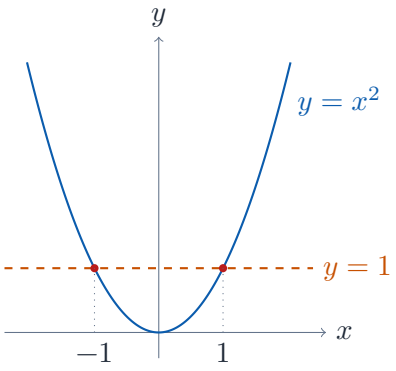
Definition 1.5.27 (Restriction)**.** Let $f\colon A \to B$ and $S \subseteq A$. The *restriction* of f to S is the function $f|_S\colon S \to B$ given by $f|_S(x) = f(x)$ for $x \in S$.

A function that is not injective has no inverse. Restricting it to a set on which it is injective and replacing the codomain by the image of that set produces a bijection that does have an inverse. Figure 1.5.2 shows the process for x^2. On $\mathbb{R}$ the line $y = 1$ meets the graph twice. On $[0, \infty)$ every horizontal line $y = c$ with $c \geq 0$ meets it once. The inverse of $x^2|_{[0,\infty)}\colon [0, \infty) \to [0, \infty)$ is $\sqrt{y}$.

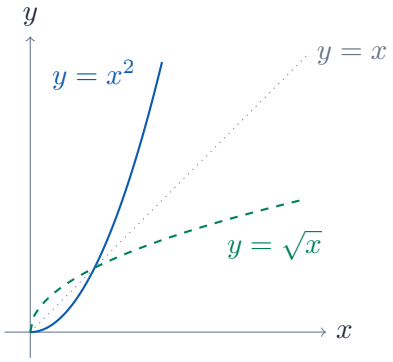
The same construction produces the inverse functions of Chapter 4. The sine function takes every value in $[-1, 1]$ infinitely often on $\mathbb{R}$. Its restriction to $[-\pi/2, \pi/2]$ is a strictly increasing bijection onto $[-1, 1]$ whose inverse is $\arcsin\colon [-1, 1] \to [-\pi/2, \pi/2]$ (Definition 4.5.14 and Corollary 5.4.13). The exponential function is a strictly increasing bijection from $\mathbb{R}$ onto $(0, \infty)$ (Theorems 4.4.17 and 5.4.12) whose inverse is the natural logarithm $\ln\colon (0, \infty) \to \mathbb{R}$ (Definition 4.4.22). In each case the inverse satisfies the two identities of Definition 1.5.24 only on the restricted sets. The identity $\arcsin(\sin x) = x$ holds for $x \in [-\pi/2, \pi/2]$ and fails at $x = \pi$ because $\arcsin(\sin \pi) = \arcsin 0 = 0$.

Common misconception

"$f^{-1}(x)$ means $1/f(x)$." The inverse function and the reciprocal are different objects. For $f(x) = 3x - 7$, $f^{-1}(x) = (x + 7)/3$ while $1/f(x) = 1/(3x - 7)$. The notation $f^{-1}(T)$ for



(a) Two intersections on $\mathbb{R}$



(b) Restriction to $[0,\infty)$ and its inverse

Figure 1.5.2: Horizontal-line test for x^2. On all of $\mathbb{R}$ the line $y=1$ meets the graph at $x=\pm1$ and x^2 is not injective. Restricted to $[0,\infty)$ it is a bijection onto $[0,\infty)$ whose inverse $\sqrt{x}$ has the reflected graph.

a preimage (Definition 1.5.10) is a third meaning and exists for every f, inverse or not. Reserve $f(x)^{-1}$ or $1/f(x)$ for the reciprocal.

Caution

A function is not determined by its formula alone. Whether x^2 is injective, surjective or invertible depends on the domain and codomain (Example 1.5.16). Before asking for an inverse, state the domain and the codomain.

Worked Example 1.5.1

Show that $f(x)=\dfrac{2x+1}{x-3}$ is a bijection from $A=\mathbb{R}\setminus\{3\}$ to $B=\mathbb{R}\setminus\{2\}$ and find its inverse.

First f maps A into B: the equation $f(x)=2$ would give $2x+1=2x-6$ and hence the false equation $1=-6$. Next solve $y=f(x)$ for x, with $y\in B$:

$$y(x-3)=2x+1 \iff yx-2x=3y+1 \iff x(y-2)=3y+1 \iff x=\frac{3y+1}{y-2},$$

the last step valid because $y\neq 2$. The first step is reversible because $x\neq 3$. The solution lies in A because $\frac{3y+1}{y-2}=3$ would give $3y+1=3y-6$. Each $y\in B$ therefore has exactly one preimage in A. Hence f is bijective with

$$f^{-1}(y)=\frac{3y+1}{y-2}, \qquad y\neq 2.$$

As a check, $f(0)=-\frac{1}{3}$ and $f^{-1}(-\frac{1}{3})=\dfrac{-1+1}{-\frac{1}{3}-2}=0$.

Worked Example 1.5.2

For $f(x) = x^2 - 2x$ on $\mathbb{R}$, find $f([0,3])$ and $f^{-1}(\{3\})$, and restrict f to the largest interval containing 3 on which it is injective. Find the inverse of that restriction.

Completing the square gives $f(x) = (x-1)^2 - 1$. On $[0,3]$ the square $(x-1)^2$ takes every value in $[0,4]$, from 0 at $x=1$ to 4 at $x=3$, and therefore $f([0,3]) = [-1,3]$. The preimage of $\{3\}$ consists of the solutions of $x^2 - 2x - 3 = (x-3)(x+1) = 0$, that is, $f^{-1}(\{3\}) = \{-1, 3\}$.

The graph is a parabola with vertex at $x=1$. On $[1,\infty)$ the function is strictly increasing because $1 \le x_1 < x_2$ gives $0 \le x_1 - 1 < x_2 - 1$ and hence $(x_1-1)^2 < (x_2-1)^2$. An interval containing 3 and a point $x < 1$ contains $[x,3]$ and with it the point $c = \max\{x, -1\}$ in $[-1,1)$ and the point $2-c$ in $(1,3]$. These two points differ and $f(2-c) = f(c)$. The largest such interval is therefore $[1,\infty)$. The restriction maps $[1,\infty)$ onto $[-1,\infty)$. Solving $y = (x-1)^2 - 1$ with $x \ge 1$ gives $x = 1 + \sqrt{y+1}$. Hence

$$\left(f|_{[1,\infty)}\right)^{-1}(y) = 1 + \sqrt{y+1}, \qquad y \ge -1.$$

Study guide

Prove a set equality by two inclusions, each starting "Let x be an element of …". A function consists of a domain, a codomain and a rule, and every question about injectivity, surjectivity or inverses depends on all three. To test injectivity, assume $f(x_1) = f(x_2)$ and try to deduce $x_1 = x_2$. To test surjectivity, take y in the codomain and try to solve $f(x) = y$. To invert, solve $y = f(x)$ for x and check that the solution exists and is unique for each y. If it is not unique, restrict the domain to an interval on which f is strictly monotone. Keep $f^{-1}(y)$ for the inverse, $f^{-1}(T)$ for a preimage and $1/f(x)$ for the reciprocal.

Problems

1.5.1 Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, B^c and the number of elements of $A \times B$. ⋆ Hint · Answer · Solution

1.5.2 Decide whether each function is injective, surjective or both.

(a) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n$.

(b) $g\colon \mathbb{R} \to [0,\infty)$, $g(x) = x^2$.

(c) $h\colon \mathbb{N} \to \mathbb{N}$, $h(n) = n + 1$.

(d) $k\colon \mathbb{R} \to \mathbb{R}$, $k(x) = 5 - 2x$.

⋆ Hint · Answer · Solution

1.5.3 For $f(x) = x^2 - 4x + 3$ on $\mathbb{R}$ find $f([0,3])$, $f^{-1}(\{0\})$ and $f^{-1}([-1,0])$.

⋆ Hint · Answer · Solution

1.5.4 Show that $f(x) = \dfrac{x+2}{x-1}$ is a bijection from $\mathbb{R} \setminus \{1\}$ to itself and find f^{-1}.

⋆⋆ Hint · Answer · Solution

1.5.5 Let $f(x) = \sqrt{x}$ on $[0,\infty)$ and $g(x) = 1 - x^2$ on $\mathbb{R}$. Find $g \circ f$ and $f \circ g$ with their domains. ⋆⋆ Hint · Answer · Solution

1.5.6 Let $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is injective then f is injective. Give an example in which $g \circ f$ is injective and g is not. ⋆⋆⋆ Hint · Answer · Solution

1.5.7 Let $f\colon A \to B$ and $S, T \subseteq A$. Prove that $f(S \cup T) = f(S) \cup f(T)$ and $f(S \cap T) \subseteq f(S) \cap f(T)$. Show by an example that the inclusion can be strict. Prove that it is an equality when f is injective. ⋆⋆⋆ Hint · Answer · Solution

Chapter 2

Real Numbers

Every limit, derivative and integral in this book is a statement about real numbers. A student arrives at calculus able to compute with real numbers but usually without a list of the facts on which those computations rest. The gaps show up exactly where calculus begins: in manipulating inequalities, in reading an absolute value as a distance and in the existence claims of the major theorems. This chapter states what we assume about $\mathbb{R}$ and proves what follows from it.

The assumptions come in three groups. The field axioms govern arithmetic and the order axioms govern inequalities. Both groups are satisfied by the rational numbers as well as by the real numbers. Neither group can therefore explain why $x^2 = 2$ has a real solution and no rational one. The third group is the single completeness axiom stating that the real line has no gaps. Completeness separates $\mathbb{R}$ from $\mathbb{Q}$. Every existence theorem of calculus (the Intermediate Value Theorem, the Extreme Value Theorem, the convergence of a bounded monotone sequence) depends on it.

Section 2.1 introduces the number systems and exhibits a gap in the rationals. Section 2.2 lists the field and order axioms and derives the rules of algebra from them. Section 2.3 develops the working tools for inequalities and absolute values that every ε–δ argument in Chapter 5 uses. Section 2.4 states the completeness axiom and proves its first consequences. The proofs use the logic and methods of proof of Chapter 1, including induction.

2.1 Number Systems

Overview

Objectives: Name the number systems $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and the operations under which each is closed. Prove that no rational number has square 2 (Theorem 2.1.3) and that the same is true for every positive integer that is not a perfect square. Convert between fractions and eventually periodic decimals. Read and write interval notation (Definition 2.1.8).
Prerequisites: Proof by contradiction and the well-ordering of the positive integers from Chapter 1. The rules of arithmetic and of inequalities, which Section 2.2 derives from the axioms.
Key results: $\sqrt{2}$ is irrational (Theorem 2.1.3 and Proposition 2.1.4). A real number is rational exactly when its decimal expansions are eventually periodic (Proposition 2.1.6). Nine kinds of interval (Definition 2.1.8).

Notation

$\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R}$	the natural numbers, integers, rationals and reals
$x \in S$	x is an element of the set S
$S \subseteq T$	every element of S is an element of T
$\emptyset$	the empty set
$[a,b],\ (a,b)$	closed and open intervals (Definition 2.1.8)
$0.1\overline{36}$	a decimal whose block 36 repeats for ever

Each number system in this section was introduced to make an equation solvable. Counting produces the natural numbers, in which $x + 5 = 3$ has no solution. The integers repair subtraction and leave $2x = 3$ without a solution. The rationals repair division by a nonzero number. Geometry then asks for more. By Pythagoras' theorem the diagonal of a unit square has a length d with $d^2 = 1^2 + 1^2 = 2$. No rational number has this property (Theorem 2.1.3). A number line on which the diagonal of a square has no length is unfit for geometry and for calculus. The real numbers are the system that fills the gap.

Definition 2.1.1 (Number systems)**.** The *natural numbers* are $\mathbb{N} = \{1, 2, 3, \ldots\}$. The *integers* are $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The *rational numbers* are the quotients of integers,

$$\mathbb{Q} = \left\{ \tfrac{m}{n} : m \in \mathbb{Z},\ n \in \mathbb{N} \right\}, \tag{2.1.1}$$

where $\frac{m}{n} = \frac{m'}{n'}$ exactly when $mn' = m'n$. A real number that is not rational is *irrational*.

This book does not count 0 as a natural number. With this convention $\mathbb{N}$ is the set of indices $n = 1, 2, 3, \ldots$ used for sequences in Chapter 5. Each system contains the previous one, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, once an integer m is identified with the rational $m/1$. The rationals $-7 = -7/1$, $0.25 = 1/4$ and $1.\overline{3} = 4/3$ are examples. As a non-example, the number d with $d^2 = 2$ is irrational by Theorem 2.1.3. The proofs below choose a convenient one among the many representations of a rational number, such as $\frac{1}{2} = \frac{2}{4} = \frac{-3}{-6}$.

The proofs of this section use the familiar rules of arithmetic and of inequalities. Section 2.2 derives every one of them from the field and order axioms alone. Those derivations use nothing from this section and the order of presentation therefore creates no circularity.

2.1.2 Irrationality of square roots

An integer is *even* if it equals $2k$ for some integer k and *odd* if it equals $2k+1$ (Definition 1.3.1). By Lemma 1.3.3 every integer is exactly one of the two. By Theorem 1.3.7 an integer whose square is even is itself even because the square of an odd integer is odd:

$$(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Theorem 2.1.3 (Irrationality of $\sqrt{2}$)**.** *No rational number r satisfies $r^2 = 2$.*

Proof. Suppose for a contradiction that $r^2 = 2$ for some rational r. Since $(-r)^2 = r^2$ we may take $r > 0$. Among all ways of writing $r = m/n$ with $m, n \in \mathbb{N}$, choose one with the least denominator n. Such a choice exists because every nonempty set of natural numbers has a least element (Assumption 1.4.1). Squaring $m/n = r$ gives $m^2 = 2n^2$. Hence m^2 is even and m is even by the remark above. Write $m = 2k$ with $k \in \mathbb{N}$. Then $4k^2 = 2n^2$ gives $n^2 = 2k^2$. The

same remark shows that n is even. Writing $n = 2l$ gives $r = k/l$ with denominator $l = n/2 < n$. This contradicts the choice of n. Hence no such r exists. ∎

The parity argument relies on the special behaviour of the prime 2. A different argument using only the order of the integers settles every positive integer that is not a perfect square at once.

Proposition 2.1.4 (Irrationality of $\sqrt{p}$)**.** *Let p be a positive integer that is not the square of an integer. Then no rational number r satisfies $r^2 = p$. In particular this is true for every prime p.*

Proof. Suppose $r^2 = p$ with r rational. As before we may take $r > 0$ and write $r = m/n$ with $m, n \in \mathbb{N}$ and n as small as possible. Let k be the largest integer with $k^2 \leq p$, namely $j - 1$ for the least $j \in \mathbb{N}$ with $j^2 > p$ (Assumption 1.4.1). Since p is not a perfect square, $k^2 < r^2 = p < (k+1)^2$ and therefore $k < r < k + 1$ by Proposition 2.2.5(vi) of Section 2.2. Multiplying by n,

$$kn < m < (k+1)n, \qquad \text{so} \qquad 0 < m - kn < n. \tag{2.1.2}$$

Now $m^2 = pn^2$ by hypothesis. Therefore

$$m(m - kn) = m^2 - kmn = pn^2 - kmn = n(pn - km). \tag{2.1.3}$$

Dividing (2.1.3) by $n(m - kn) > 0$ gives $r = \dfrac{m}{n} = \dfrac{pn - km}{m - kn}$. The numerator $pn - km$ is a positive integer because $r > 0$. The denominator $m - kn$ is a natural number smaller than n, in contradiction to the choice of n.

A prime p is not a perfect square. The number 1 is excluded from the primes by definition. A square $p = k^2$ with $k \geq 2$ would have the divisor k with $1 < k < p$. ∎

Worked Example 2.1.1

Let r be rational and t irrational. Prove that $r + t$ is irrational and that rt is irrational when $r \neq 0$.

Suppose $r + t = q$ with q rational. Then $t = q - r$ is a difference of two rationals. Writing $q = a/b$ and $r = c/d$ gives the rational number $t = (ad - bc)/(bd)$ in contradiction to the irrationality of t. Hence $r + t$ is irrational.

Suppose now $r \neq 0$ and $rt = q$ with q rational. Then $t = q/r$. With $q = a/b$ and $r = c/d$ (where $c \neq 0$) this is $t = ad/(bc)$, again rational after moving any minus sign into the numerator. The contradiction shows that rt is irrational. Combined with Theorem 2.1.3 this gives a supply of irrational numbers: $3 + d$, $-d$ and $d/5$ are all irrational when $d^2 = 2$.

Common misconception

A sum of two irrational numbers need not be irrational. If $d^2 = 2$ then d and $-d$ are both irrational and $d + (-d) = 0$. Likewise $d \cdot d = 2$ is rational. The irrationals are closed under neither addition nor multiplication. Only the mixed statements of Worked Example 2.1.1 are valid.

2.1.5 Decimal expansions

Every nonnegative real number has a decimal expansion such as $\pi = 3.14159\ldots$, a nonnegative integer part followed by an infinite string of digits. What an infinite string of digits means is made precise in Section 2.4, where the value of a decimal is defined as the least number at or

above all of its truncations. A decimal $d_0.d_1d_2\ldots$ is *eventually periodic* if there are integers $j \geq 0$ and $k \geq 1$ with $d_{i+k} = d_i$ for every $i > j$. The block $d_{j+1}\ldots d_{j+k}$ then repeats for ever and k is a *period*. Here we record which decimals are rational.

Lemma 2.4.15 of Section 2.4 supplies two facts used below. A real number has at most two decimal expansions, and two different ones consist of a terminating expansion and an expansion ending in an unbroken string of nines, such as $0.25 = 0.24999\ldots$. The digits of m/n produced by long division form the expansion of m/n that does not end in nines. That lemma rests on the completeness axiom, the Archimedean property, Bernoulli's inequality and the integer part. None of these uses anything from this section.

Proposition 2.1.6 (Decimals of rational numbers)**.** *Let $x \geq 0$ be rational. Every decimal expansion of x is eventually periodic. For $x = m/n$ with $m \in \mathbb{Z}$, $m \geq 0$ and $n \in \mathbb{N}$ the expansion produced by long division has a period of at most n digits. Conversely every eventually periodic decimal is a rational number.*

Proof. **1.** Long division. Write $x = m/n$. Division with remainder (Theorem 1.4.12) gives $m = d_0n + \rho_0$ with $\rho_0 \in \{0, 1, \ldots, n-1\}$, and then for each $k \geq 1$

$$10\rho_{k-1} = d_kn + \rho_k, \qquad 0 \leq \rho_k \leq n-1. \tag{2.1.4}$$

The digit d_k and the remainder ρ_k depend only on ρ_{k-1}. The $n+1$ remainders $\rho_0, \ldots, \rho_n$ take at most n values. Two of them therefore coincide, say $\rho_i = \rho_j$ with $i < j \leq n$. From step i on the division repeats itself and $d_{l+(j-i)} = d_l$ for every $l > i$. The period $j - i$ is at most n.
2. Every expansion. By Lemma 2.4.15 the long-division digits form a decimal expansion of x. Any other expansion of x ends in an unbroken string of nines by the same lemma and is eventually periodic with period 1.
3. Converse. Let $x = d_0.d_1d_2\ldots$ with $d_{i+k} = d_i$ for every $i > j$. By Proposition 2.4.14 the number 10^jx is the decimal whose integer part is the integer $10^jd_0 + 10^{j-1}d_1 + \cdots + d_j$ and whose digits after the point are $d_{j+1}d_{j+2}\ldots$. Likewise $10^{j+k}x$ has the digits $d_{j+k+1}d_{j+k+2}\ldots$ after the point. These coincide with $d_{j+1}d_{j+2}\ldots$ by periodicity. By Proposition 2.4.14 the two decimals differ by an integer N. Hence $10^j(10^k - 1)x = N$ and $x = N/(10^j(10^k - 1))$ is rational. ∎

The decimal expansions of an irrational number therefore never settle into a repeating block. The decimal $x = 0.101001000100001\ldots$, in which the runs of zeros grow by one each time, is irrational. Suppose it had a period k from position $j+1$ on. A run of more than k zeros begins after position j, and periodicity copies its first k zeros along the rest of the expansion. Every later digit would then be 0. The digit 1 recurs after every run. No period therefore exists and Proposition 2.1.6 shows that x is irrational. The numbers π and e are irrational as well. Lambert proved the first in 1761 by an argument beyond this book. The second is Problem 7.4.10, proved with Taylor's theorem.

Worked Example 2.1.2

Write $x = 0.1\overline{36}$ as a fraction in lowest terms and find the decimal expansion of $3/7$.
Because one digit precedes the repeating block 36 of length 2, take $j = 1$ and $k = 2$ in the proof of Proposition 2.1.6:

$$1000x = 136.\overline{36}, \qquad 10x = 1.\overline{36}, \qquad 990x = 135, \qquad x = \frac{135}{990} = \frac{3}{22}.$$

For $3/7$, long division gives the remainders $3, 2, 6, 4, 5, 1, 3, \ldots$ and the digits $4, 2, 8, 5, 7, 1$ before the remainder 3 recurs. Hence $3/7 = 0.\overline{428571}$, with period $6 \leq 7$ as the proposition predicts. As a check, $428571/999999 = 3/7$ because $999999 = 7 \cdot 142857$ and $428571 = 3 \cdot 142857$.

2.1.7 Intervals

A *set* is a collection of objects called its *elements*. It is described by a list of its elements or by a defining property, as in $\{x \in \mathbb{R} : x^2 < 4\}$, read "the set of real x with $x^2 < 4$". A set is *nonempty* if it has at least one element. The *empty set* $\emptyset$ has none. The sets of real numbers met most often in calculus are intervals.

Definition 2.1.8 (Intervals)**.** For real numbers $a < b$ the *bounded intervals* with endpoints a and b are

$$\begin{aligned} [a,b] &= \{x : a \leq x \leq b\}, \quad (a,b) = \{x : a < x < b\}, \\ [a,b) &= \{x : a \leq x < b\}, \quad (a,b] = \{x : a < x \leq b\}. \end{aligned} \tag{2.1.5}$$

The first is *closed*, the second *open* and the last two *half-open*. A square bracket marks an included endpoint and a round bracket an excluded one. The *unbounded intervals* are

$$[a,\infty) = \{x : x \geq a\}, \quad (a,\infty) = \{x : x > a\}, \quad (-\infty,b] = \{x : x \leq b\},$$
$$(-\infty,b) = \{x : x < b\}, \quad (-\infty,\infty) = \mathbb{R}.$$

The symbol ∞ is not a real number. It marks the absence of an endpoint and always takes a round bracket.

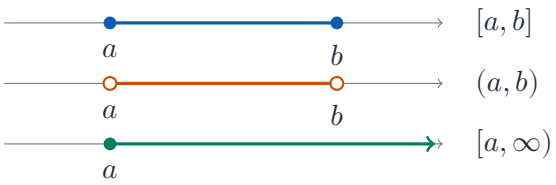


Figure 2.1.1: Intervals on the number line. A filled dot marks an included endpoint and a hollow dot an excluded one. An arrow marks an interval with no right endpoint.

Figure 2.1.1 draws three of the nine kinds. An interval I has no holes: whenever $u, v \in I$ and $u < x < v$ the number x is in I as well. The set $\{x : x^2 < 4\}$ is the interval $(-2, 2)$. As a non-example, $\{x : x^2 > 4\} = (-\infty, -2) \cup (2, \infty)$ contains -3 and 3 and misses 0. It is therefore no interval. Neither is $\mathbb{N}$. When $a = b$ the formulas of Definition 2.1.8 give the single point $[a, a] = \{a\}$ and the empty set $(a, a) = \emptyset$. Theorems about intervals therefore assume $a < b$.

Study guide

Memorise the chain $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ and the equation that each step makes solvable. Learn the proof of Theorem 2.1.3 well enough to reproduce it. Choose the representation with least denominator, show that numerator and denominator are both even and contradict the choice. Problem 2.1.4 follows the same plan with 3 in place of 2. Proposition 2.1.4 keeps the least denominator and replaces the parity step by a descent that produces a smaller denominator directly.

To convert a repeating decimal to a fraction, multiply by two powers of 10 that align the repeating blocks and subtract. When reading an interval, look at each bracket separately and ask whether that endpoint satisfies the defining inequality.

Problems

2.1.1 Write $0.\overline{27}$ as a fraction in lowest terms. ⋆ Hint · Answer · Solution

2.1.2 Find the decimal expansion of $\dfrac{5}{12}$. ⋆ Hint · Answer · Solution

2.1.3 Write each set in interval notation or explain why it is not an interval: (a) $\{x : x \geq -3 \text{ and } x < 5\}$, (b) $\{x : x \neq 0\}$, (c) $\{x : 3 - x > 1\}$. ⋆ Hint · Answer · Solution

2.1.4 Prove that no rational number has square 3 by adapting the proof of Theorem 2.1.3, using that every integer has one of the forms $3k$, $3k+1$ or $3k+2$ (Theorem 1.4.12). ⋆⋆ Hint · Answer · Solution

2.1.5 Let s and t be positive real numbers with $s^2 = 2$ and $t^2 = 3$ (they exist by Theorem 2.4.8). Prove that $s + t$ is irrational. ⋆⋆ Hint · Answer · Solution

2.1.6 Find the decimal expansion of $\dfrac{1}{7}$ and check the bound on the period in Proposition 2.1.6. ⋆⋆ Hint · Answer · Solution

2.1.7 Let $n \in \mathbb{N}$ have no prime factors other than 2 and 5. Prove that for every integer $m \geq 0$ the decimal expansion of m/n terminates. ⋆⋆⋆ Hint · Answer · Solution

2.2 Field and Order Axioms

Overview

Objectives: State the field and order axioms for $\mathbb{R}$. Derive from them the familiar rules of signs and inequalities, including $a \cdot 0 = 0$, $(-1)a = -a$, the reversal of an inequality multiplied by a negative number, the positivity of nonzero squares and $0 < 1$. Recognise which axiom fails in a structure that is not an ordered field.
Prerequisites: The number systems and intervals of Section 2.1. Direct proof and proof by contradiction from Chapter 1.
Key results: The field axioms (Assumption 2.2.1) and order axioms (Assumption 2.2.2). Rules of algebra (Proposition 2.2.4) and of inequalities (Proposition 2.2.5).

Notation

$-a,\ a^{-1}$	the additive and multiplicative inverses of a
$a-b,\ a/b$	$a+(-b)$ and $a\,b^{-1}$
$a \le b$	$a < b$ or $a = b$
$a > b$	$b < a$

You have used rules such as $(-a)(-b) = ab$ and "a negative factor reverses an inequality" for years. A proof in calculus must say which of these rules are assumed and which are consequences because a step that cannot be traced to an assumption is a gap in the argument. This section lists a short set of axioms and derives the rest from them. The list is deliberately small: nine properties of $+$ and $\cdot$ and four of $<$.

Assumption 2.2.1 (Field axioms)**.** The real numbers $\mathbb{R}$ form a set with two operations $+$ and $\cdot$ and two elements $0 \neq 1$ such that for all $a, b, c \in \mathbb{R}$:

(F1) $a + b = b + a$ and $ab = ba$ (commutativity);

(F2) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (associativity);

(F3) $a(b + c) = ab + ac$ (distributivity);

(F4) $a + 0 = a$ and $a \cdot 1 = a$ (identities);

(F5) there is a number $-a$ with $a + (-a) = 0$ and, when $a \neq 0$, a number a^{-1} with $a\,a^{-1} = 1$ (inverses).

Assumption 2.2.2 (Order axioms)**.** There is a relation $<$ on $\mathbb{R}$ such that for all $a, b, c \in \mathbb{R}$:

(O1) exactly one of $a < b$, $a = b$, $b < a$ is true (trichotomy);

(O2) if $a < b$ and $b < c$ then $a < c$ (transitivity);

(O3) if $a < b$ then $a + c < b + c$ (addition);

(O4) if $a < b$ and $0 < c$ then $ac < bc$ (multiplication by a positive number).

A number a is *positive* if $0 < a$ and *negative* if $a < 0$.

Subtraction and division are defined from the inverses as $a - b = a + (-b)$ and $a/b = a\,b^{-1}$ for $b \neq 0$. The natural numbers sit inside $\mathbb{R}$ as follows. Call a set $A \subseteq \mathbb{R}$ *inductive* if $1 \in A$ and $x + 1 \in A$ for every $x \in A$. The set $\mathbb{N}$ consists of the real numbers that lie in every inductive set.

It contains 1, $2 = 1 + 1$, $3 = 2 + 1$ and so on. The set $\mathbb{N}$ is itself inductive and is contained in every inductive set. The last property is the principle of induction (Theorem 1.4.2) for $\mathbb{N} \subseteq \mathbb{R}$: the set of n for which a statement is true is inductive when the statement is true for 1 and passes from n to $n+1$. The set $\{x \in \mathbb{R} : x \geq 1\}$ is inductive because $0 < 1$ (Proposition 2.2.5(iii) below) and (O3) give $x + 1 > x \geq 1$. Hence $n \geq 1$ for every $n \in \mathbb{N}$. The same inequality $n < n + 1$ gives $1 < 2 < 3 < \cdots$. These numbers are therefore distinct. The integers in $\mathbb{R}$ are the numbers n, 0 and $-n$ for $n \in \mathbb{N}$.

Definition 2.2.3 (Ordered field)**.** A set with operations $+$, $\cdot$ and a relation $<$ that satisfy Assumption 2.2.1 and Assumption 2.2.2 is an *ordered field*.

The rationals $\mathbb{Q}$ form an ordered field. Sums, products, negatives and reciprocals of rationals are rational. Every property in the two lists is inherited by $\mathbb{Q}$ from $\mathbb{R}$. As non-examples $\mathbb{Z}$ is not a field because 2 has no multiplicative inverse in $\mathbb{Z}$. The set $\mathbb{N}$ fails (F4) and (F5) for addition. The two-element set $\{0, 1\}$ with $1 + 1 = 0$ satisfies every field axiom but admits no order relation by Proposition 2.2.5 below.

Proposition 2.2.4 (Cancellation and sign rules for real numbers)**.** *For all $a, b, x, y \in \mathbb{R}$:*

(i) *if $a + x = a + y$ then $x = y$ (in particular $a + x = 0$ implies $x = -a$);*

(ii) *$a \cdot 0 = 0$;*

(iii) *$(-1)a = -a$;*

(iv) *$-(-a) = a$ and $(-a)(-b) = ab$;*

(v) *if $ab = 0$ then $a = 0$ or $b = 0$.*

Proof. (i) Add $-a$ on the left of both sides and use (F2), (F5), (F1) and (F4):

$$x = 0 + x = ((-a) + a) + x = (-a) + (a + x) = (-a) + (a + y) = ((-a) + a) + y = 0 + y = y.$$

If $a + x = 0$ then $a + x = a + (-a)$ and the first part gives $x = -a$.

(ii) By (F4) and (F3), $a \cdot 0 + 0 = a \cdot 0 = a(0 + 0) = a \cdot 0 + a \cdot 0$. Part (i) with $a \cdot 0$ in the role of a gives $a \cdot 0 = 0$.

(iii) By (F4), (F1), (F3), (F5) and (ii), $a + (-1)a = a \cdot 1 + a(-1) = a(1 + (-1)) = a \cdot 0 = 0$. Part (i) gives $(-1)a = -a$.

(iv) Since $(-a) + a = 0$ by (F1) and (F5), part (i) identifies a as the additive inverse $-(-a)$ of $-a$. Applying (iii) to -1 gives $(-1)(-1) = -(-1) = 1$. Then by (iii), (F1) and (F2)

$$(-a)(-b) = (-1)a\,(-1)b = (-1)(-1)\,ab = ab.$$

(v) Suppose $ab = 0$ and $a \neq 0$. Then by (F5), (F2) and (ii), $b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0$. ∎

Proposition 2.2.5 (Rules for real inequalities)**.** *For all $a, b, c, d \in \mathbb{R}$:*

(i) *$a < b$ exactly when $0 < b - a$ (and $a < b$ implies $-b < -a$);*

(ii) *if $a < b$ and $c < 0$ then $bc < ac$;*

(iii) *if $a \neq 0$ then $0 < a^2$ (in particular $0 < 1$);*

(iv) *if $0 < a$ then $0 < a^{-1}$ (and $0 < a < b$ implies $0 < b^{-1} < a^{-1}$);*

(v) if $a < b$ and $c < d$ then $a + c < b + d$;

(vi) if $0 \le a < b$ then $a^2 < b^2$ (and conversely $a, b \ge 0$ with $a^2 < b^2$ implies $a < b$).

Proof. (i) Adding $-a$ to $a < b$ by (O3) gives $0 < b - a$. Adding a to $0 < b - a$ recovers $a < b$. Adding $(-a) + (-b)$ to both sides of $a < b$ gives $-b < -a$.

(ii) From $c < 0$ part (i) gives $0 < 0 - c = -c$. Axiom (O4) applied to $a < b$ with the positive number $-c$ gives $a(-c) < b(-c)$. By Proposition 2.2.4(iii), (F1) and (F2), $a(-c) = a((-1)c) = (-1)(ac) = -(ac)$ and likewise $b(-c) = -(bc)$. Hence $-(ac) < -(bc)$. The second half of (i) and Proposition 2.2.4(iv) then give $bc < ac$.

(iii) If $0 < a$ then (O4) with $c = a$ turns $0 < a$ into $0 \cdot a < a \cdot a$, that is $0 < a^2$. If $a < 0$ then (ii) with $b = 0$ and $c = a$ turns $a < 0$ into the same inequality $0 \cdot a < a \cdot a$. Since $1 \neq 0$ by Assumption 2.2.1 and $1 = 1^2$, we get $0 < 1$.

(iv) Let $0 < a$. The inverse a^{-1} is not 0 because $a \cdot 0 = 0 \neq 1$. If $a^{-1} < 0$, then (O4) applied to $a^{-1} < 0$ with $c = a$ gives $1 < 0$ in contradiction to (iii) and (O1). Hence $0 < a^{-1}$. If $0 < a < b$ then a^{-1}, b^{-1} and their product are positive by (O4). Multiplying $a < b$ by $a^{-1}b^{-1}$ gives $b^{-1} < a^{-1}$.

(v) By (O3) twice, $a + c < b + c$ and $b + c < b + d$. Then (O2) gives $a + c < b + d$.

(vi) Let $0 \le a < b$. Then $0 < b$. If $a = 0$ then $a^2 = 0 < b^2$ by (iii). If $a > 0$ then (O4) gives $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$. Then (O2) gives $a^2 < b^2$. For the second statement let $a, b \ge 0$ with $a^2 < b^2$. If $b \le a$ then the first statement (or equality when $b = a$) gives $b^2 \le a^2$, contradicting (O1). Hence $a < b$. ∎

Part (iii) settles the two-element example above. In an ordered field $0 < 1$ gives $1 = 0 + 1 < 1 + 1$ by (O3) and then $0 < 1 + 1$ by (O2). The equation $1 + 1 = 0$ is therefore impossible in an ordered field. The set $\{0, 1\}$ with $1 + 1 = 0$ cannot be ordered. Each rule in Proposition 2.2.5 has a counterpart for $\le$, proved by treating the case of equality separately.

Caution

The symbol $-a$ means the additive inverse of a and says nothing about sign: for $a = -3$ the number $-a = 3$ is positive. A statement such as "$-x < 0$" is true only when $x > 0$.

Common misconception

Multiplying both sides of an inequality by an expression whose sign is unknown is the most common error in solving inequalities. From $\dfrac{x}{x-1} < 2$ it does not follow that $x < 2(x - 1)$ because Proposition 2.2.5(ii) reverses the inequality whenever $x - 1 < 0$. The value $x = 0$ satisfies the original inequality and violates the second. Section 2.3 gives the method that avoids this error.

Worked Example 2.2.1

Prove that if $a < b$ then $a < \dfrac{a+b}{2} < b$.

First, $2 = 1 + 1$ is positive. Adding 1 to the inequality $0 < 1$ of Proposition 2.2.5(iii) by (O3) gives $1 < 2$. Then (O2) gives $0 < 2$. By Proposition 2.2.5(iv) the number $\frac{1}{2} = 2^{-1}$ is positive. Adding a to both sides of $a < b$ gives $2a < a + b$. Adding b gives $a + b < 2b$. Multiplying both by $\frac{1}{2}$ under (O4) gives

$$a < \frac{a+b}{2} < b.$$

Between any two distinct real numbers there is therefore a third and hence infinitely many. The same is true in $\mathbb{Q}$ because the midpoint of two rationals is rational. Having no next number differs from having no gaps. Gaps are the subject of Section 2.4.

Worked Example 2.2.2

Prove that $0 < a < 1$ implies $a^2 < a$ and that $1 < a$ implies $a < a^2$.
If $0 < a < 1$, then (O4) applied to $a < 1$ with the positive number a gives $a \cdot a < 1 \cdot a$, that is $a^2 < a$. If $1 < a$ then a is positive by (O2) and $0 < 1$. Then (O4) applied to $1 < a$ with $c = a$ gives $a < a^2$. For example $0.3^2 = 0.09 < 0.3$ and $1.5^2 = 2.25 > 1.5$. Squaring shrinks numbers between 0 and 1 and enlarges numbers above 1.

Study guide

You need not memorise the numbering of the axioms. You should be able to justify any step of an algebraic or inequality manipulation by naming the rule it uses. In particular you should be able to say why a step that multiplies an inequality is valid. The multiplier must be known to be positive. A multiplier known to be negative reverses the inequality. The field $\mathbb{Q}$ satisfies every axiom of this section. No argument that uses only these axioms can produce a number whose square is 2. The extra ingredient appears in Section 2.4.

Problems

2.2.1 Prove from the axioms and Proposition 2.2.4 that $a(-b) = -(ab)$.
⋆ Hint · Answer · Solution

2.2.2 For each of $\mathbb{N}$ and $\mathbb{Z}$ with the usual operations, name an axiom of Assumption 2.2.1 that fails and give an equation that has no solution in the set.
⋆ Hint · Answer · Solution

2.2.3 Prove that $(a + b)^2 = a^2 + 2ab + b^2$ citing the axioms used. Point out where the definition $2 = 1 + 1$ enters. ⋆⋆ Hint · Answer · Solution

2.2.4 Prove that if $0 \le a < b$ then $a^3 < b^3$. ⋆⋆ Hint · Answer · Solution

2.2.5 Show by an example that $a < b$ and $c < d$ do not imply $ac < bd$. Then prove that the implication is valid when a and c are nonnegative. ⋆⋆ Hint · Answer · Solution

2.2.6 Prove that $\dfrac{a}{b} + \dfrac{b}{a} \ge 2$ for all positive a and b, with equality exactly when $a = b$.
⋆⋆ Hint · Answer · Solution

2.2.7 Prove that if $a \le b + \varepsilon$ for every $\varepsilon > 0$ then $a \le b$. ⋆⋆ Hint · Answer · Solution

2.3 Inequalities and Absolute Value

Overview

Objectives: Solve polynomial and rational inequalities with a sign chart. Read $|x-a|$ as the distance from x to a and convert $|x-a|<\delta$ to an interval. Prove and apply the triangle inequality (Lemma 2.3.8) and its reverse form. Prove Bernoulli's inequality (Theorem 2.3.11) by induction and apply the inequality of arithmetic and geometric means.
Prerequisites: The order axioms and their consequences (Section 2.2). Induction (Theorem 1.4.2).
Key results: $|x-a|<\delta \iff a-\delta<x<a+\delta$ (Eq. (2.3.2)). $|a+b|\le|a|+|b|$ (Eq. (2.3.3)). $(1+x)^n\ge 1+nx$ for $x\ge -1$ (Theorem 2.3.11). $ab\le\left(\frac{a+b}{2}\right)^2$ (Theorem 2.3.12).

Notation

$\lvert a\rvert$	the absolute value of a (Definition 2.3.5)
$\lvert a-b\rvert$	the distance between a and b
$\delta,\ \varepsilon$	positive numbers used as tolerances

The definition of a limit in Chapter 5 is an inequality. Every proof about limits manipulates inequalities. The typical step bounds a quantity such as $|x^2-4|$ by a multiple of $|x-2|$. This section develops the tools for such steps: sign charts to solve inequalities exactly, the absolute value to measure distance and three inequalities (triangle, Bernoulli, arithmetic and geometric means) used throughout the book.

2.3.1 Sign charts

To solve an inequality is to find the set of all real x for which it is true. The answer is usually an interval or a union of intervals. For a product or quotient of factors the method rests on one observation about a single linear factor.

Lemma 2.3.2 (Sign of a linear factor)**.** *For a real number r, the factor $x-r$ is negative for $x<r$, zero at $x=r$ and positive for $x>r$. A product of nonzero factors is positive when an even number of the factors is negative and negative when an odd number is negative.*

Proof. The first statement is Proposition 2.2.5(i). For the second, induct on the number of factors (Theorem 1.4.2). One factor is the definition of positive and negative. Multiplying a product P by one more factor c keeps the sign of P when $c>0$ by (O4) and reverses it when $c<0$ by Proposition 2.2.5(ii). The parity of the count of negative factors therefore determines the sign. ∎

A quadratic factor $x^2+\beta x+\gamma$ with $\beta^2<4\gamma$ is positive for every x. Completing the square writes it as the sum of the nonnegative number $\left(x+\frac{\beta}{2}\right)^2$ and the positive number $\gamma-\frac{\beta^2}{4}$. Such factors never change sign and may be ignored in a sign chart.

Theorem 2.3.3 (Sign of a factored rational function)**.** *Let $f(x)=p(x)/q(x)$ with p and q products of linear factors $x-r$ and of quadratic factors $x^2+\beta x+\gamma$ with $\beta^2<4\gamma$, times a nonzero constant. Let $c_1<c_2<\cdots<c_k$ be the zeros of the linear factors of p and q. Then f has constant sign on each of the open intervals $(-\infty,c_1),(c_1,c_2),\dots,(c_k,\infty)$.*

Proof. On each of these intervals no linear factor vanishes. By Lemma 2.3.2 each linear factor keeps one sign there. The quadratic factors are positive everywhere by the completed square

above. The number of negative factors in numerator and denominator together is therefore the same at every point of the interval. By Lemma 2.3.2 the sign of f is the same as well. A quotient u/v has the sign of the product uv because v^{-1} has the sign of v by Proposition 2.2.5(iv). ∎

Sign chart method

To solve an inequality of the form $f(x) < 0$, $f(x) \le 0$, $f(x) > 0$ or $f(x) \ge 0$:

1. Move every term to one side and combine into a single quotient $p(x)/q(x)$. Never multiply through by an expression of unknown sign.
2. Factor p and q and mark the zeros of every factor on a number line.
3. Find the sign of each factor on each interval and combine by Lemma 2.3.2.
4. Read off the intervals with the required sign. Include a zero of p when the inequality is weak and never include a zero of q.

		-2		1		3	
$x+2$	$-$		$+$		$+$		$+$
$x-1$	$-$		$-$		$+$		$+$
$x-3$	$-$		$-$		$-$		$+$
$f(x)$	$-$	0	$+$	0	$-$	undef.	$+$

Figure 2.3.1: Sign chart of $f(x) = \dfrac{(x+2)(x-1)}{x-3}$. Each row records the sign of one factor on the four intervals cut out by -2, 1 and 3. The bottom row combines them by Lemma 2.3.2.

Worked Example 2.3.1

Solve $\dfrac{(x+2)(x-1)}{x-3} \ge 0$.

The zeros -2, 1 and 3 of the factors cut the line into four intervals. Figure 2.3.1 records the sign of each factor on each interval and the resulting sign of the quotient: negative on $(-\infty, -2)$, positive on $(-2, 1)$, negative on $(1, 3)$ and positive on $(3, \infty)$. Because the inequality is weak the zeros -2 and 1 of the numerator belong to the solution set. The zero 3 of the denominator does not. The solution set is

$$[-2, 1] \cup (3, \infty).$$

Test values confirm the pieces. At $x = 0$ the quotient is $(2)(-1)/(-3) = \frac{2}{3} \ge 0$. At $x = 2$ it is $(4)(1)/(-1) = -4 < 0$.

Worked Example 2.3.2

Solve $\dfrac{x}{x-1} < 2$.

Following the method move 2 to the left and combine:

$$\frac{x}{x-1} - 2 = \frac{x - 2(x-1)}{x-1} = \frac{2-x}{x-1} < 0 \iff \frac{x-2}{x-1} > 0,$$

where the last step multiplies by -1 and reverses the inequality. The factors vanish at 1 and 2. The quotient $(x-2)/(x-1)$ is positive on $(-\infty,1)$ where both factors are negative, negative on $(1,2)$ and positive on $(2,\infty)$. The solution set is $(-\infty,1)\cup(2,\infty)$. Multiplying the original inequality by $x-1$ would have produced only $(2,\infty)$ and lost every solution with $x<1$, such as $x=0$.

2.3.4 Absolute value

Definition 2.3.5 (Absolute value)**.** The *absolute value* of a real number a is

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases} \tag{2.3.1}$$

The number $|a-b|$ is the *distance* between a and b.

For example $|-4|=4$, $|x^2+1|=x^2+1$ for every x and $|x-5|$ equals $x-5$ for $x\geq 5$ and $5-x$ for $x<5$. As a non-example the plausible rule $|a+b|=|a|+|b|$ is false: $a=3$ and $b=-3$ give 0 on the left and 6 on the right. The correct statement is the triangle inequality (2.3.3) below. The distance between -2 and 5 is $|-2-5|=7$. The number $|x|=|x-0|$ is the distance from x to 0.

Proposition 2.3.6 (Properties of the absolute value)**.** *For all real a, b, x and every $r\geq 0$:*

(i) $|a|\geq 0$, *with* $|a|=0$ *exactly when* $a=0$;

(ii) $|-a|=|a|$, $|ab|=|a|\,|b|$ *and* $|a/b|=|a|/|b|$ *for* $b\neq 0$;

(iii) $|a|^2=a^2$ *and* $-|a|\leq a\leq |a|$;

(iv) $|x|\leq r$ *exactly when* $-r\leq x\leq r$ *(and* $|x|<r$ *exactly when* $-r<x<r$*).*

Proof. (i) If $a\geq 0$ then $|a|=a\geq 0$. If $a<0$ then $|a|=-a>0$ by Proposition 2.2.5(i). The value 0 occurs only in the first case with $a=0$.

(ii) Both $|a|$ and $|-a|$ are the nonnegative one of the two numbers a and $-a$. Hence $|-a|=|a|$. For the product, ab equals $\pm|a|\,|b|$ by Proposition 2.2.4(iv). The numbers $|ab|$ and $|a|\,|b|$ are both nonnegative and are therefore equal. For $b\neq 0$ the product rule gives $|a|=|b\cdot(a/b)|=|b|\,|a/b|$. Dividing by $|b|>0$ gives $|a/b|=|a|/|b|$.

(iii) $|a|^2$ is a^2 or $(-a)^2=a^2$. If $a\geq 0$ then $-|a|\leq 0\leq a=|a|$. If $a<0$ then $-|a|=a<0<|a|$.

(iv) Suppose $|x|\leq r$. By (iii), $-r\leq -|x|\leq x\leq |x|\leq r$. Conversely suppose $-r\leq x\leq r$. If $x\geq 0$ then $|x|=x\leq r$. If $x<0$ then $|x|=-x\leq r$ because $-r\leq x$. The strict version is proved the same way. ∎

Applying (iv) to $x-a$ in place of x gives the form in which the absolute value appears in every limit argument: for $\delta>0$,

$$|x-a|<\delta \iff a-\delta<x<a+\delta \iff x\in(a-\delta,a+\delta). \tag{2.3.2}$$

The set of x within distance δ of a is the open interval of radius δ centred at a (Fig. 2.3.2). Removing the centre gives $0<|x-a|<\delta$, the set $(a-\delta,a)\cup(a,a+\delta)$ that appears in the definition of a limit (Definition 5.2.1). Similarly $|x-a|>\delta$ means $x<a-\delta$ or $x>a+\delta$.

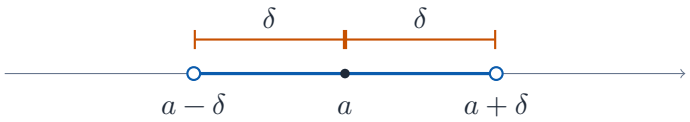


Figure 2.3.2: Solution set of $|x-a|<\delta$. It is the open interval $(a-\delta, a+\delta)$ of radius δ about a with both endpoints excluded.

Worked Example 2.3.3

Solve $|2x-3|<5$ and $|2x-3|\geq 5$.
By Proposition 2.3.6(iv),

$$|2x-3|<5 \iff -5<2x-3<5 \iff -2<2x<8 \iff -1<x<4.$$

The solution set is $(-1,4)$. Writing the inequality as $\left|x-\frac{3}{2}\right|<\frac{5}{2}$ shows it as the interval of radius $\frac{5}{2}$ about $\frac{3}{2}$, in agreement with Eq. (2.3.2). The second inequality is the negation of the first. Its solution set is the complement $(-\infty,-1]\cup[4,\infty)$.

2.3.7 Triangle inequality

Many proofs below combine two small quantities into one by a single inequality about absolute values.

Lemma 2.3.8 (Triangle inequality)**.** *For all real numbers a and b,*

$$|a+b|\leq |a|+|b|. \tag{2.3.3}$$

Proof. By Proposition 2.3.6(iii), $-|a|\leq a\leq |a|$ and $-|b|\leq b\leq |b|$. Adding these by Proposition 2.2.5(v) gives $-(|a|+|b|)\leq a+b\leq |a|+|b|$. Proposition 2.3.6(iv) with $r=|a|+|b|$ gives Eq. (2.3.3). The name comes from the geometry of Fig. 2.3.3. ∎

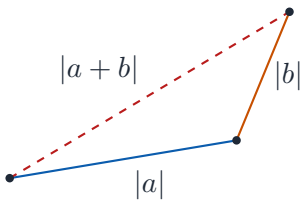


Figure 2.3.3: Triangle inequality. The direct side $|a+b|$ is never longer than the two-step path $|a|+|b|$. On the number line the triangle flattens and Eq. (2.3.3) still holds.

Two forms follow at once. Replacing a by $a-c$ and b by $c-b$ gives the form for distances,

$$|a-b|\leq |a-c|+|c-b|. \tag{2.3.4}$$

A detour through c is never shorter than the direct route. The second form bounds a difference from below.

Corollary 2.3.9 (Reverse triangle inequality)**.** *For all real a and b, $\big|\,|a|-|b|\,\big|\leq |a-b|$.*

Proof. By Eq. (2.3.3), $|a|=|(a-b)+b|\leq |a-b|+|b|$ and hence $|a|-|b|\leq |a-b|$. Exchanging a and b gives $|b|-|a|\leq |b-a|=|a-b|$. The two inequalities together say $-|a-b|\leq |a|-|b|\leq |a-b|$. Proposition 2.3.6(iv) finishes the proof. ∎

Worked Example 2.3.4

Show that if $|x-2|<1$ then $|x^2-4|\le 5|x-2|$.
The factorisation $x^2-4=(x-2)(x+2)$ gives $|x^2-4|=|x-2|\,|x+2|$ by Proposition 2.3.6(ii). The second factor is bounded with the triangle inequality after writing $x+2$ in terms of $x-2$:

$$|x+2|=|(x-2)+4|\le |x-2|+4<1+4=5.$$

Hence $|x^2-4|\le 5|x-2|$. In particular if $|x-2|<\delta$ with $\delta\le 1$ then $|x^2-4|<5\delta$. To make $|x^2-4|$ smaller than a given $\varepsilon>0$ it therefore suffices to take δ to be the smaller of 1 and $\varepsilon/5$. This is the pattern of the limit proofs in Chapter 5.

Common misconception

The triangle inequality does not give $|a-b|\le |a|-|b|$. For $a=1$ and $b=-1$ the left side is 2 and the right side is 0. The correct bound for a difference is $|a-b|\le |a|+|b|$, obtained from Eq. (2.3.3) with $-b$ in place of b. A lower bound for $|a-b|$ comes from Corollary 2.3.9.

2.3.10 Bernoulli and arithmetic and geometric means

The next inequality compares a power with a linear function and is the first proof in this book by induction on an exponent.

Theorem 2.3.11 (Bernoulli's inequality)**.** *For every $n\in\mathbb{N}$ and every real $x\ge -1$,*

$$(1+x)^n\ge 1+nx. \tag{2.3.5}$$

Proof. We use induction on n (Theorem 1.4.2). For $n=1$ both sides equal $1+x$. Suppose $(1+x)^n\ge 1+nx$ for some $n\in\mathbb{N}$. Because $x\ge -1$ the number $1+x$ is nonnegative. Multiplying the inequality by it preserves the inequality:

$$(1+x)^{n+1}=(1+x)^n(1+x)\ge (1+nx)(1+x)=1+(n+1)x+nx^2\ge 1+(n+1)x,$$

where the last step uses $nx^2\ge 0$. This is the statement for $n+1$. Induction completes the proof. ∎

The proof uses the hypothesis $x\ge -1$ exactly once to multiply an inequality by $1+x$. The hypothesis cannot be dropped. For $n=3$ and $x=-4$ the left side is $(-3)^3=-27$ and the right side is $1-12=-11$. Bernoulli's inequality (2.3.5) bounds a power below by a linear function that is easy to estimate. Taking $x=0.01$ and $n=100$ gives $1.01^{100}\ge 2$. Taking $x=9$ gives $10^n\ge 1+9n>n$ for every $n\in\mathbb{N}$. The last inequality is used in Section 2.4.

Theorem 2.3.12 (Arithmetic and geometric means)**.** *For all real $a,b\ge 0$,*

$$ab\le \left(\frac{a+b}{2}\right)^2, \tag{2.3.6}$$

with equality exactly when $a=b$.

Proof. Expanding gives $\left(\frac{a+b}{2}\right)^2-ab=\frac{1}{4}(a^2+2ab+b^2)-ab=\frac{1}{4}(a-b)^2$. This number is nonnegative by Proposition 2.2.5(iii) and zero exactly when $a=b$. ∎

Once square roots are available (Theorem 2.4.8), taking the nonnegative square root of both sides of (2.3.6) by Proposition 2.2.5(vi) gives the familiar form $\sqrt{ab}\le\frac{1}{2}(a+b)$: the geometric

mean of two nonnegative numbers is at most their arithmetic mean. Because the proof uses only $a^2 + b^2 \geq 2ab$, the squared form is true for all real a and b.

Study guide

Solve every polynomial or rational inequality by bringing it to the form $p(x)/q(x) \gtrless 0$ and drawing a sign chart. Check one test value in each interval and decide each endpoint separately.

Translate absolute values into distances. Read $|x - a| < \delta$ as "x lies within δ of a" and convert it to $a - \delta < x < a + \delta$ by Eq. (2.3.2). To bound $|f(x)|$ near a point a, rewrite each factor in terms of $x - a$ and apply Eq. (2.3.3), as in Worked Example 2.3.4. This technique recurs in every ε–δ proof of Chapter 5.

When applying Theorem 2.3.11, check the hypothesis $x \geq -1$ first.

Problems

2.3.1 Solve $x^2 - x - 6 < 0$. ⋆ Hint · Answer · Solution

2.3.2 Solve $|3x - 2| > 7$. ⋆ Hint · Answer · Solution

2.3.3 Solve $\dfrac{(x-1)^2(x+3)}{x-2} \leq 0$. ⋆⋆ Hint · Answer · Solution

2.3.4 Solve $\dfrac{1}{x} < x$. ⋆⋆ Hint · Answer · Solution

2.3.5 Find the largest $\delta > 0$ such that $|x - 3| < \delta$ implies $|2x - 6| < 0.1$. ⋆⋆ Hint · Answer · Solution

2.3.6 Prove that if $|x - 1| < \frac{1}{2}$ then $\left|\dfrac{1}{x} - 1\right| \leq 2|x - 1|$. ⋆⋆ Hint · Answer · Solution

2.3.7 Prove by induction that $(1 + x)^n \geq 1 + nx + \dfrac{n(n-1)}{2}x^2$ for every $n \in \mathbb{N}$ and every $x \geq 0$. ⋆⋆⋆ Hint · Answer · Solution

2.3.8 Among all rectangles of area 16, find the least perimeter and the rectangle that achieves it. ⋆⋆ Hint · Answer · Solution

2.4 Completeness

Overview

Objectives: Find upper and lower bounds, maxima and minima, suprema and infima of sets of real numbers (Definitions 2.4.1 and 2.4.3). State the completeness axiom (Assumption 2.4.5) and use it to prove the existence of square roots, the Archimedean property (Theorem 2.4.10), the density of the rationals and irrationals (Theorem 2.4.12) and the nested interval property. Explain why calculus fails over $\mathbb{Q}$.
Prerequisites: The ordered field axioms (Section 2.2). Inequalities and Bernoulli's inequality (Section 2.3). Negating quantified statements from Chapter 1.
Key results: Every nonempty set bounded above has a supremum (Assumption 2.4.5). The approximation property of the supremum (Lemma 2.4.4). $\mathbb{N}$ is unbounded and every interval contains rationals and irrationals (Theorems 2.4.10 and 2.4.12). Nested closed intervals have a common point (Theorem 2.4.17).

Notation

$\sup S,\ \inf S$	supremum and infimum of S (Definition 2.4.3)
$\max S,\ \min S$	greatest and least element of S (Definition 2.4.2)
$\lfloor x \rfloor$	the integer part of x (Lemma 2.4.11)
$\sqrt{a}$	the nonnegative real number whose square is $a \geq 0$ (Theorem 2.4.8)

The rationals satisfy every axiom of Section 2.2, yet the equation $x^2 = 2$ has no rational solution (Theorem 2.1.3). On the rational number line the set $\{x \in \mathbb{Q} : x > 0,\ x^2 < 2\}$ is bounded above by 2. Its elements creep up towards a point where there is no rational number. The real line is assumed to have no such gaps. This section states the assumption precisely and draws its first consequences.

Definition 2.4.1 (Bounds)**.** Let $S \subseteq \mathbb{R}$. A number M is an *upper bound* of S if $x \leq M$ for every $x \in S$. A set with an upper bound is *bounded above*. Likewise m is a *lower bound* of S if $m \leq x$ for every $x \in S$. A set with a lower bound is *bounded below*. A set bounded above and below is *bounded*.

The interval $[0, 1)$ is bounded. Its upper bounds are the numbers $M \geq 1$, such as 1, 2 and 17. As a non-example, $(0, \infty)$ is not bounded above. For any M the number $M + 1$ (when $M > 0$) or 1 (when $M \leq 0$) lies in $(0, \infty)$ and exceeds M.

Definition 2.4.2 (Maximum and minimum)**.** An element of S that is an upper bound of S is the *maximum* $\max S$. An element of S that is a lower bound of S is the *minimum* $\min S$.

For example $\max[0, 1] = 1$ and $\min[0, 1] = 0$. As a non-example, $[0, 1)$ has no maximum. For any $s \in [0, 1)$ the midpoint $\dfrac{s+1}{2}$ lies in $[0, 1)$ and exceeds s by Worked Example 2.2.1. The number 1 is still the best upper bound. The next definition gives it a name.

Definition 2.4.3 (Supremum and infimum)**.** A number β is the *supremum* of S, written $\beta = \sup S$, if β is an upper bound of S and no number smaller than β is an upper bound of S. Equivalently, β is an upper bound with $\beta \leq M$ for every upper bound M of S (the *least upper bound*). The *infimum* $\inf S$ is the greatest lower bound, defined symmetrically.

A set has at most one supremum. If β and β' are both least upper bounds, then $\beta \leq \beta'$ and $\beta' \leq \beta$ give $\beta = \beta'$. If S has a maximum then $\sup S = \max S$ because the element $\max S$ exceeds any smaller number. As an example $\sup[0,1) = 1$. The number 1 is an upper bound. If $M < 1$ were an upper bound then $M \geq 0$ (since $0 \in [0,1)$) and $\dfrac{M+1}{2} \in [0,1)$ would exceed M. As a non-example, 2 is an upper bound of $[0,1)$ but is not its supremum because the smaller number 1 is also an upper bound.

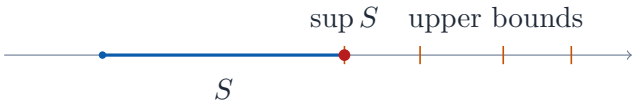


Figure 2.4.1: Upper bounds of S forming a ray to the right with leftmost point $\sup S$. This point may or may not belong to S itself.

Figure 2.4.1 shows the picture to keep in mind. The upper bounds of a set bounded above fill a ray whose left end is the supremum. The next lemma restates the definition in the form used in proofs.

Lemma 2.4.4 (Approximation property of the supremum)**.** *Let S be nonempty and let β be an upper bound of S. Then $\beta = \sup S$ if and only if for every $\varepsilon > 0$ there is an element $s \in S$ with $\beta - \varepsilon < s$. Symmetrically, if α is a lower bound of S then $\alpha = \inf S$ if and only if for every $\varepsilon > 0$ there is an element $s \in S$ with $s < \alpha + \varepsilon$.*

Proof. **1.** Supremum. Suppose $\beta = \sup S$ and $\varepsilon > 0$. The number $\beta - \varepsilon$ is smaller than β and is therefore not an upper bound. Some $s \in S$ thus satisfies $s > \beta - \varepsilon$. Conversely suppose the condition is true and let $M < \beta$. With $\varepsilon = \beta - M > 0$ there is $s \in S$ with $s > \beta - \varepsilon = M$. Hence M is not an upper bound. Hence no number smaller than β is an upper bound and $\beta = \sup S$. **2.** Infimum. Suppose $\alpha = \inf S$ and $\varepsilon > 0$. The number $\alpha + \varepsilon$ is larger than α and is therefore not a lower bound. Some $s \in S$ thus satisfies $s < \alpha + \varepsilon$. Conversely suppose the condition is true and let $m > \alpha$. With $\varepsilon = m - \alpha > 0$ there is $s \in S$ with $s < \alpha + \varepsilon = m$. Hence no number larger than α is a lower bound and $\alpha = \inf S$. ∎

Assumption 2.4.5 (Completeness of $\mathbb{R}$)**.** Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

The field axioms, the order axioms and Assumption 2.4.5 together describe $\mathbb{R}$ completely: any two ordered fields satisfying the completeness axiom are copies of one another with the same arithmetic and order. This uniqueness theorem belongs to a course in real analysis and is not needed here. The axiom mentions suprema only because infima follow.

Proposition 2.4.6 (Existence of infima)**.** *Every nonempty subset S of $\mathbb{R}$ that is bounded below has the infimum $\inf S = -\sup(-S)$ with $-S = \{-x : x \in S\}$.*

Proof. By Proposition 2.2.5(i) a number m is a lower bound of S exactly when $-m$ is an upper bound of $-S$. The set $-S$ is therefore nonempty and bounded above. Its supremum $\beta = \sup(-S)$ exists by Assumption 2.4.5. Then $-\beta$ is a lower bound of S. If m is any lower bound of S, then $-m$ is an upper bound of $-S$ with $\beta \leq -m$ and hence $m \leq -\beta$. Hence $-\beta$ is the greatest lower bound. ∎

2.4.7 Square roots

Theorem 2.4.8 (Existence of square roots)**.** *For every real $a \geq 0$ there is exactly one real $s \geq 0$ with $s^2 = a$. It is written $\sqrt{a}$.*

Proof. **1.** Uniqueness. Let $s, t \geq 0$ with $s^2 = t^2 = a$. If $t < s$ then $t^2 < s^2$ and if $s < t$ then $s^2 < t^2$ by Proposition 2.2.5(vi). Both contradict $s^2 = t^2$. Hence $s = t$ by (O1).
2. The case $a = 0$. The number $s = 0$ satisfies $s^2 = 0$.
3. A supremum for $a > 0$. Let $a > 0$ and $S = \{x \in \mathbb{R} : x > 0,\ x^2 < a\}$. The number $a/(a+1)$ is positive and lies in S. The inequality $\big(a/(a+1)\big)^2 < a$ is equivalent to $a^2 < a(a+1)^2$ after multiplying by $(a+1)^2 > 0$, and to $a < (a+1)^2 = a^2 + 2a + 1$ after dividing by $a > 0$. The last inequality is true because $a^2 + a + 1 > 0$. The set S is bounded above by $a + 1$ because $x \geq a+1$ implies $x^2 \geq (a+1)^2 > a$ by Proposition 2.2.5(vi). By Assumption 2.4.5 the supremum $s = \sup S$ exists with $s \geq a/(a+1) > 0$. We rule out $s^2 < a$ and $s^2 > a$.
(i) $s^2 < a$. Let h be half the smaller of 1 and $\dfrac{a - s^2}{2s + 1}$. Then $0 < h < 1$ and $h < \dfrac{a - s^2}{2s + 1}$. Since $h < 1$ gives $h^2 < h$ (Worked Example 2.2.2),

$$(s + h)^2 = s^2 + 2sh + h^2 < s^2 + (2s + 1)h < s^2 + (a - s^2) = a. \tag{2.4.1}$$

By (2.4.1), $s + h \in S$ and $s + h > s$. This contradicts that s is an upper bound.
(ii) $s^2 > a$. The number $h = \dfrac{s^2 - a}{2s}$ is positive and smaller than s because $s^2 - a < s^2 < 2s^2$. Then

$$(s - h)^2 = s^2 - 2sh + h^2 = a + h^2 > a. \tag{2.4.2}$$

If some $x \in S$ satisfied $x > s - h > 0$, then Proposition 2.2.5(vi) would give the impossible $x^2 > (s - h)^2 > a$ by (2.4.2). Hence $s - h$ is an upper bound of S smaller than s. This contradicts that s is the least upper bound.
By trichotomy $s^2 = a$. ∎

The number $\sqrt{2}$ is irrational by Theorem 2.1.3. Hence $\mathbb{R}$ contains numbers that $\mathbb{Q}$ lacks. For $a = 2$ the proof uses the element $\frac{2}{3}$ of S and the upper bound 3. The element 1 and the bound 2 serve equally well. We use square roots freely from now on. The same supremum argument gives nth roots (Theorem 4.4.6) and the Intermediate Value Theorem gives them again in Corollary 5.4.3.

2.4.9 Archimedean property and density

Theorem 2.4.10 (Archimedean property)**.** *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$. Equivalently, every real x is exceeded by some $n \in \mathbb{N}$. Consequently every $\varepsilon > 0$ exceeds $\dfrac{1}{n}$ for some $n \in \mathbb{N}$.*

Proof. Suppose $\mathbb{N}$ were bounded above. Its supremum $\beta = \sup \mathbb{N}$ would exist by Assumption 2.4.5. By Lemma 2.4.4 with $\varepsilon = 1$ there is $n \in \mathbb{N}$ with $n > \beta - 1$. Then $n + 1 \in \mathbb{N}$ and $n + 1 > \beta$. This contradicts that β is an upper bound of $\mathbb{N}$. Hence for every x some $n \in \mathbb{N}$ exceeds x. Given $\varepsilon > 0$ choose $n > 1/\varepsilon$. Then $\dfrac{1}{n} < \varepsilon$ by Proposition 2.2.5(iv). ∎

The statement looks obvious. It is nevertheless a theorem because some ordered fields contain an element exceeding every natural number. One example is the field of quotients $p(t)/q(t)$ of polynomials with real coefficients, where a quotient counts as positive when the leading coefficients of p and q have the same sign. The order axioms can be checked there. The element t exceeds every natural number n because $t - n$ has leading coefficient 1. The

completeness axiom excludes such fields. The last form says that $\frac{1}{n}$ can be made smaller than any prescribed tolerance. This is the first limit of Chapter 5.

Lemma 2.4.11 (Existence of the integer part)**.** *For every real x there is exactly one integer m with $m \le x < m+1$. It is written $\lfloor x \rfloor$. Every integer $k \le x$ satisfies $k \le \lfloor x \rfloor$.*

Proof. **1.** Existence. By Theorem 2.4.10 there is $n \in \mathbb{N}$ with $n > |x|$ and hence $-n < x < n$. The set $J = \{j \in \mathbb{N} : j > x + n\}$ contains $2n$ because $x + n < 2n$. By Assumption 1.4.1 it has a least element j. Put $m = j - 1 - n$. Then $m + 1 = j - n > x$. If $j = 1$ then $m = -n < x$. If $j \ge 2$ then $j - 1 \in \mathbb{N}$ and $j - 1 \notin J$ by minimality. Then $j - 1 \le x + n$ and $m \le x$.
2. Uniqueness and maximality. Let m be as above and let $k \le x$ be an integer. If $k > m$ then $k - m$ is a positive integer. The integers are the numbers n, 0 and $-n$ with $n \in \mathbb{N}$, and the only positive ones are the natural numbers. Every natural number is at least 1 (Section 2.2). Hence $k - m \ge 1$ and $k \ge m + 1 > x$. This contradicts $k \le x$. Hence $k \le m$. A second integer m' with $m' \le x < m' + 1$ satisfies $m' \le m$ by this argument and $m \le m'$ by the same argument with the roles exchanged. Hence $m' = m$. ∎

Theorem 2.4.12 (Density of $\mathbb{Q}$ and of the irrationals)**.** *If $a < b$ are real numbers, there is a rational number q and an irrational number t with $a < q < b$ and $a < t < b$.*

Proof. By Theorem 2.4.10 choose $n \in \mathbb{N}$ with $\frac{1}{n} < b - a$, that is $nb - na > 1$. Let $m = \lfloor na \rfloor + 1$. Then $m - 1 \le na < m$ by Lemma 2.4.11. The inequality $na < m$ gives $a < \frac{m}{n}$. The inequality $m \le na + 1 < nb$ gives $\frac{m}{n} < b$. The rational number $q = \frac{m}{n}$ lies in (a, b).

For the irrational number apply the first part to $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ to get a rational q_1 in between. If $q_1 = 0$, apply the first part again to $0 < \frac{b}{\sqrt{2}}$ to replace it by a nonzero rational in $(0, \frac{b}{\sqrt{2}})$. Multiplying by $\sqrt{2} > 0$ gives $a < q_1\sqrt{2} < b$. The number $t = q_1\sqrt{2}$ is irrational by Worked Example 2.1.1 and Theorem 2.1.3. ∎

Repeated application of the theorem inside (a, q) and (q, b) shows that every open interval contains infinitely many rationals and infinitely many irrationals. Density also shows that $\mathbb{Q}$ itself is not complete. The set $T = \{x \in \mathbb{Q} : x > 0,\ x^2 < 2\}$ is nonempty and bounded above in $\mathbb{Q}$. A positive rational $u < \sqrt{2}$ is no upper bound of T because Theorem 2.4.12 supplies a rational in $(u, \sqrt{2})$ that belongs to T. The rational upper bounds of T are therefore the rationals $u > \sqrt{2}$. For each such u the theorem supplies a smaller rational upper bound u' with $\sqrt{2} < u' < u$. Hence T has no least upper bound in $\mathbb{Q}$.

Worked Example 2.4.1

Find the supremum and infimum of $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ and decide whether they are a maximum and a minimum.
The elements are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. Since $\frac{1}{n} \le 1$, every element is at least $0 = 1 - \frac{1}{1} \in S$. Hence $\inf S = \min S = 0$. The number 1 exceeds every element and is therefore an upper bound. Given $\varepsilon > 0$, Theorem 2.4.10 gives n with $\frac{1}{n} < \varepsilon$ and then $1 - \frac{1}{n} > 1 - \varepsilon$. By Lemma 2.4.4, $\sup S = 1$. The supremum is not a maximum because $1 - \frac{1}{n} = 1$ would require $\frac{1}{n} = 0$.

2.4.13 Decimal expansions revisited

Section 2.1 used decimals without defining the value of an infinite decimal. With the supremum available the definition is short. For a nonnegative integer d_0 and digits $d_1, d_2, \ldots \in \{0, 1, \ldots, 9\}$, the decimal $d_0.d_1d_2d_3\ldots$ is the number

$$\sup\{t_0, t_1, t_2, \ldots\}, \qquad t_k = d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k}. \tag{2.4.3}$$

The number t_k is the kth *truncation*. The truncations are nondecreasing because $t_{k+1} = t_k + d_{k+1}/10^{k+1} \geq t_k$. The identity $(10-1)(10^{l-1} + \cdots + 10 + 1) = 10^l - 1$ gives

$$\frac{9}{10^{k+1}} + \frac{9}{10^{k+2}} + \cdots + \frac{9}{10^{k+l}} = 10^{-k} - 10^{-(k+l)}. \tag{2.4.4}$$

Taking $k = 0$ in (2.4.4) gives $t_l \leq d_0 + 1 - 10^{-l} < d_0 + 1$ for every l. The supremum therefore exists by Assumption 2.4.5.

Proposition 2.4.14 (Suprema of translated and scaled sets)**.** *Let S be nonempty and bounded above.*

(i) For real c the set $c + S = \{c + x : x \in S\}$ has $\sup(c + S) = c + \sup S$.

(ii) For $c > 0$ the set $cS = \{cx : x \in S\}$ has $\sup(cS) = c \sup S$.

(iii) If $S = \{s_0, s_1, s_2, \ldots\}$ with $s_0 \leq s_1 \leq s_2 \leq \cdots$, then $\sup\{s_k : k \geq j\} = \sup S$ for every $j \geq 0$.

(iv) Two decimals with the same digits after the point differ by the integer obtained by subtracting their integer parts.

(v) For $j \geq 0$ the number $10^j \cdot d_0.d_1d_2\ldots$ is the decimal with integer part $10^jd_0 + 10^{j-1}d_1 + \cdots + d_j$ and digits $d_{j+1}d_{j+2}\ldots$ after the point.

Proof. (i) By (O3), $x \leq M$ exactly when $c + x \leq c + M$. The upper bounds of $c + S$ are therefore the numbers $c + M$ with M an upper bound of S. The least of them is $c + \sup S$.

(ii) Let $c > 0$. Then $c^{-1} > 0$ by Proposition 2.2.5(iv). Multiplying by c or by c^{-1} under (O4) shows that $x \leq M$ exactly when $cx \leq cM$. The upper bounds of cS are therefore the numbers cM with M an upper bound of S. The least of them is $c \sup S$.

(iii) The set $T = \{s_k : k \geq j\}$ is a nonempty subset of S. Every upper bound of S is an upper bound of T. Conversely let M be an upper bound of T. For $k < j$ we have $s_k \leq s_j \leq M$. Hence M is an upper bound of S. The two sets have the same upper bounds and therefore the same least upper bound.

(iv) The truncations of $d_0.d_1d_2\ldots$ are d_0 plus the truncations of $0.d_1d_2\ldots$. By (i) the decimal equals d_0 plus the decimal $0.d_1d_2\ldots$. Two decimals with the same fractional digits therefore differ by the difference of their integer parts.

(v) Let $x = d_0.d_1d_2\ldots$ have truncations t_k and let y be the decimal described in (v), with truncations u_k. Multiplying out,

$$10^j t_{j+k} = 10^j d_0 + 10^{j-1}d_1 + \cdots + d_j + \frac{d_{j+1}}{10} + \cdots + \frac{d_{j+k}}{10^k} = u_k.$$

Hence $y = \sup_{k \geq 0} 10^j t_{j+k} = 10^j \sup_{k \geq j} t_k = 10^j x$ by (ii) and by (iii) applied to the nondecreasing truncations. ∎

Lemma 2.4.15 (Uniqueness of decimal expansions)**.** *(i) Let $y = e_0.e_1e_2\ldots$ have truncations s_k. For every $k \geq 0$,*

$$s_k \leq y \leq s_k + 10^{-k}, \tag{2.4.5}$$

and $y = s_k + 10^{-k}$ exactly when $e_i = 9$ for every $i > k$.

(ii) Every real $x \geq 0$ has the decimal expansion with digits $d_0 = \lfloor x \rfloor$ and $d_k = \lfloor 10^k x \rfloor - 10\lfloor 10^{k-1}x \rfloor$ for $k \geq 1$. Its truncations are $t_k = \lfloor 10^k x \rfloor / 10^k$ and it does not end in an unbroken string of nines. It is the standard expansion *of x.*

(iii) An expansion of x that does not end in an unbroken string of nines is the standard expansion. If an expansion $e_0.e_1e_2\ldots$ of x has $e_i = 9$ for every $i > N$, with N the least such index, then $d_k = e_k$ for $k < N$, $d_N = e_N + 1$ and $d_k = 0$ for $k > N$. A real number therefore has at most two decimal expansions. Two different expansions of one number consist of a terminating expansion and an expansion ending in nines.

(iv) For integers $m \geq 0$ and $n \in \mathbb{N}$ the digits of m/n produced by long division (2.1.4) form the standard expansion of m/n.

Proof. **1.** The bracket. Let $l > k$. By (2.4.4),

$$s_l - s_k = \frac{e_{k+1}}{10^{k+1}} + \cdots + \frac{e_l}{10^l} \leq 10^{-k} - 10^{-l} < 10^{-k}.$$

For $l \leq k$ we have $s_l \leq s_k$. Hence $s_k + 10^{-k}$ is an upper bound of the truncations and $s_k \leq y \leq s_k + 10^{-k}$. Suppose $e_i = 9$ for every $i > k$. Then $s_l = s_k + 10^{-k} - 10^{-l}$ for $l > k$. Given $\varepsilon > 0$ choose $l > k$ with $l > 1/\varepsilon$ (Theorem 2.4.10). Bernoulli's inequality gives $10^l > l$. Hence $10^{-l} < \varepsilon$ and $s_l > s_k + 10^{-k} - \varepsilon$. By Lemma 2.4.4, $y = s_k + 10^{-k}$. Suppose instead $e_i \leq 8$ for some $i > k$. For $l \geq i$ the term $e_i/10^i$ falls short of $9/10^i$ by at least 10^{-i}, and $s_l \leq s_k + 10^{-k} - 10^{-i}$. For $l < i$ the same bound follows from $s_l \leq s_i$. Hence $y \leq s_k + 10^{-k} - 10^{-i} < s_k + 10^{-k}$.
2. The standard expansion. Let $k \geq 1$ and $p = \lfloor 10^{k-1}x \rfloor$. Then $10p \leq 10^k x < 10p + 10$. The integer $10p$ is at most $10^k x$ and therefore at most $\lfloor 10^k x \rfloor$ by Lemma 2.4.11. The integer $10p + 10 - \lfloor 10^k x \rfloor$ is positive and hence at least 1. Together $0 \leq d_k \leq 9$. The sum $d_0 + d_1/10 + \cdots + d_k/10^k$ telescopes to $t_k = \lfloor 10^k x \rfloor / 10^k$. Dividing $\lfloor 10^k x \rfloor \leq 10^k x < \lfloor 10^k x \rfloor + 1$ by 10^k gives $t_k \leq x < t_k + 10^{-k}$. Given $\varepsilon > 0$ choose $k > 1/\varepsilon$ by Theorem 2.4.10. Bernoulli's inequality gives $10^k > k$ and hence $t_k > x - 10^{-k} > x - \varepsilon$. Then Lemma 2.4.4 gives $x = \sup_k t_k$. If $d_i = 9$ for every $i > k$, the bracket of step 1 would give $x = t_k + 10^{-k}$, against $x < t_k + 10^{-k}$.
3. Uniqueness. Let $x = e_0.e_1e_2\ldots$ with truncations s_k. The number $10^k s_k$ is an integer. Suppose that $e_i \neq 9$ for some $i > k$. Step 1 gives $s_k \leq x < s_k + 10^{-k}$, that is $10^k s_k \leq 10^k x < 10^k s_k + 1$. Lemma 2.4.11 gives $10^k s_k = \lfloor 10^k x \rfloor$ and hence $s_k = t_k$. If the expansion does not end in nines this is true for every k. Then $e_0 = s_0 = t_0 = d_0$ and $e_k = 10^k(s_k - s_{k-1}) = d_k$ for $k \geq 1$. Otherwise let N be as in (iii). For $k < N$ the digit e_N with $N > k$ is not 9 by the choice of N. Hence $s_k = t_k$ and $e_k = d_k$ for $k < N$. Step 1 gives $x = s_N + 10^{-N}$. For $k \geq N$ the number $10^k x = 10^{k-N}(10^N s_N + 1)$ is an integer. Then $t_k = x$ for $k \geq N$ and $d_k = 10^k(t_k - t_{k-1}) = 0$ for $k > N$. If $N \geq 1$ then $d_N = 10^N(x - s_{N-1}) = 10^N(s_N - s_{N-1}) + 1 = e_N + 1$. If $N = 0$ then $d_0 = t_0 = x = e_0 + 1$. The standard expansion therefore determines N as the position of its last nonzero digit and determines every digit of a second expansion. A second expansion exists only when the standard one terminates.

4. Long division. Let $d_0, d_1, \ldots$ and $\rho_0, \rho_1, \ldots$ be as in (2.1.4), with $m = d_0 n + \rho_0$, and put $P_k = 10^k d_0 + 10^{k-1} d_1 + \cdots + d_k$. We show $10^k m = P_k n + \rho_k$ by induction on k. The case $k = 0$ is $m = d_0 n + \rho_0$. If it is true for k then

$$10^{k+1} m = 10 P_k n + 10\rho_k = (10P_k + d_{k+1})n + \rho_{k+1} = P_{k+1} n + \rho_{k+1}.$$

Dividing by n gives $10^k m/n = P_k + \rho_k/n$ with $0 \le \rho_k/n < 1$. Lemma 2.4.11 gives $P_k = \lfloor 10^k m/n \rfloor$. Hence $d_0 = P_0$ and $d_k = P_k - 10P_{k-1}$ are the standard digits of m/n. ∎

Worked Example 2.4.2

Show that $0.999\ldots = 1$.

The truncations are $t_k = \dfrac{9}{10} + \cdots + \dfrac{9}{10^k} = 1 - 10^{-k}$ by the identity $(10-1)(10^{k-1} + \cdots + 1) = 10^k - 1$. The number 1 exceeds each t_k and is therefore an upper bound. Given $\varepsilon > 0$ choose k with $k > 1/\varepsilon$ by Theorem 2.4.10. Bernoulli's inequality (2.3.5) gives $10^k \ge 1 + 9k > k$. Hence $10^{-k} < \varepsilon$ and $t_k > 1 - \varepsilon$. By Lemma 2.4.4, $0.999\ldots = \sup_k t_k = 1$. The two decimals $0.999\ldots$ and $1.000\ldots$ name the same real number. In the same way every nonzero terminating decimal has a second expansion ending in nines, such as $0.25 = 0.24999\ldots$. By Lemma 2.4.15 it has no third.

2.4.16 Nested intervals

Theorem 2.4.17 (Nested intervals)**.** *Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$ be closed bounded intervals. Then some real number x lies in every $[a_n, b_n]$. If moreover for every $\varepsilon > 0$ some n has $b_n - a_n < \varepsilon$, then exactly one such x exists.*

Proof. Nesting means $a_n \le a_{n+1} \le b_{n+1} \le b_n$. Hence $a_m \le b_n$ for all m and n. If $m \le n$ then $a_m \le a_n \le b_n$. If $m > n$ then $a_m \le b_m \le b_n$. Every b_n is thus an upper bound of the nonempty set $A = \{a_1, a_2, \ldots\}$. By Assumption 2.4.5 the number $x = \sup A$ exists. It satisfies $a_n \le x$ because it is an upper bound of A and $x \le b_n$ because it is the least upper bound. Hence $x \in [a_n, b_n]$ for every n.

If $x < y$ both lie in every interval, then $y - x \le b_n - a_n$ for every n. The hypothesis with $\varepsilon = y - x$ contradicts this. ∎

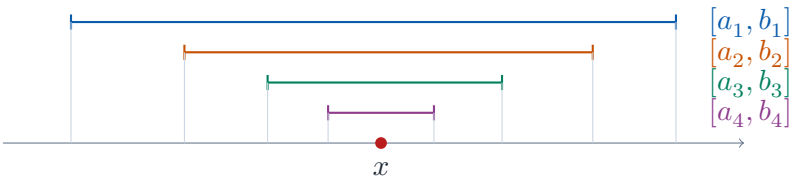


Figure 2.4.2: Nested closed intervals. Each interval lies inside the one above it. Completeness supplies a point x common to all of them.

Figure 2.4.2 illustrates the theorem. Both hypotheses matter. The intervals $[n, \infty)$ are nested, closed and unbounded. No real number lies in all of them by Theorem 2.4.10. The intervals $(0, \frac{1}{n}]$ are nested and bounded but not closed. Their intersection is empty because any $x > 0$ in all of them would satisfy $x \le \frac{1}{n}$ for every n. This contradicts Theorem 2.4.10. Over $\mathbb{Q}$ the theorem fails as well. The intervals $[t_k, t_k + 10^{-k}]$ built from the decimal truncations of $\sqrt{2}$ have rational endpoints and the irrational number $\sqrt{2}$ as their only common point.

2.4.18 Completeness in calculus

The examples above show what goes wrong on a number line with gaps. The function $f(x) = x^2 - 2$ on the rationals in $[1, 2]$ has $f(1) = -1 < 0$ and $f(2) = 2 > 0$ and takes the value 0 at no rational point. The Intermediate Value Theorem (Theorem 5.4.2) is therefore false over $\mathbb{Q}$. Its proof in Chapter 5 takes a supremum. The nondecreasing sequence of truncations $1,\ 1.4,\ 1.41,\ 1.414, \ldots$ of $\sqrt{2}$ is bounded above by 2 and has no rational limit. The Monotone Convergence Theorem (Theorem 5.1.18) is therefore false over $\mathbb{Q}$ as well. Its proof applies the approximation property Lemma 2.4.4 to the set of terms. The Extreme Value Theorem, the Mean Value Theorem and the existence of the integral all rest on these two results and through them on Assumption 2.4.5.

Caution

A supremum need not be an element of the set. A statement such as "let $\beta = \sup S \in S$" is false for $S = [0, 1)$. Lemma 2.4.4 supplies elements of S as close to β as required.

Study guide

Read $\sup S$ as the lowest ceiling above all of S (Fig. 2.4.1) and $\inf S$ as the highest floor below all of S. To prove that a candidate β is the supremum, check that β is an upper bound and that for each $\varepsilon > 0$ some element of S exceeds $\beta - \varepsilon$ (Lemma 2.4.4). The second step almost always calls on Theorem 2.4.10.

Learn the proof of Theorem 2.4.8. Its pattern, assuming the supremum is too small or too large and moving by a carefully chosen h to reach a contradiction, is the pattern of the proof of the Intermediate Value Theorem.

Problems

2.4.1 Find the supremum, infimum, maximum and minimum of $(-1, 3]$, where they exist.

⋆ Hint · Answer · Solution

2.4.2 Find $\sup S$ and $\inf S$ for $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ and decide which is attained.

⋆ Hint · Answer · Solution

2.4.3 Find $\sup S$ and $\inf S$ for $S = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$. ⋆⋆ Hint · Answer · Solution

2.4.4 Let $S \subseteq T$ be nonempty sets with T bounded above. Prove that $\sup S \leq \sup T$.

⋆⋆ Hint · Answer · Solution

2.4.5 Let A and B be nonempty sets of real numbers with $a \leq b$ for every $a \in A$ and $b \in B$. Prove that $\sup A$ and $\inf B$ exist and that $\sup A \leq \inf B$.

⋆⋆⋆ Hint · Answer · Solution

2.4.6 Let $a, b \geq 0$. Prove that $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ and that $a < b$ implies $\sqrt{a} < \sqrt{b}$. Deduce that $\sqrt{ab} \leq \frac{1}{2}(a + b)$. ⋆⋆ Hint · Answer · Solution

2.4.7 Use the definition of a decimal as the supremum of its truncations to show that $0.\overline{3} = \frac{1}{3}$.

⋆⋆⋆ Hint · Answer · Solution

Chapter 3

Vectors and Linear Maps

Calculus in one variable uses a small amount of linear algebra at every turn. The derivative of a function at a point is the slope of the linear map that best approximates the increment of the function (Theorem 6.1.9). Section 6.2 reads the differential $\mathrm{d}x$ as a linear functional on the displacements at a point and builds every integrand $f(x)\,\mathrm{d}x$ from it. Chapter 7 treats differentiation as a linear map between spaces of functions with a kernel, an image and a matrix on polynomials. The definite integral of Chapter 8 is a linear functional on the continuous functions of an interval. Each of these statements needs the words vector, linear map and linear functional with exact meanings. This chapter supplies them with complete proofs.

The chapter also supplies the geometry of the plane that the trigonometry of Section 4.5 and the proofs of Appendix A use: the dot product and the length of a vector, the Cauchy–Schwarz inequality, rotations and reflections as matrices, the isometries of the plane that fix the origin, and the determinant of a 2×2 matrix as a signed area. Vectors in several variables together with their calculus (partial derivatives, gradients, integrals over regions) belong to multivariable calculus and are not treated in this book. The plane appears here as a setting for geometry and as the first example of a vector space beyond the line.

Section 3.1 introduces vectors as displacements in the line and the plane and separates them from points. Section 3.2 states the axioms of a vector space and develops span, linear independence, bases and dimension. Section 3.3 studies linear maps, their kernels and images, the rank–nullity theorem and matrices. Section 3.4 specialises to linear maps with values in $\mathbb{R}$, the linear functionals on which the 1-forms of Section 6.2 rest. Section 3.5 returns to the plane with lengths, angles, isometries and determinants. Section 3.6 replaces the Euclidean length by any norm, compares the norms on $\mathbb{R}^n$ with the sup norm on functions, and proves that a linear map is continuous exactly when it is bounded. The later chapters state limits, continuity and convergence in normed spaces, with the absolute value on $\mathbb{R}$ as the first example. Section 6.2 defines the derivative of a map between normed spaces (Definition 6.2.14). The proofs use only the field, order and completeness axioms of Chapter 2, the existence of square roots (Theorem 2.4.8) and the language of sets and functions of Chapter 1. A proof that cites a theorem of a later chapter says why the reference is not circular. The treatment follows the one-variable parts of Hubbard and Hubbard [1].

3.1 Vectors in the Line and the Plane

Overview

Objectives: Describe a displacement in the line or the plane by a vector; add vectors and multiply them by numbers both geometrically and in coordinates; distinguish a point from a vector and decide which combinations of points have a meaning; describe the tangent space $T_p\mathbb{R}$ of displacements based at a point p.

Prerequisites: The field axioms (Assumption 2.2.1); ordered pairs and functions (Definition 1.5.8).
Key results: Vectors of the plane and their operations (Definition 3.1.3); the rules of vector algebra (Proposition 3.1.4); affine combinations of points (Proposition 3.1.6); the tangent space $T_p\mathbb{R}$ (Definition 3.1.8).

Notation

$\mathbf{u}, \mathbf{v}, \mathbf{w}$	vectors (bold); numbers are written in italic
$\mathbb{R}^2$	the plane: ordered pairs (x, y) of real numbers
$\mathbf{v} = (v_1, v_2)$	a vector of the plane and its components
$\mathbf{0}$	the zero vector
$\overrightarrow{PQ}$	the displacement from the point P to the point Q
$T_p\mathbb{R}$	the displacements based at the point p of the line (Definition 3.1.8)

A number can record a position or a change of position. On a road marked in kilometres the statement "the car is at 5" gives a position. The statement "the car moved 3 kilometres forward" gives a change of position with no reference to where the car started. Starting at 5 the car ends at 8. Starting at -2 it ends at 1. A change of position is a *displacement*. Calculus constantly compares the value of a function at a point p with its value at a displaced point $p+h$, and the derivative is a statement about how the value responds to the displacement h. This section keeps positions and displacements apart.

3.1.1 Displacements in the line

The displacement from a point p of the line to a point q is the number $q - p$. It is positive when q lies to the right of p. Two displacements performed one after the other give their sum: moving by u and then by v moves by $u + v$. A displacement v multiplied by a number c is the displacement cv. It is $|c|$ times as long and points the other way when $c < 0$. A displacement of the line is called a *vector* of the line and is written in bold, $\mathbf{v}$, when its role as a displacement matters. The underlying object is a real number. The bold type records how the number is used.

3.1.2 Vectors in the plane

The plane $\mathbb{R}^2$ is the set of ordered pairs (x, y) of real numbers. The displacement from the point $P = (p_1, p_2)$ to the point $Q = (q_1, q_2)$ is the pair of coordinate changes

$$\overrightarrow{PQ} = (q_1 - p_1,\ q_2 - p_2). \tag{3.1.1}$$

Two pairs of points give the same displacement exactly when one pair is the other moved without turning. The displacement from $(0, 0)$ to $(2, 1)$ and the displacement from $(3, -1)$ to $(5, 0)$ are both $(2, 1)$.

Definition 3.1.3 (Vectors in the plane)**.** A *vector* of the plane is an ordered pair $\mathbf{v} = (v_1, v_2)$ of real numbers, its *components*. For vectors $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and a real number c, the *sum* and the *scalar multiple* are

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2), \qquad c\mathbf{v} = (cv_1,\ cv_2). \tag{3.1.2}$$

The *zero vector* is $\mathbf{0} = (0, 0)$. The *negative* of $\mathbf{v}$ is $-\mathbf{v} = (-v_1, -v_2)$, and $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Real numbers used to multiply vectors are called *scalars*.

The definition is coordinatewise. Its geometric content appears in Fig. 3.1.1. Drawing $\mathbf{u}$ as an arrow and then $\mathbf{v}$ as an arrow starting at the tip of $\mathbf{u}$ produces the arrow $\mathbf{u} + \mathbf{v}$ from the start of the first to the end of the second. Drawing both from a common point produces a parallelogram whose diagonal is $\mathbf{u} + \mathbf{v}$. The multiple $2\mathbf{v}$ points the same way as $\mathbf{v}$ and is twice as long. The multiple $-\mathbf{v}$ has the length of $\mathbf{v}$ and the opposite direction. The vector $(3, 1)$ is an example. As non-examples, the pair of points $\{(0, 0), (3, 1)\}$ and the triple $(3, 1, 0)$ are no vectors of the plane: the first is a set of two points and the second has three components.

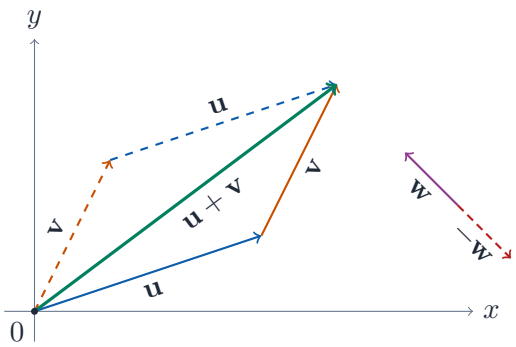


Figure 3.1.1: Sum of $\mathbf{u} = (3, 1)$ and $\mathbf{v} = (1, 2)$: tip to tail along either pair of sides of the parallelogram, with the diagonal $\mathbf{u} + \mathbf{v} = (4, 3)$. On the right a vector $\mathbf{w}$ and its negative $-\mathbf{w}$.

Proposition 3.1.4 (Rules of vector algebra)**.** *For all vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *of the plane and all scalars* c, d*:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *and* $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$*;*
2. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ *and* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$*;*
3. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ *and* $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$*;*
4. $c(d\mathbf{v}) = (cd)\mathbf{v}$ *and* $1\mathbf{v} = \mathbf{v}$*.*

Proof. Each identity compares two pairs. Two pairs are equal exactly when their first components agree and their second components agree. In each identity the first components agree by the corresponding field axiom of Assumption 2.2.1 applied to real numbers. For example the first component of $c(\mathbf{u} + \mathbf{v})$ is $c(u_1 + v_1)$ and that of $c\mathbf{u} + c\mathbf{v}$ is $cu_1 + cv_1$. They agree by distributivity. The second components agree for the same reason. ∎

The same definitions and the same proof apply to the line with one component and to triples or n-tuples with n components. Section 3.2 takes the eight rules of Proposition 3.1.4 as the definition of a vector space.

Worked Example 3.1.1

Let $P = (1, 2)$ and $Q = (4, -2)$. Find $\overrightarrow{PQ}$, the midpoint M of the segment PQ and the point S two thirds of the way from P to Q.
The displacement is $\overrightarrow{PQ} = (4-1, -2-2) = (3, -4)$. Moving from P by half of it reaches the midpoint:

$$M = P + \tfrac{1}{2}\overrightarrow{PQ} = (1, 2) + (\tfrac{3}{2}, -2) = (\tfrac{5}{2}, 0).$$

Moving by two thirds of it reaches $S = (1, 2) + (2, -\frac{8}{3}) = (3, -\frac{2}{3})$. As a check, $\overrightarrow{SQ} = (1, -\frac{4}{3})$ is one third of $\overrightarrow{PQ}$.

3.1.5 Points and vectors

A point and a vector of the plane are both ordered pairs. They play different roles. A point is a position measured from a chosen origin. A vector is a displacement and needs no origin. Adding a vector to a point gives a point ($P + \mathbf{v}$ is the point reached from P by the displacement $\mathbf{v}$). Subtracting two points gives a vector, $Q - P = \overrightarrow{PQ}$. Adding two points gives a pair whose meaning depends on the origin. If the origin is moved so that every point acquires new coordinates $P' = P - \mathbf{a}$, then $Q' - P' = Q - P$ is unchanged and $P' + Q' = P + Q - 2\mathbf{a}$ describes a different position. The next proposition identifies the combinations of points that do not depend on the origin.

Proposition 3.1.6 (Affine combinations)**.** *Let $P_1, \dots, P_k$ be points and $c_1, \dots, c_k$ scalars, and move the origin so that each point P acquires the coordinates $P' = P - \mathbf{a}$. Then*

$$\sum_{i=1}^{k} c_i P_i' = \sum_{i=1}^{k} c_i P_i - \Big(\sum_{i=1}^{k} c_i\Big)\mathbf{a}. \tag{3.1.3}$$

If $\sum c_i = 1$ the combination is the same point in both coordinate systems. If $\sum c_i = 0$ it is the same vector in both.

Proof. By Proposition 3.1.4, $\sum c_i(P_i - \mathbf{a}) = \sum c_i P_i - \sum c_i \mathbf{a} = \sum c_i P_i - (\sum c_i)\mathbf{a}$. When $\sum c_i = 1$ the new coordinates are the old ones minus $\mathbf{a}$, exactly as the coordinates of a fixed point change. When $\sum c_i = 0$ the coordinates do not change, exactly as a displacement behaves. ∎

The midpoint $\frac{1}{2}P + \frac{1}{2}Q$ and the centroid $\frac{1}{3}(P+Q+R)$ of a triangle are points, with coefficients adding to 1. The difference $Q - P$ is a vector, with coefficients adding to 0. The sum $P + Q$, with coefficients adding to 2, is neither.

Common misconception

Points and vectors look alike because both are written as pairs, and it is tempting to add two points. The sum $P + Q$ of two positions has no meaning independent of the origin. Combine points only with coefficients adding to 1 (giving a point) or to 0 (giving a vector), as in Proposition 3.1.6.

Worked Example 3.1.2

Prove that the diagonals of a parallelogram bisect each other.

Let the parallelogram have vertices A, $B = A + \mathbf{u}$, $D = A + \mathbf{v}$ and $C = A + \mathbf{u} + \mathbf{v}$, as in Fig. 3.1.1 with A at the origin. The midpoint of the diagonal AC is

$$\tfrac{1}{2}A + \tfrac{1}{2}C = A + \tfrac{1}{2}(\mathbf{u} + \mathbf{v}).$$

The midpoint of the diagonal BD is $\frac{1}{2}(A+\mathbf{u})+\frac{1}{2}(A+\mathbf{v}) = A+\frac{1}{2}(\mathbf{u}+\mathbf{v})$ by Proposition 3.1.4. The two midpoints coincide. Each diagonal therefore passes through the midpoint of the other.

3.1.7 Displacements based at a point

The derivative compares $f(p + h)$ with $f(p)$ for displacements h of one fixed point p. The displacements of p form a copy of the line attached to p. Two displacements of the same point can be added and scaled. A displacement of p and a displacement of a different point q belong to different copies. Recording the base point keeps them apart.

Definition 3.1.8 (Tangent space of the line)**.** Let $p \in \mathbb{R}$. The *tangent space* of $\mathbb{R}$ at p is the set

$$T_p\mathbb{R} = \{(p, v) : v \in \mathbb{R}\} \tag{3.1.4}$$

of displacements based at p, with the operations $(p, u) + (p, v) = (p, u + v)$ and $c\,(p, v) = (p, cv)$. When the base point is fixed by the context the element (p, v) is written $\mathbf{v}$ and called a displacement at p. Its *endpoint* is the point $p + v$. The tangent space $T_P\mathbb{R}^2$ of the plane at a point P is defined in the same way from the vectors of the plane.

The operations on $T_p\mathbb{R}$ are those of the line, transported to the point p. The displacement $(3, 0.5) \in T_3\mathbb{R}$ has endpoint 3.5. The pair $(3, 0.5) + (4, 1)$ is not defined, since the summands lie in different tangent spaces $T_3\mathbb{R}$ and $T_4\mathbb{R}$ (Fig. 3.1.2). For the line the distinction looks pedantic because every $T_p\mathbb{R}$ is a copy of $\mathbb{R}$. It stops being pedantic as soon as an object assigns to each point p something that acts on $T_p\mathbb{R}$ and changes with p. The differential $\mathrm{d}f$ of Section 6.2 is such an object. At each point p it is a linear functional on $T_p\mathbb{R}$ (Section 3.4).

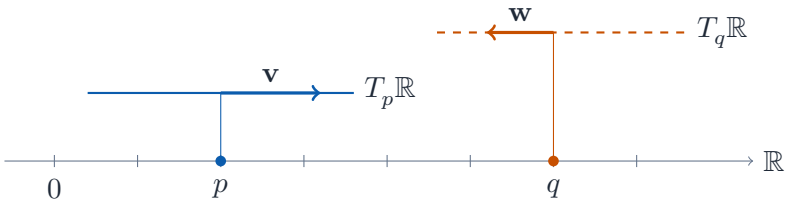


Figure 3.1.2: Tangent spaces $T_p\mathbb{R}$ and $T_q\mathbb{R}$ drawn as copies of the line attached to two points. A displacement $\mathbf{v}$ at p and a displacement $\mathbf{w}$ at q lie in different copies and are not added.

Study guide

A vector is a displacement. In the plane it is a pair of coordinate changes, added and scaled component by component, and drawn as an arrow that may start anywhere. Practise the two pictures of the sum (tip to tail and the parallelogram) and the effect of a negative scalar. Keep points and vectors apart: point plus vector is a point, point minus point is a vector, and a combination of points makes sense only when its coefficients add to 1 or to 0. Learn the definition of $T_p\mathbb{R}$ well enough to say why displacements at different points are not added.

Problems

3.1.1 Let $\mathbf{u} = (2, -1)$ and $\mathbf{v} = (-3, 4)$. Compute $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} - 2\mathbf{v}$, and find the vector $\mathbf{w}$ with $\mathbf{u} + \mathbf{w} = \mathbf{v}$. ⋆ Hint · Answer · Solution

3.1.2 Let $P = (-1, 3)$ and $Q = (5, 0)$. Find the two points that divide the segment PQ into three equal parts. ⋆ Hint · Answer · Solution

3.1.3 Find the centroid $\frac{1}{3}(A + B + C)$ of the triangle with vertices $A = (0, 0)$, $B = (4, 0)$ and $C = (2, 6)$, and show that it lies two thirds of the way from A to the midpoint of BC. ⋆ Hint · Answer · Solution

3.1.4 Prove the rule $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ of Proposition 3.1.4 from the field axioms, naming each axiom used. ⋆⋆ Hint · Answer · Solution

3.1.5 Let A, B, C be the vertices of a triangle and let M and N be the midpoints of AB and AC. Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$. Conclude that the segment MN is parallel to BC and half as long. ⋆⋆ Hint · Answer · Solution

3.1.6 Let P and Q be points and $\mathbf{v}$ a vector of the plane. Decide for each expression whether it is a point, a vector or neither when the origin is moved: (a) $P + \mathbf{v}$, (b) $2P - Q$, (c) $P + Q$, (d) $3(Q - P) + \mathbf{v}$. ⋆⋆ Hint · Answer · Solution

3.1.7 At the point $p = 2$ of the line, find the displacement $\mathbf{v} \in T_2\mathbb{R}$ whose endpoint is 5 and the endpoint of the displacement $-3\mathbf{v}$. Explain why the expression $(2, 3) + (5, 1)$ with $(2, 3) \in T_2\mathbb{R}$ and $(5, 1) \in T_5\mathbb{R}$ is not defined. ⋆ Hint · Answer · Solution

3.2 Vector Spaces

Overview

Objectives: State the axioms of a real vector space and derive their first consequences; recognise $\mathbb{R}^n$, the polynomial spaces $\mathcal{P}_n$, spaces of functions and spaces of sequences as vector spaces; test a subset for being a subspace; compute spans; decide linear independence; prove that any two bases of a finite-dimensional space have the same number of elements; find coordinates in a basis.
Prerequisites: Vectors in the plane (Section 3.1); induction (Theorem 1.4.2); functions (Definition 1.5.8).
Key results: The axioms (Definition 3.2.2) and their consequences (Proposition 3.2.3); the subspace test (Theorem 3.2.13); the exchange lemma (Lemma 3.2.24); invariance of the number of elements of a basis (Theorem 3.2.25) and the dimension (Definition 3.2.26); extension to a basis (Theorem 3.2.27); coordinates (Theorem 3.2.23).

Notation

V, W	real vector spaces
$\mathbb{R}^n$	the n-tuples $(x_1, \dots, x_n)$ of real numbers
$\mathcal{P}_n$	the real polynomials of degree at most n together with 0
$\mathcal{F}(I)$, $C(I)$	all real functions on the interval I; the continuous ones
$\mathbb{R}^{\mathbb{N}}$	the real sequences $(a_1, a_2, \dots)$
span(...)	the set of linear combinations of a list (Definition 3.2.16)
$\dim V$	the dimension of V (Definition 3.2.26)
$\mathbf{e}_1, \dots, \mathbf{e}_n$	the standard basis of $\mathbb{R}^n$
$[\mathbf{v}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{v}$ in the basis $\mathcal{B}$

Vectors of the plane are added and scaled under the eight rules of Proposition 3.1.4. Polynomials obey the same rules: the sum of $1 + x$ and $2x^2$ and the multiple $3(1 + x)$ are polynomials. So do real functions on an interval, added and scaled value by value, and real sequences, added and scaled term by term. A theorem proved from the eight rules alone is a theorem about all of these objects at once. That economy motivates the abstract definition.

3.2.1 Axioms

Definition 3.2.2 (Real vector space)**.** A *real vector space* is a set V with an *addition* that assigns to $\mathbf{u}, \mathbf{v} \in V$ an element $\mathbf{u} + \mathbf{v} \in V$ and a *scalar multiplication* that assigns to $c \in \mathbb{R}$ and $\mathbf{v} \in V$ an element $c\mathbf{v} \in V$, such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in \mathbb{R}$:

(V1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;

(V2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;

(V3) there is an element $\mathbf{0} \in V$, the *zero vector*, with $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for every $\mathbf{v}$;

(V4) for each $\mathbf{v}$ there is an element $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$;

(V5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$;

(v6) $(c+d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$;

(v7) $c(d\mathbf{v}) = (cd)\mathbf{v}$;

(v8) $1\mathbf{v} = \mathbf{v}$.

The elements of V are called *vectors*. We write $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

Vectors are written in bold. Functions and polynomials keep their usual letters f, g, p even when they are treated as vectors.

Proposition 3.2.3 (Cancellation and zero rules in a vector space)**.** *In every real vector space V, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$:*

1. *if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ then $\mathbf{u} = \mathbf{v}$;*
2. *the zero vector is unique, and each $\mathbf{v}$ has exactly one negative;*
3. *$0\mathbf{v} = \mathbf{0}$ and $c\mathbf{0} = \mathbf{0}$;*
4. *$(-1)\mathbf{v} = -\mathbf{v}$;*
5. *if $c\mathbf{v} = \mathbf{0}$ then $c = 0$ or $\mathbf{v} = \mathbf{0}$.*

Proof. (1) Add $-\mathbf{w}$ to both sides and use (v2), (v4) and (v3): $\mathbf{u} = \mathbf{u} + (\mathbf{w} + (-\mathbf{w})) = (\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w}) = \mathbf{v}$.

(2) If $\mathbf{0}$ and $\mathbf{0}'$ both satisfy (v3), then $\mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}$ by (v1). If $\mathbf{v} + \mathbf{x} = \mathbf{0} = \mathbf{v} + (-\mathbf{v})$, then $\mathbf{x} = -\mathbf{v}$ by (1) after commuting the sums.

(3) By (v6), $0\mathbf{v} + 0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} = 0\mathbf{v} + \mathbf{0}$. Cancelling $0\mathbf{v}$ by (1) gives $0\mathbf{v} = \mathbf{0}$. Likewise $c\mathbf{0} + c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} = c\mathbf{0} + \mathbf{0}$ by (v5) and gives $c\mathbf{0} = \mathbf{0}$.

(4) By (v8), (v6) and (3), $\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1-1)\mathbf{v} = \mathbf{0}$. The negative is unique by (2). Hence $(-1)\mathbf{v} = -\mathbf{v}$.

(5) Suppose $c\mathbf{v} = \mathbf{0}$ and $c \neq 0$. Then by (v8), (v7) and (3), $\mathbf{v} = 1\mathbf{v} = (c^{-1}c)\mathbf{v} = c^{-1}(c\mathbf{v}) = c^{-1}\mathbf{0} = \mathbf{0}$. ∎

3.2.4 Examples of vector spaces

Example 3.2.5 (Coordinate spaces)**.** For $n \geq 1$ the set $\mathbb{R}^n$ of n-tuples $\mathbf{x} = (x_1, \dots, x_n)$ with componentwise addition and scalar multiplication is a vector space. The proof of Proposition 3.1.4 applies to each component. The cases $n = 1$ and $n = 2$ are the line and the plane of Section 3.1. The zero vector is $(0, \dots, 0)$.

Example 3.2.6 (Polynomials)**.** For $n \geq 0$ let $\mathcal{P}_n$ be the set of polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients, including the zero polynomial. Sums and scalar multiples are taken coefficient by coefficient. With these operations $\mathcal{P}_n$ is a vector space because the operations on the coefficient lists $(a_0, \dots, a_n)$ are those of $\mathbb{R}^{n+1}$. The set $\mathcal{P}$ of all polynomials is a vector space in the same way.

Example 3.2.7 (Functions on an interval)**.** Let I be an interval and $\mathcal{F}(I)$ the set of all functions $f \colon I \to \mathbb{R}$. Define $f + g$ and cf pointwise:

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = c\,f(x) \qquad (x \in I). \tag{3.2.1}$$

Each axiom is an identity between two functions. Two functions on I are equal when their values agree at every $x \in I$, and at each x the identity is the corresponding field axiom for real numbers. The zero vector is the function with value 0 at every point. The negative of f is the function $x \mapsto -f(x)$.

Example 3.2.8 (Continuous functions)**.** The set $C(I)$ of continuous functions on I is a vector space with the operations of $\mathcal{F}(I)$. Continuity is defined in Chapter 5 (Definition 5.3.1), where Theorem 5.3.6 proves that sums and constant multiples of continuous functions are continuous. That proof uses limits and the order of $\mathbb{R}$ and nothing from this chapter. The forward reference is therefore not circular. By Theorem 3.2.13 below, these two facts together with the continuity of the zero function make $C(I)$ a vector space.

Example 3.2.9 (Sequences)**.** The set $\mathbb{R}^{\mathbb{N}}$ of real sequences $(a_1, a_2, \ldots)$, added and scaled term by term, is a vector space. It is the space $\mathcal{F}(\mathbb{N})$ of functions on $\mathbb{N}$ with the pointwise operations, and the proof for $\mathcal{F}(I)$ applies unchanged.

Example 3.2.10 (Non-examples)**.** The set $\mathbb{R}^2$ with the usual addition and the scalar multiplication $c \odot (x, y) = (cx, 0)$ is not a vector space because $1 \odot (0, 1) = (0, 0) \neq (0, 1)$ violates (v8). The set of positive real numbers with the usual addition and multiplication is not a vector space: it contains no zero vector, and $(-1) \cdot 2$ lies outside it. The set of polynomials of degree exactly 2 is not a vector space because $(x^2 + x) + (-x^2) = x$ has degree 1.

3.2.11 Subspaces

Definition 3.2.12 (Subspace)**.** A *subspace* of a vector space V is a subset $W \subseteq V$ that contains $\mathbf{0}$ and is *closed* under addition and scalar multiplication: $\mathbf{u} + \mathbf{v} \in W$ and $c\mathbf{v} \in W$ whenever $\mathbf{u}, \mathbf{v} \in W$ and $c \in \mathbb{R}$.

Theorem 3.2.13 (Subspace test)**.** *A subspace W of V is a vector space with the addition and scalar multiplication of V.*

Proof. The operations of V send elements of W to elements of W by closure. Axioms (v1), (v2) and (v5)–(v8) are identities that are true for all elements of V and hence for those of W. The zero vector lies in W by assumption. For $\mathbf{v} \in W$ the negative $-\mathbf{v} = (-1)\mathbf{v}$ (Proposition 3.2.3(4)) lies in W by closure under scalar multiplication. ∎

Every line through the origin of the plane, such as $\{(x, y) : 2x - y = 0\}$, is a subspace of $\mathbb{R}^2$. The chain $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots$ consists of subspaces of $\mathcal{P}$. An interval has endpoints $a < b$ (Definition 2.1.8) and so infinitely many points. Two polynomials that define the same function on I therefore agree at more points than their degrees and are equal by Corollary 4.2.8. Identifying a polynomial with the function it defines on I thus makes $\mathcal{P}_n$ a subspace of $C(I)$ and $C(I)$ is a subspace of $\mathcal{F}(I)$. For a fixed $a \in I$ the functions with $f(a) = 0$ form a subspace of $\mathcal{F}(I)$.

Example 3.2.14 (Subsets that are not subspaces)**.** The line $\{(x, y) : x + y = 1\}$ misses $\mathbf{0}$. The first quadrant $\{(x, y) : x \geq 0,\ y \geq 0\}$ contains $(1, 1)$ and not $(-1)(1, 1)$. The unit circle $\{(x, y) : x^2 + y^2 = 1\}$ contains $(1, 0)$ and $(0, 1)$ and not their sum. The functions on $[0, 1]$ with $f(0) = 1$ omit the zero function, and the sum of two of them takes the value 2 at 0.

Common misconception

A line in the plane is a subspace only when it passes through the origin. The zero vector of $\mathcal{F}(I)$ is the zero function, and a set of functions that omits it is no subspace. A set closed under scalar multiplication can still fail closure under addition. The pair of diagonals $\{(x, y) : |x| = |y|\}$ contains $\mathbf{0}$ and every multiple of each of its points. It contains $(1, 1)$ and $(1, -1)$ and misses their sum $(2, 0)$.

3.2.15 Span and linear independence

Definition 3.2.16 (Linear combination and span)**.** A *linear combination* of the list $\mathbf{v}_1, \dots, \mathbf{v}_k$ of vectors of V is a vector $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ with scalars $c_1, \dots, c_k$. The *span* $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is the set of all such combinations. The span of the empty list is $\{\mathbf{0}\}$. The list *spans* V when its span is V.

Proposition 3.2.17 (Span is a subspace)**.** *The span of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a subspace of V that contains each $\mathbf{v}_i$. Every subspace containing each $\mathbf{v}_i$ contains the span.*

Proof. The span contains $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k$ and each $\mathbf{v}_i$ (coefficient 1 on $\mathbf{v}_i$ and 0 on the others). By the axioms $\sum c_i\mathbf{v}_i + \sum d_i\mathbf{v}_i = \sum (c_i + d_i)\mathbf{v}_i$ and $a \sum c_i\mathbf{v}_i = \sum (ac_i)\mathbf{v}_i$. Both lie in the span. A subspace containing each $\mathbf{v}_i$ contains every multiple $c_i\mathbf{v}_i$ and every sum of such multiples by closure. ∎

The span of the single vector $(1, 2)$ is the line $y = 2x$ through the origin. The span of $(1, 2)$ and $(3, 1)$ is the whole plane (Fig. 3.2.1). The span of $1, x, x^2$ is $\mathcal{P}_2$. As a non-example, the span of $(1, 2)$ and $(2, 4)$ is only the line $y = 2x$, since $(2, 4) = 2(1, 2)$ adds nothing new.

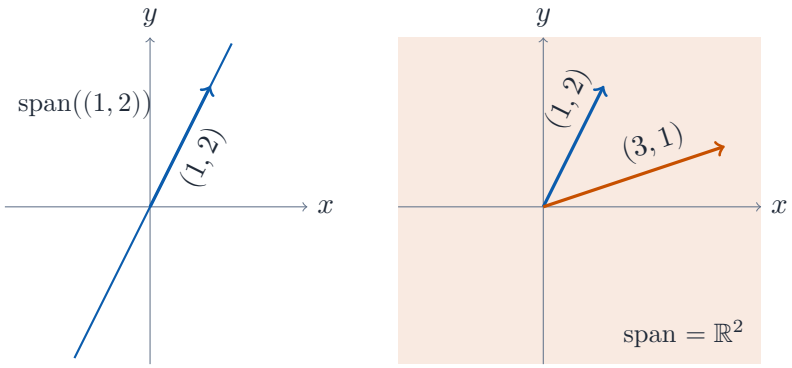


Figure 3.2.1: Span of one nonzero vector of the plane (a line through the origin, left) and of two vectors that are not multiples of each other (the whole plane, right).

Definition 3.2.18 (Linear independence)**.** A list $\mathbf{v}_1, \dots, \mathbf{v}_k$ in V is *linearly independent* if the only scalars with

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \tag{3.2.2}$$

are $c_1 = \cdots = c_k = 0$. Otherwise the list is *linearly dependent*. The empty list is linearly independent.

The vectors $(1, 2)$ and $(3, 1)$ are independent. From $a(1, 2) + b(3, 1) = \mathbf{0}$ the equations $a + 3b = 0$ and $2a + b = 0$ give $b = -2a$ and then $-5a = 0$. Hence $a = b = 0$. The vectors $(1, 2)$ and $(2, 4)$ are dependent because $2(1, 2) - (2, 4) = \mathbf{0}$. Every list containing $\mathbf{0}$ is dependent $(1 \cdot \mathbf{0} = \mathbf{0})$.

Example 3.2.19 (Monomials)**.** The monomials $1, x, \dots, x^n$ are linearly independent in $\mathcal{F}(\mathbb{R})$. If $c_0 + c_1x + \cdots + c_nx^n = 0$ for every real x, this polynomial agrees with the zero polynomial at the $n + 1$ distinct points $0, 1, \dots, n$. By Corollary 4.2.8 all its coefficients are 0. That corollary is proved in Section 4.2 from the division algorithm for polynomials and uses nothing from this chapter.

Lemma 3.2.20 (Linear dependence lemma)**.** *If $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly dependent, there is an index j with $\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$. Removing $\mathbf{v}_j$ from the list does not change its span.*

Proof. Choose scalars, not all 0, with $\sum c_i\mathbf{v}_i = \mathbf{0}$, and let j be the largest index with $c_j \neq 0$. Then $c_j\mathbf{v}_j = -\sum_{i<j} c_i\mathbf{v}_i$ and $\mathbf{v}_j = \sum_{i<j}(-c_i/c_j)\mathbf{v}_i$. (If $j = 1$ this reads $\mathbf{v}_1 = \mathbf{0}$, the span of the empty list.) In any linear combination of the full list, replace $\mathbf{v}_j$ by this expression. The result is a combination of the list without $\mathbf{v}_j$. ∎

3.2.21 Bases and dimension

Definition 3.2.22 (Basis)**.** A *basis* of V is a linearly independent list that spans V. The vectors $\mathbf{e}_1 = (1, 0, \dots, 0)$, …, $\mathbf{e}_n = (0, \dots, 0, 1)$ form the *standard basis* of $\mathbb{R}^n$. The monomials $1, x, \dots, x^n$ form the *monomial basis* of $\mathcal{P}_n$.

Every $\mathbf{x} \in \mathbb{R}^n$ equals $x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, and this combination is $\mathbf{0}$ only when every $x_i = 0$. The standard list is a basis. The monomials span $\mathcal{P}_n$ by definition and are independent by Example 3.2.19. As a non-example, $(1, 2)$ alone is independent and does not span $\mathbb{R}^2$. The three vectors $(1, 0), (0, 1), (1, 1)$ span $\mathbb{R}^2$ and are dependent.

Theorem 3.2.23 (Coordinates in a basis)**.** *A list $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a basis of V exactly when every $\mathbf{v} \in V$ can be written as $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ in exactly one way. The numbers c_i are the* coordinates *of $\mathbf{v}$ in $\mathcal{B}$, and $[\mathbf{v}]_{\mathcal{B}} = (c_1, \dots, c_n) \in \mathbb{R}^n$ is its* coordinate vector*.*

Proof. Existence of the representation for every $\mathbf{v}$ is the statement that the list spans V. If the list is independent and $\sum c_i\mathbf{v}_i = \sum d_i\mathbf{v}_i$, then $\sum(c_i - d_i)\mathbf{v}_i = \mathbf{0}$ and every $c_i = d_i$. Conversely, if representations are unique, the representation $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$ is the only one and the list is independent. ∎

Coordinates respect the operations. If $\mathbf{u} = \sum c_i\mathbf{v}_i$ and $\mathbf{v} = \sum d_i\mathbf{v}_i$, then $\mathbf{u} + a\mathbf{v} = \sum(c_i + ad_i)\mathbf{v}_i$ by the axioms, and this representation is the only one by Theorem 3.2.23. Hence

$$[\mathbf{u} + a\mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + a\,[\mathbf{v}]_{\mathcal{B}} \qquad (\mathbf{u}, \mathbf{v} \in V,\ a \in \mathbb{R}). \tag{3.2.3}$$

The coordinate map $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$ is therefore linear in the sense of Section 3.3. It is a bijection by Theorem 3.2.23: its inverse sends $(c_1, \dots, c_n)$ to $\sum c_i\mathbf{v}_i$.

The next lemma compares the lengths of independent lists and spanning lists.

Lemma 3.2.24 (Exchange lemma)**.** *If $\mathbf{u}_1, \dots, \mathbf{u}_k$ is linearly independent in V and $\mathbf{w}_1, \dots, \mathbf{w}_m$ spans V, then $k \leq m$.*

Proof. We prove by induction on $j = 0, 1, \dots, k$ that $j \leq m$ and that, after the $\mathbf{w}_i$ are renumbered, the list

$$\mathbf{u}_1, \dots, \mathbf{u}_j,\ \mathbf{w}_{j+1}, \dots, \mathbf{w}_m$$

spans V. For $j = 0$ this is the hypothesis. Suppose it is true for some $j < k$. Since the list spans V,

$$\mathbf{u}_{j+1} = a_1\mathbf{u}_1 + \cdots + a_j\mathbf{u}_j + b_{j+1}\mathbf{w}_{j+1} + \cdots + b_m\mathbf{w}_m$$

for some scalars. If all the b_i were 0 (in particular if $j = m$ and no $\mathbf{w}$ remains), then $\mathbf{u}_{j+1} - a_1\mathbf{u}_1 - \cdots - a_j\mathbf{u}_j = \mathbf{0}$ would be a dependence among the $\mathbf{u}$'s. Hence $j+1 \leq m$ and some $b_i \neq 0$. Renumber so that $b_{j+1} \neq 0$. Solving for $\mathbf{w}_{j+1}$ expresses it as a combination of $\mathbf{u}_1, \dots, \mathbf{u}_{j+1}, \mathbf{w}_{j+2}, \dots, \mathbf{w}_m$. Every vector of V is a combination of the old list. Replacing $\mathbf{w}_{j+1}$ in that combination by its expression shows that the new list $\mathbf{u}_1, \dots, \mathbf{u}_{j+1}, \mathbf{w}_{j+2}, \dots, \mathbf{w}_m$ spans V. This completes the induction. The case $j = k$ gives $k \leq m$. ∎

A vector space is *finite-dimensional* if some finite list spans it.

Theorem 3.2.25 (Existence of a basis and its size)**.** *Let V be finite-dimensional. Every finite list that spans V contains a basis of V. In particular V has a basis. Any two bases of V have the same number of elements.*

Proof. Start from a spanning list. While it is dependent, Lemma 3.2.20 removes one vector without changing the span. Because the list shortens at each step, the process ends with an independent list that still spans V: a basis. If $\mathcal{B}$ has k elements and $\mathcal{B}'$ has m, then Lemma 3.2.24 applied with $\mathcal{B}$ independent and $\mathcal{B}'$ spanning gives $k \leq m$. Exchanging the roles gives $m \leq k$. ∎

Definition 3.2.26 (Dimension)**.** The *dimension* $\dim V$ of a finite-dimensional vector space V is the number of elements of any basis of V. A vector space that is not finite-dimensional is *infinite-dimensional*.

The definition rests on Theorem 3.2.25. The standard and monomial bases give $\dim \mathbb{R}^n = n$ and $\dim \mathcal{P}_n = n + 1$. The space $\{\mathbf{0}\}$ has the empty basis and dimension 0. A line through the origin in $\mathbb{R}^2$ has dimension 1. The spaces $\mathcal{P}$, $\mathcal{F}(I)$, $C(I)$ and $\mathbb{R}^{\mathbb{N}}$ are infinite-dimensional. For every n each contains an independent list of $n+1$ elements: the monomials $1, x, \ldots, x^n$, or in $\mathbb{R}^{\mathbb{N}}$ the first $n+1$ sequences with a single term equal to 1 and all other terms 0. In $\mathcal{F}(I)$ and $C(I)$ a combination of monomials that vanishes on I vanishes at $n+1$ distinct points of I, and the argument of Example 3.2.19 with these points gives independence. A spanning list of length m would contradict Lemma 3.2.24 once $n + 1 > m$.

Theorem 3.2.27 (Extension to a basis)**.** *Let* $\dim V = n$.

1. *Every linearly independent list in V has at most n elements and extends to a basis of V.*
2. *A list of exactly n vectors is a basis as soon as it is linearly independent or as soon as it spans V.*
3. *Every subspace W of V is finite-dimensional with* $\dim W \leq n$*, and* $\dim W = n$ *only when $W = V$.*

Proof. (1) The bound is Lemma 3.2.24 with a basis as the spanning list. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be independent. If it spans V it is a basis. Otherwise choose $\mathbf{v} \in V$ outside its span. The list $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}$ is independent: by Lemma 3.2.20 a dependence would put one of its vectors in the span of the earlier ones. The independence of the $\mathbf{u}_i$ rules this out for them and the choice of $\mathbf{v}$ rules it out for $\mathbf{v}$. Repeat. Because the length never exceeds n, after at most $n - k$ steps the list spans V.

(2) An independent list of length n extends to a basis by (1), and the basis has n elements by Theorem 3.2.25. Nothing was added. A spanning list of length n contains a basis by Theorem 3.2.25, again of length n, and nothing was removed.

(3) Run the procedure of (1) inside W, starting from the empty list and adding vectors of W outside the current span. The lists stay independent in V and have at most n elements. The procedure stops at a basis of W with at most n elements. If it has n elements it is a basis of V by (2), and then $W = V$. ∎

Worked Example 3.2.1

Show that $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ form a basis of $\mathbb{R}^3$ and find the coordinates of $(2, 3, 5)$ in it.

By Theorem 3.2.27(2) three vectors in the three-dimensional space $\mathbb{R}^3$ form a basis as soon as they span it. Given (p,q,r), we look for a,b,c with $a(1,1,0)+b(0,1,1)+c(1,0,1) = (a+c,\ a+b,\ b+c) = (p,q,r)$. Adding the three equations gives $a+b+c = \frac{1}{2}(p+q+r)$. Subtracting each equation in turn gives

$$b = \tfrac{1}{2}(-p+q+r), \qquad c = \tfrac{1}{2}(p-q+r), \qquad a = \tfrac{1}{2}(p+q-r).$$

These numbers solve the system for every (p,q,r). The list spans $\mathbb{R}^3$ and is a basis. For $(p,q,r) = (2,3,5)$ the coordinates are $a=0$, $b=3$, $c=2$. As a check, $3(0,1,1)+2(1,0,1) = (2,3,5)$.

Worked Example 3.2.2

Show that $1,\ x-1,\ (x-1)^2$ is a basis of $\mathcal{P}_2$ and find the coordinates of $q(x) = 3+2x+x^2$ in it.
Expanding in powers of $x-1$ uses $x = (x-1)+1$ and $x^2 = \big((x-1)+1\big)^2 = (x-1)^2+2(x-1)+1$. Hence every $a_0+a_1x+a_2x^2$ equals

$$(a_0+a_1+a_2)+(a_1+2a_2)(x-1)+a_2(x-1)^2,$$

and the three polynomials span $\mathcal{P}_2$. Since $\dim \mathcal{P}_2 = 3$, they form a basis by Theorem 3.2.27(2). For q the coordinates are $(3+2+1,\ 2+2,\ 1) = (6,4,1)$. As a check, $6+4(x-1)+(x-1)^2 = 6+4x-4+x^2-2x+1 = 3+2x+x^2$. Coordinates in powers of $x-a$ return in Taylor's theorem (Section 7.4).

Caution

A basis is a list, and its order matters for coordinates: the coordinate vector of $(2,3)$ is $(2,3)$ in $\mathbf{e}_1,\mathbf{e}_2$ and $(3,2)$ in $\mathbf{e}_2,\mathbf{e}_1$. Dimension counts the vectors in a basis and never the number of components needed to write one of them: the line $y=2x$ lies in $\mathbb{R}^2$ and has dimension 1.

Study guide

Learn the eight axioms by grouping them: four about addition, two distributive laws, and two about scalars. To show a subset is a subspace, check three things: it contains $\mathbf{0}$ and is closed under both operations. To refute it, exhibit one failure. To decide independence, set a linear combination equal to $\mathbf{0}$ and solve for the coefficients. Theorem 3.2.25, the definition of dimension and Theorem 3.2.27 all rest on the exchange lemma. Reproduce its induction. In a space of known dimension n a list of n vectors needs only one of the two properties of a basis.

Problems

3.2.1 Which of the following subsets of $\mathbb{R}^2$ are subspaces? (a) $\{(x,y) : x+2y = 0\}$, (b) $\{(x,y) : x+2y = 1\}$, (c) $\{(x,y) : xy \geq 0\}$, (d) $\{(x,y) : x = y^2\}$.

⋆ Hint · Answer · Solution

3.2.2 Decide whether $(1,2,3)$, $(4,5,6)$, $(7,8,9)$ are linearly independent.

⋆ Hint · Answer · Solution

3.2.3 Find the coordinates of $5 + x - 2x^2$ in the basis 1, $1 + x$, $1 + x + x^2$ of $\mathcal{P}_2$.

⋆ Hint · Answer · Solution

3.2.4 Let U and W be subspaces of V. Prove that $U \cap W$ is a subspace. Show by an example in $\mathbb{R}^2$ that $U \cup W$ need not be one.

⋆⋆ Hint · Answer · Solution

3.2.5 Prove that the functions 1, x, $|x|$ are linearly independent in $\mathcal{F}(\mathbb{R})$.

⋆⋆ Hint · Answer · Solution

3.2.6 Let $W = \{p \in \mathcal{P}_3 : p(1) = 0\}$. Prove that W is a subspace, show that $x - 1$, $x^2 - 1$, $x^3 - 1$ is a basis of W and find $\dim W$.

⋆⋆ Hint · Answer · Solution

3.2.7 Let V be a vector space with a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let $\mathbf{w} = \sum c_i \mathbf{v}_i$ with $c_1 \neq 0$. Prove without Theorem 3.2.27 that $\mathbf{w}, \mathbf{v}_2, \dots, \mathbf{v}_n$ is again a basis.

⋆⋆ Hint · Answer · Solution

3.2.8 Find a basis of $\mathcal{P}_2$ whose three polynomials all have degree exactly 2. Then prove that no basis of $\mathcal{P}_2$ consists of polynomials p with $p(0) = 0$.

⋆⋆ Hint · Answer · Solution

3.3 Linear Maps and Matrices

Overview

Objectives: Decide whether a map between vector spaces is linear; compute kernels and images; prove that a linear map is injective exactly when its kernel is trivial; state and prove the rank–nullity theorem; write the matrix of a linear map in given bases and compose maps by multiplying matrices; invert a 2×2 matrix; write rotations and reflections of the plane, differentiation of polynomials and multiplication operators as linear maps.
Prerequisites: Vector spaces, bases and dimension (Section 3.2); injective, surjective and inverse functions (Definitions 1.5.13, 1.5.14 and 1.5.24).
Key results: Linear maps (Definition 3.3.2); kernel and image (Definitions 3.3.7 and 3.3.8); injectivity and the kernel (Theorem 3.3.10); rank–nullity (Theorem 3.3.11); composition as matrix product (Theorem 3.3.17); the 2×2 inverse (Theorem 3.3.19); rotations and reflections (Definitions 3.3.21 and 3.3.22).

Notation

$T\colon V \to W$	a linear map from V to W
$\ker T$, $\operatorname{im} T$	kernel and image of T (Definitions 3.3.7 and 3.3.8)
$A = (a_{ij})$	a matrix with entry a_{ij} in row i and column j
$A\mathbf{x}$, AB	matrix times vector, matrix product
I, id	the identity matrix, the identity map
A^{-1}, T^{-1}	inverse matrix, inverse map
$R_{c,s}$, $F_{c,s}$	rotation and reflection of the plane (Definitions 3.3.21 and 3.3.22)
M_m	multiplication by the function m

A vector space comes with two operations. The maps between vector spaces that matter are those that respect both. Differentiation respects them because the derivative of a sum is the sum of the derivatives. Evaluation at a point and the integral over a fixed interval respect them as well. A rotation of the plane about the origin respects them because rotating a parallelogram gives a parallelogram.

3.3.1 Linear maps

Definition 3.3.2 (Linear map)**.** Let V and W be real vector spaces. A function $T\colon V \to W$ is a *linear map* if

$$T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v} \qquad \text{and} \qquad T(c\mathbf{v}) = cT\mathbf{v} \tag{3.3.1}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$. A linear map from V to itself is a *linear operator* on V.

A linear map sends $\mathbf{0}$ to $\mathbf{0}$, since $T\mathbf{0} = T(0\mathbf{0}) = 0\,T\mathbf{0} = \mathbf{0}$ by Proposition 3.2.3(3). Induction on the number of terms gives $T(\sum c_i \mathbf{v}_i) = \sum c_i T\mathbf{v}_i$.

Example 3.3.3 (Linear maps)**.** Multiplication by a fixed number, $v \mapsto av$, is a linear operator on $\mathbb{R}$. The map $(x, y) \mapsto (2x + y,\, x - 3y)$ is a linear operator on $\mathbb{R}^2$ because each component is a fixed combination of x and y. Evaluation $p \mapsto p(2)$ is linear from $\mathcal{P}_n$ to $\mathbb{R}$. On sequences the shift $(a_1, a_2, a_3, \ldots) \mapsto (a_2, a_3, \ldots)$ is linear.

Example 3.3.4 (Maps that are not linear)**.** The translation $v \mapsto v+1$ of $\mathbb{R}$ sends 0 to 1 and is not linear. The map $(x,y) \mapsto (xy,x)$ sends $(1,1)$ to $(1,1)$ and $2(1,1)$ to $(4,2) \neq 2(1,1)$. The map $f \mapsto f^2$ on $\mathcal{F}(I)$ sends $2f$ to $4f^2$. The function $4f^2$ differs from $2f^2$ whenever f is not the zero function.

Common misconception

School algebra calls $x \mapsto ax+b$ a linear function because its graph is a line. In linear algebra it is a linear map only when $b=0$. For $b \neq 0$ it is called *affine*: a linear map followed by a translation.

Theorem 3.3.5 (Linear maps on a basis)**.** *Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V and let $\mathbf{w}_1, \dots, \mathbf{w}_n$ be any vectors of W. There is exactly one linear map $T\colon V \to W$ with $T\mathbf{v}_i = \mathbf{w}_i$ for each i.*

Proof. Each $\mathbf{v} \in V$ has unique coordinates c_i with $\mathbf{v} = \sum c_i \mathbf{v}_i$ (Theorem 3.2.23). Define $T\mathbf{v} = \sum c_i \mathbf{w}_i$. The coordinates of $\mathbf{u}+\mathbf{v}$ and of $a\mathbf{v}$ are the sums and the multiples of the coordinates by (3.2.3). Hence T is linear with $T\mathbf{v}_i = \mathbf{w}_i$. Any linear S with $S\mathbf{v}_i = \mathbf{w}_i$ satisfies $S\mathbf{v} = \sum c_i S\mathbf{v}_i = T\mathbf{v}$. ∎

3.3.6 Kernel and image

Definition 3.3.7 (Kernel)**.** The *kernel* of a linear map $T\colon V \to W$ is $\ker T = \{\mathbf{v} \in V : T\mathbf{v} = \mathbf{0}\}$.

Definition 3.3.8 (Image)**.** The *image* of a linear map $T\colon V \to W$ is $\operatorname{im} T = \{T\mathbf{v} : \mathbf{v} \in V\}$, its image as a function (Definition 1.5.10).

Proposition 3.3.9 (Kernel and image are subspaces)**.** *For a linear map $T\colon V \to W$, $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.*

Proof. Both contain $\mathbf{0}$ since $T\mathbf{0} = \mathbf{0}$. If $T\mathbf{u} = T\mathbf{v} = \mathbf{0}$ then $T(\mathbf{u}+\mathbf{v}) = \mathbf{0}$ and $T(c\mathbf{u}) = \mathbf{0}$. For the image, $T\mathbf{u} + T\mathbf{v} = T(\mathbf{u}+\mathbf{v})$ and $cT\mathbf{u} = T(c\mathbf{u})$ are again values of T. ∎

Theorem 3.3.10 (Injectivity and the kernel)**.** *A linear map T is injective exactly when $\ker T = \{\mathbf{0}\}$. If $T\mathbf{v}_0 = \mathbf{w}$, the solutions of $T\mathbf{v} = \mathbf{w}$ are exactly the vectors $\mathbf{v}_0 + \mathbf{k}$ with $\mathbf{k} \in \ker T$.*

Proof. By linearity $T\mathbf{v} = T\mathbf{v}_0$ is equivalent to $T(\mathbf{v}-\mathbf{v}_0) = \mathbf{0}$, that is $\mathbf{v}-\mathbf{v}_0 \in \ker T$. This proves the second statement. If $\ker T = \{\mathbf{0}\}$, the only solution of $T\mathbf{v} = T\mathbf{v}_0$ is $\mathbf{v} = \mathbf{v}_0$ and T is injective. If T is injective, the only solution of $T\mathbf{v} = \mathbf{0} = T\mathbf{0}$ is $\mathbf{v} = \mathbf{0}$. ∎

The second statement of Theorem 3.3.10 is the shape of every solution set of a linear equation in this book: one particular solution plus the kernel. Chapter 7 applies it to differentiation, where the kernel on an interval consists of the constant functions and the statement becomes "antiderivatives differ by a constant".

Theorem 3.3.11 (Rank–nullity)**.** *Let V be finite-dimensional and $T\colon V \to W$ linear. Then $\operatorname{im} T$ is finite-dimensional and*

$$\dim \ker T + \dim \operatorname{im} T = \dim V. \tag{3.3.2}$$

Proof. Let $n = \dim V$. The kernel is a subspace of V (Proposition 3.3.9) and has a basis $\mathbf{k}_1, \dots, \mathbf{k}_r$ with $r \leq n$ by Theorem 3.2.27(3). Extend it to a basis $\mathbf{k}_1, \dots, \mathbf{k}_r, \mathbf{v}_1, \dots, \mathbf{v}_{n-r}$ of V (Theorem 3.2.27(1)). We show that $T\mathbf{v}_1, \dots, T\mathbf{v}_{n-r}$ is a basis of $\operatorname{im} T$. Then $\dim \operatorname{im} T = n-r$ and the formula follows.

Spanning: every $\mathbf{v} \in V$ is $\sum a_i \mathbf{k}_i + \sum b_j \mathbf{v}_j$, and $T\mathbf{v} = \sum a_i T\mathbf{k}_i + \sum b_j T\mathbf{v}_j = \sum b_j T\mathbf{v}_j$ because each $T\mathbf{k}_i = \mathbf{0}$.

Independence: suppose $\sum b_j T\mathbf{v}_j = \mathbf{0}$. Then $T(\sum b_j \mathbf{v}_j) = \mathbf{0}$ and $\sum b_j \mathbf{v}_j \in \ker T$. Hence $\sum b_j \mathbf{v}_j = \sum a_i \mathbf{k}_i$ for some scalars a_i. The vector $\sum a_i \mathbf{k}_i - \sum b_j \mathbf{v}_j = \mathbf{0}$ is a combination of the basis of V, and all its coefficients vanish. In particular every $b_j = 0$. ∎

Corollary 3.3.12 (Operators on a finite-dimensional space)**.** *Let $\dim V = n$ and let $T \colon V \to V$ be linear. Then T is injective exactly when it is surjective, and in that case it is bijective.*

Proof. By Theorem 3.3.10 T is injective exactly when $\dim \ker T = 0$. By Theorem 3.3.11 this happens exactly when $\dim \operatorname{im} T = n$, which by Theorem 3.2.27(3) happens exactly when $\operatorname{im} T = V$. ∎

Caution

Corollary 3.3.12 fails in infinite dimensions. On the sequence space $\mathbb{R}^{\mathbb{N}}$ the shift $(a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots)$ is surjective and has the nonzero sequence $(1, 0, 0, \ldots)$ in its kernel. On $\mathcal{P}$ the map $p \mapsto xp$ is injective and misses the constant polynomial 1.

3.3.13 Matrices

An $m \times n$ *matrix* $A = (a_{ij})$ is a rectangular array of real numbers with m rows and n columns. For $\mathbf{x} \in \mathbb{R}^n$, written as a column, the product $A\mathbf{x} \in \mathbb{R}^m$ is the column whose ith entry is $\sum_{j=1}^{n} a_{ij} x_j$. Equivalently $A\mathbf{x}$ is the combination $x_1 \mathbf{a}_1 + \cdots + x_n \mathbf{a}_n$ of the columns $\mathbf{a}_j$ of A. The map $\mathbf{x} \mapsto A\mathbf{x}$ is linear from $\mathbb{R}^n$ to $\mathbb{R}^m$, and its value on $\mathbf{e}_j$ is the jth column.

Definition 3.3.14 (Matrix of a linear map)**.** Let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be a basis of V, $\mathcal{C}$ a basis of W with m elements and $T \colon V \to W$ linear. The *matrix of T* in these bases is the $m \times n$ matrix whose jth column is the coordinate vector $[T\mathbf{v}_j]_{\mathcal{C}}$.

Theorem 3.3.15 (Matrix of a map acts on coordinates)**.** *If A is the matrix of T in the bases $\mathcal{B}$ and $\mathcal{C}$, then $[T\mathbf{v}]_{\mathcal{C}} = A\,[\mathbf{v}]_{\mathcal{B}}$ for every $\mathbf{v} \in V$.*

Proof. Write $\mathbf{v} = \sum x_j \mathbf{v}_j$. Then $T\mathbf{v} = \sum x_j T\mathbf{v}_j$. Taking coordinates in $\mathcal{C}$ is linear by (3.2.3). Hence $[T\mathbf{v}]_{\mathcal{C}} = \sum x_j [T\mathbf{v}_j]_{\mathcal{C}}$. The right side is the combination of the columns of A with coefficients x_j and equals $A[\mathbf{v}]_{\mathcal{B}}$. ∎

For a map $\mathbb{R}^n \to \mathbb{R}^m$ and the standard bases, the matrix has the columns $T\mathbf{e}_1, \ldots, T\mathbf{e}_n$. The map $(x, y) \mapsto (2x + y,\ x - 3y)$ has matrix $\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$.

Definition 3.3.16 (Matrix product)**.** For an $m \times n$ matrix A and an $n \times p$ matrix B the *product* AB is the $m \times p$ matrix with entries $(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$. Its kth column is A times the kth column of B. The $n \times n$ *identity matrix* I has entries 1 on the diagonal and 0 elsewhere.

Theorem 3.3.17 (Composition is the matrix product)**.** *Let $T \colon U \to V$ and $S \colon V \to W$ be linear, with bases $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of U, V, W. If B is the matrix of T and A the matrix of S, then $S \circ T$ is linear with matrix AB.*

Proof. Linearity of $S \circ T$: $S(T(\mathbf{u} + c\mathbf{v})) = S(T\mathbf{u} + cT\mathbf{v}) = S(T\mathbf{u}) + c\,S(T\mathbf{v})$. By Theorem 3.3.15 applied twice, the kth column of the matrix of $S \circ T$ is $[S(T\mathbf{u}_k)]_{\mathcal{C}} = A[T\mathbf{u}_k]_{\mathcal{B}}$, that is A times the kth column of B. This is the kth column of AB. ∎

Because composition of functions is associative, so is the matrix product: $(AB)C = A(BC)$. The product is not commutative. With $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ one finds $AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

3.3.18 Invertibility

A linear map $T\colon V \to W$ is *invertible* if it is bijective. Its inverse function T^{-1} (Definition 1.5.24) is then linear: for $\mathbf{w}, \mathbf{w}' \in W$, the vector $T^{-1}\mathbf{w} + cT^{-1}\mathbf{w}'$ is sent by T to $\mathbf{w} + c\mathbf{w}'$ and therefore equals $T^{-1}(\mathbf{w} + c\mathbf{w}')$. A square matrix A is *invertible* if there is a matrix A^{-1} with $AA^{-1} = A^{-1}A = I$. By Theorem 3.3.17 this happens exactly when the map $\mathbf{x} \mapsto A\mathbf{x}$ is invertible, and A^{-1} is the matrix of the inverse map.

Theorem 3.3.19 (Inverse of a 2×2 matrix)**.** *The matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is invertible exactly when* $ad - bc \neq 0$*, and then*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{3.3.3}$$

Proof. Multiplying out, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (ad - bc)I$, and the product in the other order is also $(ad - bc)I$. If $ad - bc \neq 0$, dividing by it gives the stated inverse. Suppose $ad - bc = 0$. Then $A(d, -c) = (ad - bc,\, cd - dc) = \mathbf{0}$ and $A(-b, a) = (-ab + ba,\, -bc + ad) = \mathbf{0}$. If one of a, b, c, d is nonzero, one of these two vectors is nonzero and lies in $\ker A$. Then A is not injective (Theorem 3.3.10). If all four are zero, A sends every vector to $\mathbf{0}$. In both cases A is not invertible. ∎

Section 3.5 names the number $ad - bc$ the determinant of A and reads it as a signed area.

Worked Example 3.3.1

Let $T(x, y) = (x + 2y,\, 3x + 6y)$. Find the matrix of T, its kernel and its image, and check Theorem 3.3.11.

The columns are $T\mathbf{e}_1 = (1, 3)$ and $T\mathbf{e}_2 = (2, 6)$. Hence

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

$T(x, y) = \mathbf{0}$ means $x + 2y = 0$ (the second component is three times the first). Hence $\ker T = \operatorname{span}((-2, 1))$, of dimension 1. Every value is $(x + 2y)(1, 3)$. Hence $\operatorname{im} T = \operatorname{span}((1, 3))$, also of dimension 1. The dimensions add to $2 = \dim \mathbb{R}^2$. Since $1 \cdot 6 - 2 \cdot 3 = 0$, Theorem 3.3.19 confirms that A is not invertible.

3.3.20 Rotations and reflections of the plane

The trigonometry of Section 4.5 and the proofs of Appendix A.1 use two families of linear operators on $\mathbb{R}^2$. They are defined here from a pair of numbers (c, s) with $c^2 + s^2 = 1$ and use no trigonometric function.

Definition 3.3.21 (Rotation)**.** For real c, s with $c^2 + s^2 = 1$ the *rotation* $R_{c,s}$ is the linear operator on $\mathbb{R}^2$ with matrix

$$R_{c,s} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \qquad R_{c,s}(x, y) = (cx - sy,\ sx + cy). \tag{3.3.4}$$

Definition 3.3.22 (Reflection)**.** For real c, s with $c^2 + s^2 = 1$ the *reflection* $F_{c,s}$ in the line through $\mathbf{0}$ and $\mathbf{u} = (c, s)$ is the linear operator on $\mathbb{R}^2$ with matrix

$$F_{c,s} = \begin{pmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{pmatrix}. \tag{3.3.5}$$

The rotation $R_{c,s}$ sends $\mathbf{e}_1$ to (c, s) and $\mathbf{e}_2$ to $(-s, c)$. The quarter turn $R_{0,1}(x, y) = (-y, x)$ and the half turn $R_{-1,0} = -I$ are examples. The reflection $F_{c,s}$ fixes $\mathbf{u}$ and reverses the perpendicular vector $\mathbf{u}^\perp = (-s, c)$:

$$F_{c,s}\mathbf{u} = ((c^2 - s^2)c + 2cs^2,\ 2c^2 s + (s^2 - c^2)s) = (c, s), \qquad F_{c,s}\mathbf{u}^\perp = (s, -c) = -\mathbf{u}^\perp,$$

using $c^2 + s^2 = 1$. Since $\mathbf{u}$ and $\mathbf{u}^\perp$ form a basis, these two values determine $F_{c,s}$ (Theorem 3.3.5). The reflections in the x-axis, in the y-axis and in the line $y = x$ are $F_{1,0}(x, y) = (x, -y)$, $F_{0,1}(x, y) = (-x, y)$ and $F_{c,c}(x, y) = (y, x)$ with $c = 1/\sqrt{2}$. Figure 3.3.1 shows a rotation and a reflection acting on the same vector. Once Definition 4.5.5 is available, $R_{\cos\theta,\sin\theta}$ is the rotation through the angle θ. As a non-example, the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has the form of $R_{c,s}$ with $c = 2$, $s = 0$, but $c^2 + s^2 = 4 \neq 1$ and it doubles every vector: a scaling, no rotation.

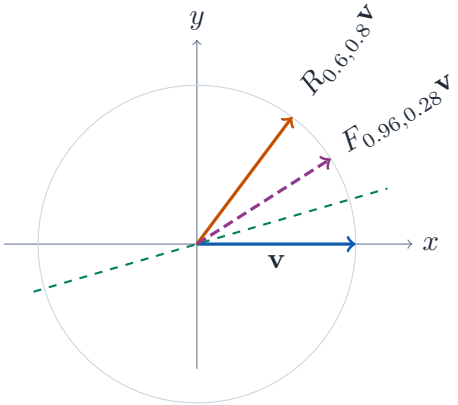


Figure 3.3.1: Rotation $R_{0.6,0.8}$ and reflection $F_{0.96,0.28}$ applied to $\mathbf{v} = (1.4, 0)$. The reflection fixes the dashed mirror line through $\mathbf{0}$ and $\mathbf{u} = (0.96, 0.28)$. Both images lie on the circle through $\mathbf{v}$.

Proposition 3.3.23 (Products of rotations and reflections)**.** *Let $c^2 + s^2 = 1$ and $c'^2 + s'^2 = 1$. Then*

1. $R_{c,s}R_{c',s'} = R_{cc'-ss',\,sc'+cs'}$, *and in particular* $R_{c,s}R_{c,-s} = I$;
2. $F_{c,s}F_{c,s} = I$;
3. $F_{c,s} = R_{c^2-s^2,\,2cs}\,F_{1,0}$.

Proof. (1) Multiplying the matrices gives

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix} = \begin{pmatrix} cc'-ss' & -(sc'+cs') \\ sc'+cs' & cc'-ss' \end{pmatrix}, \tag{3.3.6}$$

the matrix of $R_{C,S}$ with $C = cc' - ss'$ and $S = sc' + cs'$. Expanding, $C^2 + S^2 = (c^2+s^2)(c'^2 + s'^2) = 1$. With $(c', s') = (c, -s)$ the product is $R_{1,0} = I$. (2) The diagonal entries of $F_{c,s}^2$ are $(c^2-s^2)^2+4c^2s^2 = (c^2+s^2)^2 = 1$, and the off-diagonal entries are $2cs(c^2-s^2)+2cs(s^2-c^2) = 0$. (3) With $C = c^2 - s^2$ and $S = 2cs$, where $C^2 + S^2 = 1$ by (2),

$$R_{C,S}F_{1,0} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} = F_{c,s}. \qquad \blacksquare$$

The product (3.3.6) is the angle-sum formula of Theorem 4.5.9 in matrix form: with $(c, s) = (\cos\alpha, \sin\alpha)$ and $(c', s') = (\cos\beta, \sin\beta)$ the entries of the product are $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. Section 3.5 proves that rotations and reflections preserve lengths and that they are the only linear operators of the plane that do so.

3.3.24 Differentiation and multiplication on polynomials

Chapter 7 studies differentiation as a linear map. On polynomials the map is defined now from the coefficients alone. For $p(x) = a_0 + a_1x + \cdots + a_nx^n$ in $\mathcal{P}_n$ put

$$p'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}. \tag{3.3.7}$$

Because the coefficients of p are unique (Example 3.2.19), this defines a map $p \mapsto p'$ from $\mathcal{P}_n$ to itself. It is linear because the coefficients of $p + q$ and cp are the sums and multiples of those of p and q. Sections 6.1 and 6.3 prove that p' is the derivative of p in the sense of limits.

Worked Example 3.3.2

Find the matrix of $p \mapsto p'$ on $\mathcal{P}_3$ in the basis $1, x, x^2, x^3$, its kernel and its image. Verify rank–nullity and apply the matrix to $p(x) = 2 - x + 5x^3$.

The images of the basis are 0, 1, $2x$, $3x^2$, with coordinate vectors $(0,0,0,0)$, $(1,0,0,0)$, $(0,2,0,0)$, $(0,0,3,0)$. These are the columns of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$A(a_0, a_1, a_2, a_3) = (a_1, 2a_2, 3a_3, 0)$ is zero exactly when $a_1 = a_2 = a_3 = 0$. The kernel consists of the constants and has dimension 1. The image is $\mathcal{P}_2$, since $b_0 + b_1x + b_2x^2$ is the image of $b_0x + \tfrac{1}{2}b_1x^2 + \tfrac{1}{3}b_2x^3$. Its dimension is 3, and $1 + 3 = 4 = \dim \mathcal{P}_3$. For p, $A(2, -1, 0, 5) = (-1, 0, 15, 0)$, the coordinate vector of $p'(x) = -1 + 15x^2$.

Applying the map four times to any element of $\mathcal{P}_3$ gives 0, since each application lowers the degree of a nonconstant polynomial by one. Chapter 7 returns to this property under the name nilpotence.

For a fixed function m on an interval I, the *multiplication operator* $M_m f = mf$ is linear on $\mathcal{F}(I)$, since $m(f + cg) = mf + c\,mg$. Its kernel is the set of f with $f(x) = 0$ at every x where

$m(x) \neq 0$. It is injective exactly when m has no zero on I. Multiplication by x maps $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$.

Worked Example 3.3.3

Let $(c, s) = (\frac{3}{5}, \frac{4}{5})$. Compute $R^2_{c,s}$ and $F_{c,s}$, and compare $R^2_{c,s}\mathbf{e}_1$ with $F_{c,s}\mathbf{e}_1$.
By Proposition 3.3.23(1), $R^2_{c,s} = R_{c^2-s^2,\,2cs} = R_{-7/25,\,24/25}$. By Definition 3.3.22,

$$F_{c,s} = \frac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}.$$

Both send $\mathbf{e}_1$ to $(-\frac{7}{25}, \frac{24}{25})$, as Proposition 3.3.23(3) predicts because $F_{1,0}\mathbf{e}_1 = \mathbf{e}_1$. The two maps differ on $\mathbf{e}_2$: $R^2_{c,s}\mathbf{e}_2 = (-\frac{24}{25}, -\frac{7}{25})$ and $F_{c,s}\mathbf{e}_2 = (\frac{24}{25}, \frac{7}{25})$.

Study guide

Test linearity on sums and on scalar multiples, and test $T\mathbf{0} = \mathbf{0}$ first since it refutes most non-examples. To find a matrix, apply the map to each basis vector and write the coordinates of the result as a column. The kernel comes from solving $T\mathbf{v} = \mathbf{0}$ and the image from the span of the columns. Rank–nullity checks the two answers against each other. Memorise $R_{c,s}$ and $F_{c,s}$ together with what they do to $\mathbf{e}_1$ and $\mathbf{e}_2$, and the 2×2 inverse formula.

Problems

3.3.1 Which maps $\mathbb{R}^2 \to \mathbb{R}^2$ are linear? (a) $(x, y) \mapsto (y, x)$, (b) $(x, y) \mapsto (x + 1, y)$, (c) $(x, y) \mapsto (x^2, y)$, (d) $(x, y) \mapsto (3x - y, 0)$. ⋆ Hint · Answer · Solution

3.3.2 Let $T(x, y, z) = (x + y,\, y + z)$. Find its matrix, its kernel and its image, and check Theorem 3.3.11. ⋆ Hint · Answer · Solution

3.3.3 Invert $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and use the inverse to solve $2x + y = 4$, $5x + 3y = 7$. ⋆ Hint · Answer · Solution

3.3.4 Let T be the map $p \mapsto p'$ on $\mathcal{P}_2$. Find the matrices of T and of $T \circ T$ in the basis $1, x, x^2$, and the kernel of $T \circ T$. ⋆⋆ Hint · Answer · Solution

3.3.5 Let M be multiplication by x from $\mathcal{P}_2$ to $\mathcal{P}_3$. Find its matrix in the monomial bases, its kernel and its image, and check Theorem 3.3.11. ⋆⋆ Hint · Answer · Solution

3.3.6 Prove that if $T\colon V \to W$ is linear and injective and $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly independent, then $T\mathbf{v}_1, \dots, T\mathbf{v}_k$ is linearly independent. ⋆⋆ Hint · Answer · Solution

3.3.7 Prove that the product $F_{c,s}F_{c',s'}$ of two reflections is a rotation. ⋆⋆⋆ Hint · Answer · Solution

3.3.8 Let $T(x, y) = (y, 0)$ on $\mathbb{R}^2$. Show that $\ker T = \operatorname{im} T$. Then prove that no linear operator S on $\mathbb{R}^3$ has $\ker S = \operatorname{im} S$. ⋆⋆ Hint · Answer · Solution

3.4 Linear Functionals and the Dual Space

Overview

Objectives: Recognise linear functionals on $\mathbb{R}$, on $\mathbb{R}^2$, on polynomial spaces and on function spaces; prove that every linear functional on $\mathbb{R}$ is multiplication by a number and that every linear functional on $\mathbb{R}^2$ is $(x, y) \mapsto ax + by$; construct the dual basis; read evaluation at a point and the definite integral as linear functionals; describe a 1-form as a choice of linear functional on each tangent space $T_p\mathbb{R}$.
Prerequisites: Linear maps and bases (Sections 3.2 and 3.3); tangent spaces (Definition 3.1.8).
Key results: Linear functionals (Definition 3.4.2); the dual space (Definition 3.4.7); functionals on $\mathbb{R}$ and $\mathbb{R}^2$ (Theorem 3.4.5); the dual basis and $\dim V^* = \dim V$ (Theorem 3.4.8); evaluation functionals (Definition 3.4.10).

Notation

$\lambda,\ \mu$	linear functionals
V^*	the dual space of V (Definition 3.4.7)
$\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$	the dual basis of a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ (Theorem 3.4.8)
δ_{ij}	1 if $i = j$ and 0 otherwise
E_a	evaluation at a, $E_a f = f(a)$ (Definition 3.4.10)

Many quantities in calculus assign a single number to a vector and respect sums and scalar multiples. The value $f(a)$ of a function at a fixed point is one. The integral of a continuous function over a fixed interval is another. On the tangent space $T_p\mathbb{R}$ the rise of a tangent line over a displacement $\mathbf{v}$, namely $f'(p)\,\mathbf{v}$, is a third. A linear map with values in $\mathbb{R}$ is called a linear functional. The 1-forms of Section 6.2 are built from linear functionals on the tangent spaces $T_p\mathbb{R}$, and this section develops what they need.

3.4.1 Linear functionals

Definition 3.4.2 (Linear functional)**.** A *linear functional* on a vector space V is a linear map $\lambda\colon V \to \mathbb{R}$:

$$\lambda(\mathbf{u} + \mathbf{v}) = \lambda(\mathbf{u}) + \lambda(\mathbf{v}), \qquad \lambda(c\mathbf{v}) = c\,\lambda(\mathbf{v}) \tag{3.4.1}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$.

Example 3.4.3 (Linear functionals)**.** On $\mathbb{R}^2$ the maps $(x, y) \mapsto 2x - y$, $(x, y) \mapsto x$ and $(x, y) \mapsto 0$ are linear functionals. On $\mathcal{P}_n$ the maps $p \mapsto p(1)$, $p \mapsto a_k$ (the coefficient of x^k) and $p \mapsto p'(0)$ are linear functionals. On $T_p\mathbb{R}$ the map $(p, v) \mapsto 3v$ is a linear functional.

Example 3.4.4 (Non-examples)**.** The map $(x, y) \mapsto xy$ is not linear: it sends $(1, 1)$ to 1 and $2(1, 1)$ to 4. The map $\mathbf{v} \mapsto \mathbf{v} + 1$ on $T_p\mathbb{R}$ sends $\mathbf{0}$ to 1, while every linear map sends $\mathbf{0}$ to 0. The map $\mathbf{v} \mapsto |\mathbf{v}|$ takes the value 1 at both 1 and $-1 = (-1) \cdot 1$ and therefore fails $\lambda(-\mathbf{v}) = -\lambda(\mathbf{v})$. The map $(x, y) \mapsto (x, y)$ is linear and not a functional, since its values lie in $\mathbb{R}^2$.

Theorem 3.4.5 (Linear functionals on $\mathbb{R}$ and on $\mathbb{R}^2$)**.** *1. For every linear functional λ on $\mathbb{R}$ there is exactly one number a with $\lambda(v) = av$ for all $v \in \mathbb{R}$, namely $a = \lambda(1)$.*

2. For every linear functional λ on $\mathbb{R}^2$ there is exactly one pair of numbers a, b with $\lambda(x, y) = ax + by$ for all (x, y), namely $a = \lambda(1, 0)$ and $b = \lambda(0, 1)$.

3. The same is true for each tangent space: every linear functional on $T_p\mathbb{R}$ is $(p, v) \mapsto av$ for exactly one number a, namely $a = \lambda((p, 1))$.

Proof. (1) By linearity $\lambda(v) = \lambda(v \cdot 1) = v\,\lambda(1)$. If also $\lambda(v) = a'v$ for all v, then $v = 1$ gives $a' = \lambda(1)$. (2) Writing $(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2$ gives $\lambda(x, y) = x\lambda(\mathbf{e}_1) + y\lambda(\mathbf{e}_2)$. If also $\lambda(x, y) = a'x + b'y$, then evaluating at $\mathbf{e}_1$ and $\mathbf{e}_2$ gives $a' = a$ and $b' = b$. (3) Since the operations of $T_p\mathbb{R}$ are those of $\mathbb{R}$ (Definition 3.1.8), the proof of (1) applies with $(p, 1)$ in place of 1. ∎

When the base point p is fixed, Definition 3.1.8 writes the displacement (p, v) as $\mathbf{v}$. In that notation part (3) reads $\lambda(\mathbf{v}) = a\mathbf{v}$, where $a\mathbf{v}$ on the right is the number av. The same argument shows that every linear functional on $\mathbb{R}^n$ is $\mathbf{x} \mapsto a_1x_1 + \cdots + a_nx_n$ with $a_i = \lambda(\mathbf{e}_i)$. A linear functional on $\mathbb{R}^n$ is a $1 \times n$ matrix (a row). Part (3) is the fact on which Theorem 6.2.8 rests: a linear functional on a tangent space of the line is determined by one number.

Common misconception

A linear functional on $\mathbb{R}$ is $v \mapsto av$ with no constant term, and its graph is a line through the origin. The function $v \mapsto av + b$ with $b \neq 0$ is affine and is no linear functional. A graph made of straight pieces need not belong to a linear map either. The graph of $v \mapsto |v|$ consists of two half-lines through the origin, and the map fails additivity: $|1 + (-1)| = 0$ while $|1| + |-1| = 2$.

The zero set of a nonzero functional $\lambda(x, y) = ax + by$ on $\mathbb{R}^2$ is a line through the origin, its kernel. By Theorem 3.3.11 the kernel has dimension $2 - 1 = 1$ because the image is all of $\mathbb{R}$. Each level set $\{\lambda = k\}$ is the parallel line $ax + by = k$, a translate of the kernel (Theorem 3.3.10). Figure 3.4.1 draws the level sets of $\lambda(x, y) = 2x + y$. The functional is determined by its level lines: their direction fixes the kernel, and the spacing fixes the scale.

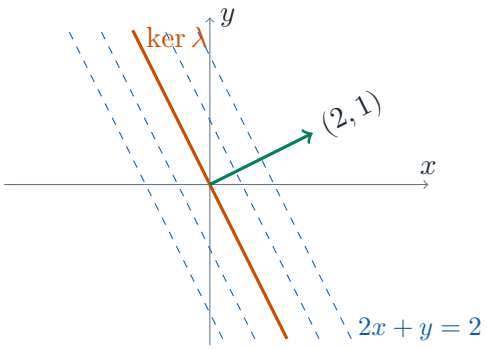


Figure 3.4.1: Level lines $2x + y = k$ of the functional $\lambda(x, y) = 2x + y$ for $k = -2, -1, 0, 1, 2$. The kernel ($k = 0$, solid) is a line through the origin and the other level lines are its translates. The vector $(2, 1)$ of coefficients is drawn at reduced scale.

Worked Example 3.4.1

Find the linear functional λ on $\mathbb{R}^2$ with $\lambda(1, 1) = 3$ and $\lambda(1, -1) = 1$, and describe its kernel.

By Theorem 3.4.5(2), $\lambda(x, y) = ax + by$. The conditions $a + b = 3$ and $a - b = 1$ give $a = 2$ and $b = 1$. Hence $\lambda(x, y) = 2x + y$, the functional of Fig. 3.4.1. Its kernel is the

line $y = -2x$, spanned by $(1, -2)$. The data determine λ because $(1, 1)$ and $(1, -1)$ form a basis of $\mathbb{R}^2$ (Theorem 3.3.5).

3.4.6 Dual space and dual basis

Definition 3.4.7 (Dual space)**.** The *dual space* V^* of a vector space V is the set of all linear functionals on V, with the pointwise operations $(\lambda + \mu)(\mathbf{v}) = \lambda(\mathbf{v}) + \mu(\mathbf{v})$ and $(c\lambda)(\mathbf{v}) = c\,\lambda(\mathbf{v})$.

The dual space is a subspace of the space of all functions from V to $\mathbb{R}$ (the proof for $\mathcal{F}(I)$ applies with V in place of I). The zero functional is linear, and sums and multiples of linear functionals are linear since $(\lambda + \mu)(\mathbf{u} + c\mathbf{v}) = \lambda(\mathbf{u}) + c\lambda(\mathbf{v}) + \mu(\mathbf{u}) + c\mu(\mathbf{v})$. By Theorem 3.2.13 the dual space is a vector space.

Theorem 3.4.8 (Dual basis)**.** *Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis of V. For each i there is exactly one linear functional $\mathbf{e}_i^*$ with $\mathbf{e}_i^*(\mathbf{e}_j) = \delta_{ij}$, and it returns the ith coordinate: $\mathbf{e}_i^*(\sum_j c_j \mathbf{e}_j) = c_i$. The functionals $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ form a basis of V^*, the* dual basis*, and every $\lambda \in V^*$ is*

$$\lambda = \lambda(\mathbf{e}_1)\,\mathbf{e}_1^* + \cdots + \lambda(\mathbf{e}_n)\,\mathbf{e}_n^*. \tag{3.4.2}$$

In particular $\dim V^* = \dim V$.

Proof. Existence and uniqueness of $\mathbf{e}_i^*$ is Theorem 3.3.5 with the values $\delta_{i1}, \dots, \delta_{in}$. By linearity $\mathbf{e}_i^*(\sum c_j \mathbf{e}_j) = \sum c_j \delta_{ij} = c_i$. The two sides of (3.4.2) are linear functionals that agree on every $\mathbf{e}_j$, since the right side gives $\sum_i \lambda(\mathbf{e}_i)\delta_{ij} = \lambda(\mathbf{e}_j)$. They are equal by the uniqueness in Theorem 3.3.5. Hence the dual basis spans V^*. If $\sum a_i \mathbf{e}_i^* = 0$, evaluating at $\mathbf{e}_j$ gives $a_j = 0$, and the list is independent. ∎

For $V = \mathbb{R}$ with basis 1 the dual basis is the single functional $v \mapsto v$, and Theorem 3.4.8 restates Theorem 3.4.5(1): every functional on $\mathbb{R}$ is a multiple of $v \mapsto v$. On each tangent space $T_p\mathbb{R}$ the same functional $\mathbf{v} \mapsto \mathbf{v}$ is the dual basis of the basis $(p, 1)$. Section 6.2 shows that it is the value at p of the differential $\mathrm{d}x$ of the coordinate function. For $\mathbb{R}^2$ the dual basis of $\mathbf{e}_1, \mathbf{e}_2$ consists of the coordinate functionals $(x, y) \mapsto x$ and $(x, y) \mapsto y$. For $\mathcal{P}_n$ with the monomial basis, $\mathbf{e}_k^*$ is the functional that returns the coefficient of x^k.

Caution

The dual basis functional $\mathbf{e}_1^*$ depends on the whole basis and not on $\mathbf{e}_1$ alone. In the basis $(1, 0), (0, 1)$ of $\mathbb{R}^2$ it is $(x, y) \mapsto x$. In the basis $(1, 0), (1, 1)$ the first vector is the same and the first dual functional is $(x, y) \mapsto x - y$. It vanishes on $(1, 1)$.

3.4.9 Functionals on function spaces

Definition 3.4.10 (Evaluation functional)**.** Let I be an interval and $a \in I$. The *evaluation at* a is the map $E_a \colon \mathcal{F}(I) \to \mathbb{R}$, $E_a f = f(a)$.

Evaluation is linear because the operations of $\mathcal{F}(I)$ are pointwise: $E_a(f + cg) = (f + cg)(a) = f(a) + c\,g(a)$. Its kernel is the subspace of functions vanishing at a. Restricted to $\mathcal{P}_n$ it is $E_a = \mathbf{e}_0^* + a\,\mathbf{e}_1^* + \cdots + a^n \mathbf{e}_n^*$ by Theorem 3.4.8, since $E_a(x^k) = a^k$.

The definite integral of Chapter 8 gives a second family. For a closed interval $[a, b]$ the map

$$f \longmapsto \int_a^b f(x)\,\mathrm{d}x$$

is a linear functional on $C([a,b])$. Its linearity is Eq. (8.2.5), proved in Section 8.2. As a non-example, $f \mapsto \int_a^b f(x)^2\,dx$ is not linear, since doubling f multiplies the value by 4. In this book the integral is the integral of a 1-form.

3.4.11 Families of functionals and 1-forms

A function f on an interval I assigns a number to each point. A 1-*form* on I assigns to each point $p \in I$ a linear functional ω_p on the tangent space $T_p\mathbb{R}$ (Definition 6.2.2). The rule

$$\omega_p(\mathbf{v}) = p^2\,\mathbf{v} \qquad (p \in \mathbb{R},\ \mathbf{v} \in T_p\mathbb{R})$$

is an example. At $p = 3$ it is the functional $\mathbf{v} \mapsto 9\mathbf{v}$ on $T_3\mathbb{R}$, and at $p = -1$ it is $\mathbf{v} \mapsto \mathbf{v}$ on $T_{-1}\mathbb{R}$. By Theorem 3.4.5(3) each ω_p is multiplication by one number $f(p) = \omega_p((p,1))$. Thus a 1-form on an interval is determined by a function. The functional changes with the point. The base point recorded in Definition 3.1.8 makes such a family meaningful. As a non-example, the rule $\mathbf{v} \mapsto \mathbf{v}^2$ at every point is no 1-form, since $\mathbf{v} \mapsto \mathbf{v}^2$ is not linear.

Worked Example 3.4.2

On $\mathcal{P}_2$ write the evaluation E_2 in the dual basis of $1, x, x^2$, and find the polynomial $p \in \mathcal{P}_2$ with $E_0 p = 1$, $E_1 p = 0$ and $E_2 p = 0$.
Since $E_2(a_0 + a_1 x + a_2 x^2) = a_0 + 2a_1 + 4a_2$,

$$E_2 = \mathbf{e}_0^* + 2\,\mathbf{e}_1^* + 4\,\mathbf{e}_2^*.$$

The polynomial p must vanish at 1 and 2. Two applications of Theorem 4.2.6 and a count of degrees give $p(x) = k(x-1)(x-2)$ for a constant k. That theorem is proved in Section 4.2 from the division algorithm and uses nothing from this chapter. Then $p(0) = 2k = 1$ gives

$$p(x) = \tfrac{1}{2}(x-1)(x-2) = 1 - \tfrac{3}{2}x + \tfrac{1}{2}x^2.$$

As a check, $p(1) = 1 - \frac{3}{2} + \frac{1}{2} = 0$ and $p(2) = 1 - 3 + 2 = 0$.

Study guide

A linear functional is a linear map with values in $\mathbb{R}$. On $\mathbb{R}$ it is multiplication by $\lambda(1)$. On $\mathbb{R}^2$ it is $ax + by$ with a and b read off from the standard basis. Its level sets are parallel lines. To find a functional from its values on a basis, solve for the coefficients. The dual basis functional $\mathbf{e}_i^*$ returns the ith coordinate. The two functionals on function spaces to remember are evaluation E_a and the definite integral. A 1-form chooses one linear functional on each tangent space $T_p\mathbb{R}$, and on the line each choice is multiplication by a number.

Problems

3.4.1 Which maps are linear functionals on $\mathbb{R}^2$? (a) $(x,y) \mapsto 3x - 5y$, (b) $(x,y) \mapsto x + y + 1$, (c) $(x,y) \mapsto xy$, (d) $(x,y) \mapsto 0$. ⋆ Hint · Answer · Solution

3.4.2 Find the dual basis of the basis $\mathbf{u}_1 = (1,1)$, $\mathbf{u}_2 = (1,2)$ of $\mathbb{R}^2$.
⋆ Hint · Answer · Solution

3.4.3 On $\mathcal{P}_3$ write the functionals E_{-1} and $p \mapsto p'(0)$ in the dual basis of $1, x, x^2, x^3$.
⋆ Hint · Answer · Solution

3.4.4 Prove that the kernel of a nonzero linear functional on $\mathbb{R}^2$ is a line through the origin and has dimension 1. ⋆⋆ Hint · Answer · Solution

3.4.5 Let $a \neq b$. Find $p, q \in \mathcal{P}_1$ with $E_a p = 1$, $E_b p = 0$, $E_a q = 0$, $E_b q = 1$, and conclude that E_a, E_b is a basis of $\mathcal{P}_1^*$. ⋆⋆ Hint · Answer · Solution

3.4.6 Let ω assign to each $p \in \mathbb{R}$ the functional $\omega_p(\mathbf{v}) = (p^2 + 1)\,\mathbf{v}$ on $T_p\mathbb{R}$. Compute $\omega_2((2, -3))$ and $\omega_0((0, 5))$, and find the function f with $\omega_p(\mathbf{v}) = f(p)\,\mathbf{v}$. ⋆ Hint · Answer · Solution

3.4.7 Let a_0, a_1, a_2 be distinct real numbers. Prove that $E_{a_0}, E_{a_1}, E_{a_2}$ is a basis of $\mathcal{P}_2^*$. ⋆⋆⋆ Hint · Answer · Solution

3.4.8 Let λ and μ be nonzero linear functionals on V with $\ker \lambda = \ker \mu$. Prove that $\mu = c\lambda$ for some number c. ⋆⋆ Hint · Answer · Solution

3.5 Inner Product and Isometries of the Plane

Overview

Objectives: Compute dot products, lengths and distances; prove the Cauchy–Schwarz and triangle inequalities; decide orthogonality and project one vector on another; prove that rotations and reflections preserve lengths and dot products and that they are the only isometries of the plane fixing the origin; compute 2×2 determinants and read them as signed areas and as orientation.
Prerequisites: Vectors of the plane (Section 3.1); rotations, reflections and the 2×2 inverse (Definitions 3.3.21 and 3.3.22 and Theorem 3.3.19); square roots (Theorem 2.4.8) and the triangle inequality for numbers (Lemma 2.3.8).
Key results: The dot product (Definition 3.5.2); the Cauchy–Schwarz inequality (Theorem 3.5.4); the triangle inequality (Theorem 3.5.5); rotations and reflections preserve the dot product (Theorem 3.5.10); the isometries fixing the origin (Theorem 3.5.11); the determinant (Definition 3.5.13) and its properties (Proposition 3.5.14); signed area (Theorem 3.5.15).

Notation

$\mathbf{u} \cdot \mathbf{v}$	the dot product (Definition 3.5.2)
$\lvert\mathbf{v}\rvert$	the length of $\mathbf{v}$
$\mathbf{u} \perp \mathbf{v}$	$\mathbf{u}$ and $\mathbf{v}$ are orthogonal
$\operatorname{proj}_{\mathbf{v}} \mathbf{u}$	the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$
$\mathbf{v}^{\perp}$	the quarter turn $R_{0,1}\mathbf{v} = (-v_2, v_1)$
$\det A$	the determinant of a 2×2 matrix (Definition 3.5.13)
$\det(\mathbf{v}, \mathbf{w})$	the determinant of the matrix with columns $\mathbf{v}, \mathbf{w}$

The vector operations of Section 3.1 know nothing about length: they treat the vectors $(1, 0)$ and $(3, 4)$ alike. Geometry needs the length of a vector, the distance between two points and the right angle. The trigonometric functions are defined in Section 4.5 from the unit circle, the set of points at distance 1 from the origin. The facts about that circle used there are proved in Appendix A.1 from the results of this section: rotations and reflections map the circle to itself, rotations preserve the direction of travel round it, and reflections reverse that direction.

3.5.1 Dot product and length

Definition 3.5.2 (Dot product and length)**.** The *dot product* of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n. \tag{3.5.1}$$

The *length* of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$, and the *distance* between points P and Q is $|Q - P|$. A vector of length 1 is a *unit vector*.

In the plane $|(v_1, v_2)| = \sqrt{v_1^2 + v_2^2}$ is the length of the hypotenuse of a right triangle with legs $|v_1|$ and $|v_2|$. The square root exists by Theorem 2.4.8. For example $|(3, 4)| = 5$ and $(3, 4)/5$ is a unit vector. In $\mathbb{R}$ the length of v is $\sqrt{v^2} = |v|$, the absolute value. As a non-example, $|(3, 4)|$ is

not $3+4$. The length equals $|v_1|+|v_2|$ only when one component is 0. Section 3.6 studies that sum as a second measure of the size of a vector.

Proposition 3.5.3 (Properties of the dot product)**.** *For all* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ *and* $c\in\mathbb{R}$*:*

1. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$*;*
2. $(\mathbf{u}+c\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+c\,\mathbf{v}\cdot\mathbf{w}$*;*
3. $\mathbf{v}\cdot\mathbf{v}\geq 0$*, with equality only for* $\mathbf{v}=\mathbf{0}$*;*
4. $|c\mathbf{v}|=|c|\,|\mathbf{v}|$*;*
5. $|\mathbf{u}+\mathbf{v}|^2=|\mathbf{u}|^2+2\,\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2$.

Proof. (1) and (2) follow from the field axioms applied to each term of the sum. (3) $\mathbf{v}\cdot\mathbf{v}=\sum v_i^2$ is a sum of squares, each nonnegative and positive unless $v_i=0$ (Proposition 2.2.5). (4) $(c\mathbf{v})\cdot(c\mathbf{v})=c^2\,\mathbf{v}\cdot\mathbf{v}$, and the nonnegative square root of c^2 is $|c|$. (5) Expand $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ by (1) and (2). ∎

By (2) and (1), for fixed $\mathbf{w}$ the map $\mathbf{u}\mapsto\mathbf{u}\cdot\mathbf{w}$ is a linear functional on $\mathbb{R}^n$. By Theorem 3.4.5 and the remark after it every linear functional on $\mathbb{R}^n$ has this form, with $\mathbf{w}$ the vector of its coefficients.

Theorem 3.5.4 (Cauchy–Schwarz inequality)**.** *For all* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$*,*

$$|\mathbf{u}\cdot\mathbf{v}|\leq|\mathbf{u}|\,|\mathbf{v}|, \tag{3.5.2}$$

with equality exactly when one of the vectors is a multiple of the other.

Proof. If $\mathbf{v}=\mathbf{0}$ both sides are 0 and $\mathbf{v}=0\mathbf{u}$. Let $\mathbf{v}\neq\mathbf{0}$ and put $t=(\mathbf{u}\cdot\mathbf{v})/|\mathbf{v}|^2$. By Proposition 3.5.3,

$$0\leq|\mathbf{u}-t\mathbf{v}|^2=|\mathbf{u}|^2-2t\,\mathbf{u}\cdot\mathbf{v}+t^2|\mathbf{v}|^2=|\mathbf{u}|^2-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2}. \tag{3.5.3}$$

Multiplying (3.5.3) by $|\mathbf{v}|^2>0$ gives $(\mathbf{u}\cdot\mathbf{v})^2\leq|\mathbf{u}|^2|\mathbf{v}|^2$, and taking nonnegative square roots gives (3.5.2) (Proposition 2.2.5). Equality forces $|\mathbf{u}-t\mathbf{v}|=0$, that is $\mathbf{u}=t\mathbf{v}$. Conversely, if $\mathbf{u}=s\mathbf{v}$ then both sides equal $|s||\mathbf{v}|^2$. ∎

Theorem 3.5.5 (Triangle inequality for vectors)**.** *For all* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$*,* $|\mathbf{u}+\mathbf{v}|\leq|\mathbf{u}|+|\mathbf{v}|$ *and* $\big||\mathbf{u}|-|\mathbf{v}|\big|\leq|\mathbf{u}-\mathbf{v}|$*. For points* P,Q,S*,* $|S-P|\leq|Q-P|+|S-Q|$.

Proof. By Proposition 3.5.3(5) and Theorem 3.5.4,

$$|\mathbf{u}+\mathbf{v}|^2\leq|\mathbf{u}|^2+2|\mathbf{u}||\mathbf{v}|+|\mathbf{v}|^2=\big(|\mathbf{u}|+|\mathbf{v}|\big)^2.$$

Both sides of the first inequality are nonnegative, and it follows by taking square roots. Applying it to $\mathbf{u}=(\mathbf{u}-\mathbf{v})+\mathbf{v}$ gives $|\mathbf{u}|-|\mathbf{v}|\leq|\mathbf{u}-\mathbf{v}|$, and exchanging the vectors gives the second inequality. For points apply the first inequality to $\mathbf{u}=Q-P$ and $\mathbf{v}=S-Q$. ∎

For $n=1$ these are the triangle inequalities of Lemma 2.3.8 and Corollary 2.3.9. In the plane they say that one side of a triangle is at most the sum of the other two.

3.5.6 Angle and orthogonality

For nonzero $\mathbf{u}, \mathbf{v}$, Theorem 3.5.4 places the number

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

in $[-1,1]$. It is 1 exactly when $\mathbf{u}$ is a positive multiple of $\mathbf{v}$, and -1 exactly when it is a negative multiple. Once the cosine is defined in Section 4.5, the *angle* between $\mathbf{u}$ and $\mathbf{v}$ is the number $\theta \in [0,\pi]$ with $\cos\theta$ equal to this ratio, that is $\theta = \arccos$ of it (Definition 4.5.14). This chapter uses only the ratio and its sign. Nothing in Section 4.5 uses the angle defined here. The forward reference is not circular.

Definition 3.5.7 (Orthogonality)**.** Vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal*, $\mathbf{u} \perp \mathbf{v}$, if $\mathbf{u}\cdot\mathbf{v} = 0$. For $\mathbf{v} \neq \mathbf{0}$ the *orthogonal projection* of $\mathbf{u}$ on $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\,\mathbf{v}. \tag{3.5.4}$$

The vectors $(1,2)$ and $(-2,1)$ are orthogonal. In the plane every vector $\mathbf{v}$ is orthogonal to its quarter turn $\mathbf{v}^{\perp} = R_{0,1}\mathbf{v} = (-v_2, v_1)$. The zero vector is orthogonal to every vector. By Proposition 3.5.3(5), $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$ exactly when $\mathbf{u} \perp \mathbf{v}$: Pythagoras' theorem in vector form. The remainder $\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to $\mathbf{v}$, since its dot product with $\mathbf{v}$ is $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} = 0$. The decomposition

$$\mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u}) \tag{3.5.5}$$

writes $\mathbf{u}$ as a vector along $\mathbf{v}$ plus a vector orthogonal to $\mathbf{v}$ (Fig. 3.5.1). The proof of Theorem 3.5.4 used exactly (3.5.5), with $t\mathbf{v} = \operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

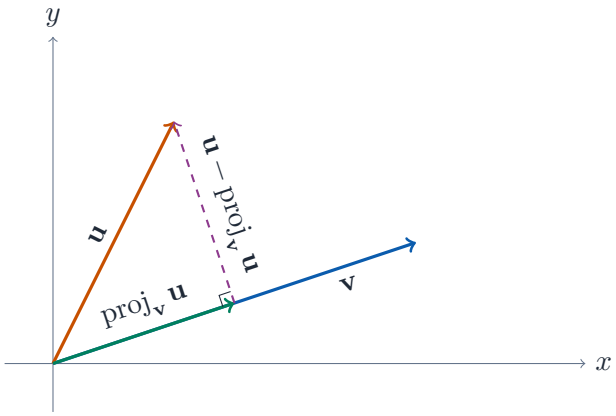


Figure 3.5.1: Orthogonal projection of $\mathbf{u} = (1,2)$ on $\mathbf{v} = (3,1)$. The projection $(\frac{3}{2}, \frac{1}{2})$ lies along $\mathbf{v}$ and the remainder $(-\frac{1}{2}, \frac{3}{2})$ is orthogonal to it.

Common misconception

The dot product of two vectors is a number, never a vector, and it is not computed by multiplying the vectors component by component into a new pair. A dot product of 0 means orthogonality and says nothing about either vector being zero: $(1,2)\cdot(-2,1) = 0$ with neither factor zero. There is no cancellation law: $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ with $\mathbf{u} \neq \mathbf{0}$ does not give $\mathbf{v} = \mathbf{w}$.

Worked Example 3.5.1

For $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 1)$ compute $\mathbf{u} \cdot \mathbf{v}$, the lengths, the ratio $\mathbf{u} \cdot \mathbf{v}/(|\mathbf{u}||\mathbf{v}|)$ and the projection of $\mathbf{u}$ on $\mathbf{v}$, and check Theorem 3.5.4.
Here $\mathbf{u} \cdot \mathbf{v} = 3 + 2 = 5$, $|\mathbf{u}| = \sqrt{5}$ and $|\mathbf{v}| = \sqrt{10}$. Cauchy–Schwarz reads $5 \leq \sqrt{50}$, true since $25 \leq 50$. The ratio is $5/\sqrt{50} = 1/\sqrt{2}$. (In Section 4.5 this identifies the angle as $\pi/4$.) The projection is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \tfrac{5}{10}(3, 1) = (\tfrac{3}{2}, \tfrac{1}{2}),$$

and the remainder $(-\frac{1}{2}, \frac{3}{2})$ has dot product $-\frac{3}{2} + \frac{3}{2} = 0$ with $\mathbf{v}$ (Fig. 3.5.1).

3.5.8 Isometries of the plane

Definition 3.5.9 (Isometry)**.** An *isometry* of the plane is a map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $|T\mathbf{v} - T\mathbf{w}| = |\mathbf{v} - \mathbf{w}|$ for all $\mathbf{v}, \mathbf{w}$.

Every translation $\mathbf{v} \mapsto \mathbf{v} + \mathbf{a}$ is an isometry. The scaling $\mathbf{v} \mapsto 2\mathbf{v}$ is linear and is not an isometry. Neither is the projection $(x, y) \mapsto (x, 0)$: it sends $(0, 0)$ and $(0, 1)$ to the same point.

Theorem 3.5.10 (Rotations and reflections preserve the dot product)**.** *Let $c^2 + s^2 = 1$, $\mathbf{u} = (c, s)$ and $T = R_{c,s}$ or $T = F_{c,s}$. Then $T\mathbf{v} \cdot T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ and $|T\mathbf{v}| = |\mathbf{v}|$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$. In particular T is an isometry and maps the unit circle onto itself. The reflection satisfies $F_{c,s}\mathbf{v} = 2(\mathbf{v} \cdot \mathbf{u})\,\mathbf{u} - \mathbf{v}$.*

Proof. For $\mathbf{v} = (x, y)$ and $\mathbf{w} = (x', y')$,

$$R_{c,s}\mathbf{v} \cdot R_{c,s}\mathbf{w} = (cx - sy)(cx' - sy') + (sx + cy)(sx' + cy') = (c^2 + s^2)(xx' + yy') = \mathbf{v} \cdot \mathbf{w},$$

since the cross terms $-csxy' - csyx' + csxy' + csyx'$ cancel. For the reflection first verify the formula. Using $2c^2 - 1 = c^2 - s^2$ and $2s^2 - 1 = s^2 - c^2$,

$$2(cx + sy)(c, s) - (x, y) = \big((c^2 - s^2)x + 2csy,\ 2csx + (s^2 - c^2)y\big) = F_{c,s}\mathbf{v}. \tag{3.5.6}$$

With $a = \mathbf{v} \cdot \mathbf{u}$, $b = \mathbf{w} \cdot \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{u} = 1$, Proposition 3.5.3 gives

$$F_{c,s}\mathbf{v} \cdot F_{c,s}\mathbf{w} = (2a\mathbf{u} - \mathbf{v}) \cdot (2b\mathbf{u} - \mathbf{w}) = 4ab - 2ab - 2ab + \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}.$$

Taking $\mathbf{w} = \mathbf{v}$ gives the statement about lengths. By linearity $|T\mathbf{v} - T\mathbf{w}| = |T(\mathbf{v} - \mathbf{w})| = |\mathbf{v} - \mathbf{w}|$. Each T maps the unit circle into itself, and onto it because T is invertible with an inverse of the same kind ($R_{c,-s}$ or $F_{c,s}$ by Proposition 3.3.23): a point $\mathbf{v}$ of the circle is the image of the point $T^{-1}\mathbf{v}$ of the circle. ∎

The converse is the main theorem of this section. A translation is an isometry and is not linear. An isometry that fixes the origin is linear and is a rotation or a reflection.

Theorem 3.5.11 (Isometries fixing the origin)**.** *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry with $T\mathbf{0} = \mathbf{0}$. Then $T = R_{c,s}$ or $T = F_{c,s}$ for some c, s with $c^2 + s^2 = 1$. Consequently every isometry of the plane is a rotation or a reflection followed by a translation.*

Proof. **1.** Dot product preserved. With $\mathbf{w} = \mathbf{0}$ the hypothesis gives $|T\mathbf{v}| = |\mathbf{v}|$. By Proposition 3.5.3(5), $|\mathbf{v} - \mathbf{w}|^2 = |\mathbf{v}|^2 - 2\,\mathbf{v} \cdot \mathbf{w} + |\mathbf{w}|^2$, and the same identity for $T\mathbf{v}$, $T\mathbf{w}$ has equal lengths in every place. Hence $T\mathbf{v} \cdot T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$.
2. Images of the standard basis. Let $\mathbf{a} = T\mathbf{e}_1 = (c, s)$ and $\mathbf{b} = T\mathbf{e}_2 = (x, y)$. By step 1, $c^2 + s^2 = 1$, $x^2 + y^2 = 1$ and $cx + sy = 0$. Put $\varepsilon = -sx + cy$. Then $\varepsilon(-s) = s^2x - csy =$

$s^2x + c^2x = x$, using $-sy = cx$, and $\varepsilon c = -csx + c^2y = s^2y + c^2y = y$, using $-cx = sy$. Hence $\mathbf{b} = \varepsilon(-s, c) = \varepsilon\mathbf{a}^\perp$, and $1 = |\mathbf{b}|^2 = \varepsilon^2$ gives $\varepsilon = \pm 1$.

3. Linearity. For any $\mathbf{w} \in \mathbb{R}^2$ a direct expansion gives $(\mathbf{w}\cdot\mathbf{a})\mathbf{a} + (\mathbf{w}\cdot\mathbf{a}^\perp)\mathbf{a}^\perp = \mathbf{w}$ (the coefficient of w_1 in the first component is $c^2 + s^2 = 1$, and so on). Since $\mathbf{b} = \varepsilon\mathbf{a}^\perp$ with $\varepsilon^2 = 1$, this reads $\mathbf{w} = (\mathbf{w}\cdot\mathbf{a})\mathbf{a} + (\mathbf{w}\cdot\mathbf{b})\mathbf{b}$. Apply it to $\mathbf{w} = T\mathbf{v}$ with $\mathbf{v} = (v_1, v_2)$. By step 1, $T\mathbf{v}\cdot\mathbf{a} = T\mathbf{v}\cdot T\mathbf{e}_1 = v_1$ and $T\mathbf{v}\cdot\mathbf{b} = v_2$. Hence $T\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b}$. Thus T is linear with matrix columns $\mathbf{a}$ and $\mathbf{b}$.

4. Identification. If $\varepsilon = 1$ the matrix is $\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = R_{c,s}$. If $\varepsilon = -1$ it is $\begin{pmatrix} c & s \\ s & -c \end{pmatrix}$. Put $c' = \sqrt{(1+c)/2}$ and $s' = \pm\sqrt{(1-c)/2}$ with the sign of s (either sign if $s = 0$). Both radicands are nonnegative because $|c| \le 1$. Then $c'^2 + s'^2 = 1$, $c'^2 - s'^2 = c$ and $2c's' = \pm\sqrt{1-c^2} = \pm|s| = s$. The matrix is $F_{c',s'}$.

For the last statement, if T is any isometry then $S\mathbf{v} = T\mathbf{v} - T\mathbf{0}$ is an isometry with $S\mathbf{0} = \mathbf{0}$, and $T\mathbf{v} = S\mathbf{v} + T\mathbf{0}$. ∎

Worked Example 3.5.2

Show that $T(x, y) = (\frac{3x+4y}{5}, \frac{4x-3y}{5})$ is a reflection and find its mirror line.
The columns $\mathbf{a} = (\frac{3}{5}, \frac{4}{5})$ and $\mathbf{b} = (\frac{4}{5}, -\frac{3}{5})$ are unit vectors with $\mathbf{a}\cdot\mathbf{b} = 0$, and $\mathbf{b} = -\mathbf{a}^\perp$. This is the case $\varepsilon = -1$ of Theorem 3.5.11 with $c = \frac{3}{5}$ and $s = \frac{4}{5}$. Step 4 gives $c' = \sqrt{4/5} = 2/\sqrt{5}$ and $s' = \sqrt{1/5} = 1/\sqrt{5}$. Then $T = F_{c',s'}$ with mirror line through $\mathbf{0}$ and $(2, 1)$. As a check, $T(2, 1) = (\frac{10}{5}, \frac{5}{5}) = (2, 1)$ and $T(-1, 2) = (\frac{5}{5}, \frac{-10}{5}) = (1, -2)$.

3.5.12 Determinant as area and orientation

Definition 3.5.13 (Determinant of a 2×2 matrix)**.** The *determinant* of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$. For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, $\det(\mathbf{v}, \mathbf{w}) = v_1w_2 - v_2w_1$ is the determinant of the matrix with columns $\mathbf{v}$ and $\mathbf{w}$.

By Theorem 3.3.19 A is invertible exactly when $\det A \neq 0$. For example $\det((1, 2), (3, 1)) = 1 - 6 = -5$ and $\det((1, 2), (2, 4)) = 0$. The second pair is dependent.

Proposition 3.5.14 (Properties of the determinant)**.** *For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, scalars t and 2×2 matrices A, B:*

1. $\det(\mathbf{u} + t\mathbf{v}, \mathbf{w}) = \det(\mathbf{u}, \mathbf{w}) + t\det(\mathbf{v}, \mathbf{w})$, *and the same in the second slot;*
2. $\det(\mathbf{w}, \mathbf{v}) = -\det(\mathbf{v}, \mathbf{w})$ *and* $\det(\mathbf{v}, \mathbf{v}) = 0$; $\det(\mathbf{e}_1, \mathbf{e}_2) = 1$;
3. $\det(\mathbf{v}, \mathbf{w}) = \mathbf{v}^\perp \cdot \mathbf{w}$;
4. $\det(A\mathbf{v}, A\mathbf{w}) = \det A \det(\mathbf{v}, \mathbf{w})$;
5. $\det(AB) = \det A \det B$;
6. $\det R_{c,s} = 1$ *and* $\det F_{c,s} = -1$;
7. $(\mathbf{v}\cdot\mathbf{w})^2 + \det(\mathbf{v}, \mathbf{w})^2 = |\mathbf{v}|^2|\mathbf{w}|^2$.

Proof. (1)–(3) follow by expanding $v_1w_2 - v_2w_1$ and $\mathbf{v}^\perp\cdot\mathbf{w} = -v_2w_1 + v_1w_2$. (4) With $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(A\mathbf{v}, A\mathbf{w}) = (av_1 + bv_2)(cw_1 + dw_2) - (cv_1 + dv_2)(aw_1 + bw_2) = (ad - bc)(v_1w_2 - v_2w_1), \quad (3.5.7)$$

since the terms in v_1w_1 and v_2w_2 cancel. (5) The columns of AB are $AB\mathbf{e}_1$ and $AB\mathbf{e}_2$. By (4) and (2), $\det(AB) = \det(AB\mathbf{e}_1, AB\mathbf{e}_2) = \det A \det(B\mathbf{e}_1, B\mathbf{e}_2) = \det A \det B$. (6) $\det R_{c,s} = c^2 + s^2 = 1$ and $\det F_{c,s} = -(c^2 - s^2)^2 - 4c^2s^2 = -(c^2 + s^2)^2 = -1$. (7) Expanding, $(v_1w_1 + v_2w_2)^2 + (v_1w_2 - v_2w_1)^2 = (v_1^2 + v_2^2)(w_1^2 + w_2^2)$, the cross terms $\pm 2v_1w_1v_2w_2$ cancelling. ∎

Identity (7), Lagrange's identity, gives a second proof of Theorem 3.5.4 in the plane, since $(\mathbf{v} \cdot \mathbf{w})^2 \le |\mathbf{v}|^2|\mathbf{w}|^2$ with equality exactly when $\det(\mathbf{v}, \mathbf{w}) = 0$.

Area has no definition at this point of the book. The elementary formula for the area of a parallelogram is base times height, and the next theorem shows that this formula gives $|\det(\mathbf{v}, \mathbf{w})|$. In that sense the determinant is a signed area.

Theorem 3.5.15 (Determinant as base times height)**.** *Let* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ *with* $\mathbf{v} \neq \mathbf{0}$*, and let* $\mathbf{h} = \mathbf{w} - \operatorname{proj}_{\mathbf{v}} \mathbf{w}$ *be the component of* $\mathbf{w}$ *orthogonal to* $\mathbf{v}$*. Then*

$$|\mathbf{v}|\,|\mathbf{h}| = |\det(\mathbf{v}, \mathbf{w})|. \tag{3.5.8}$$

The parallelogram with vertices $\mathbf{0}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *and* $\mathbf{v} + \mathbf{w}$ *has base* $|\mathbf{v}|$ *and height* $|\mathbf{h}|$ *(Fig. 3.5.2).*

Proof. By Proposition 3.5.3, the orthogonality of $\mathbf{h}$ and $\operatorname{proj}_{\mathbf{v}} \mathbf{w}$ and (7) of Proposition 3.5.14,

$$|\mathbf{h}|^2 = |\mathbf{w}|^2 - \frac{(\mathbf{v} \cdot \mathbf{w})^2}{|\mathbf{v}|^2} = \frac{\det(\mathbf{v}, \mathbf{w})^2}{|\mathbf{v}|^2}.$$

Multiplying by $|\mathbf{v}|^2$ and taking nonnegative square roots (Proposition 2.2.5) gives (3.5.8). ∎

When $\mathbf{v} = \mathbf{0}$ the parallelogram collapses to a segment and the determinant is 0. In what follows the *area* of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$ means $|\det(\mathbf{v}, \mathbf{w})|$.

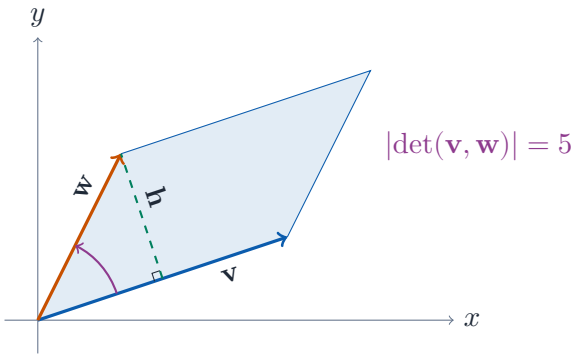


Figure 3.5.2: Parallelogram spanned by $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (1, 2)$ with base $|\mathbf{v}|$ and height $|\mathbf{h}|$. The arc turns anticlockwise from $\mathbf{v}$ to $\mathbf{w}$, and $\det(\mathbf{v}, \mathbf{w}) = 6 - 1 = 5 > 0$.

The sign of the determinant records orientation. By Proposition 3.5.14(3), $\det(\mathbf{v}, \mathbf{w}) > 0$ exactly when $\mathbf{w}$ has a positive component along the quarter turn $\mathbf{v}^\perp$ of $\mathbf{v}$. In that case $\mathbf{w}$ and $\mathbf{v}^\perp$ lie on the same side of the line through $\mathbf{0}$ and $\mathbf{v}$. An ordered pair $(\mathbf{v}, \mathbf{w})$ with $\det(\mathbf{v}, \mathbf{w}) > 0$ is called *positively oriented*. The standard pair $(\mathbf{e}_1, \mathbf{e}_2)$ is positively oriented and $(\mathbf{e}_2, \mathbf{e}_1)$ is not.

Corollary 3.5.16 (Orientation under isometries)**.** *For all* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$*,* $\det(R_{c,s}\mathbf{v}, R_{c,s}\mathbf{w}) = \det(\mathbf{v}, \mathbf{w})$ *and* $\det(F_{c,s}\mathbf{v}, F_{c,s}\mathbf{w}) = -\det(\mathbf{v}, \mathbf{w})$*. Rotations preserve orientation and area, and reflections reverse orientation and preserve area, with area as in Theorem 3.5.15.*

Proof. Combine (4) and (6) of Proposition 3.5.14 with Theorem 3.5.15. ∎

Appendix A.1 uses this corollary in the form $\det(P(t), P(u)) > 0$ for nearby parameters $t < u$ to say that a curve on the unit circle travels anticlockwise, and that a rotation keeps the direction of travel while a reflection reverses it.

Caution

The area is the absolute value of the determinant. The determinant itself can be negative: $\det(\mathbf{w}, \mathbf{v}) = -5$ for the vectors of Fig. 3.5.2. The pair $(\mathbf{w}, \mathbf{v})$ spans the same parallelogram. A rotation and a reflection both preserve lengths and areas. Only the sign of the determinant tells them apart.

Study guide

Compute dot products, lengths and projections fluently. Learn the proof of Cauchy–Schwarz from $|\mathbf{u}-t\mathbf{v}|^2 \geq 0$ with the projection coefficient t, and derive the triangle inequality from it. To classify a linear map of the plane, check that its columns are orthogonal unit vectors and then read the sign of its determinant: $+1$ for a rotation, -1 for a reflection. Remember the three readings of $\det(\mathbf{v}, \mathbf{w})$: the formula $v_1 w_2 - v_2 w_1$, the signed area of the parallelogram, and $\mathbf{v}^{\perp} \cdot \mathbf{w}$ for orientation.

Problems

3.5.1 For $\mathbf{u} = (2, -1)$ and $\mathbf{v} = (1, 3)$ compute $\mathbf{u} \cdot \mathbf{v}$, $|\mathbf{u}|$, $|\mathbf{v}|$ and $\mathbf{u} \cdot \mathbf{v}/(|\mathbf{u}||\mathbf{v}|)$. Are they orthogonal? ⋆ Hint · Answer · Solution

3.5.2 Find all unit vectors orthogonal to $(3, 4)$. ⋆ Hint · Answer · Solution

3.5.3 Find the area of the triangle with vertices $(1, 1)$, $(4, 2)$, $(2, 5)$. ⋆ Hint · Answer · Solution

3.5.4 Show that $A = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ is a rotation and $B = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$ is a reflection, and find the mirror line of B. ⋆⋆ Hint · Answer · Solution

3.5.5 Prove the parallelogram law $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$ and interpret it geometrically. ⋆⋆ Hint · Answer · Solution

3.5.6 Prove that $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for all real a, b, c, with equality exactly when $a = b = c$. ⋆⋆ Hint · Answer · Solution

3.5.7 Prove that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$ exactly when one of the vectors is a nonnegative multiple of the other. ⋆⋆⋆ Hint · Answer · Solution

3.5.8 Let T be a linear operator on $\mathbb{R}^2$ with $|T\mathbf{v}| = |\mathbf{v}|$ for every $\mathbf{v}$. Prove that $\det T = \pm 1$ and that T is a rotation when $\det T = 1$. ⋆⋆ Hint · Answer · Solution

3.5.9 Let $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that S preserves areas and orientation and does not preserve lengths. Conclude that the condition $\det T = \pm 1$ does not make a linear operator T of the plane an isometry. ⋆⋆ Hint · Answer · Solution

3.6 Normed Spaces

Overview

Objectives: State the axioms of a norm and recognise the absolute value, the Euclidean length and the 1-, 2- and ∞-norms on $\mathbb{R}^n$ as norms; draw their unit balls in the plane; compute the sup norm of a bounded function on an interval; measure distance with a norm; prove that any two norms on a finite-dimensional space are equivalent and that the statement fails in infinite dimensions; decide whether a linear map between normed spaces is bounded and compute its operator norm; prove that a linear map is continuous exactly when it is bounded.

Prerequisites: Vector spaces and linear maps (Sections 3.2 and 3.3); length and the Cauchy–Schwarz and triangle inequalities (Definition 3.5.2 and Theorems 3.5.4 and 3.5.5); suprema, completeness, nested intervals and the Archimedean property (Definition 2.4.3, Assumption 2.4.5, and Theorems 2.4.10 and 2.4.17).

Key results: Norms and normed spaces (Definitions 3.6.2 and 3.6.3); the 1-, 2- and ∞-norms (Definition 3.6.6) and their comparison (Proposition 3.6.7); the sup norm (Definition 3.6.9); equivalence of norms in finite dimensions (Theorem 3.6.15); bounded linear maps and the operator norm (Definitions 3.6.18 and 3.6.19 and Proposition 3.6.20); a linear map is continuous exactly when it is bounded (Theorem 3.6.21); linear maps on finite-dimensional spaces are bounded (Corollary 3.6.22).

Notation

$\|\mathbf{v}\|$	a norm of $\mathbf{v}$ (Definition 3.6.2)
$\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$	the 1-, 2- and ∞-norms on $\mathbb{R}^n$ (Definition 3.6.6)
$\mathcal{B}(I)$	the bounded real functions on the interval I
$\|f\|_\infty$	the sup norm of a bounded function (Definition 3.6.9)
$d(\mathbf{u},\mathbf{v})$, $B_r(\mathbf{v})$	distance and open ball (Definition 3.6.12)
$\mathcal{L}(V,W)$	the bounded linear maps from V to W
$\|T\|$	the operator norm of a bounded linear map (Definition 3.6.19)

A limit is a statement about distance. In Chapter 5 the statement $x_n \to a$ says that $|x_n - a|$ becomes and stays small. The continuity of f at a says that $|f(x) - f(a)|$ is small whenever $|x - a|$ is small. The absolute value is the only measure of size the line needs. A vector space usually has several reasonable ones. A displacement (x, y) of the plane can be measured by its Euclidean length, by the larger of $|x|$ and $|y|$, or by the distance $|x| + |y|$ travelled along a grid of streets. A continuous function on $[0, 1]$ can be measured by the largest of its absolute values. Each of these measures has the three properties that the proofs of calculus use: only the zero vector has size 0, scaling a vector by c scales its size by $|c|$, and the size of a sum is at most the sum of the sizes. A measure of size with these properties is a norm. With a norm in place of the absolute value, limits, continuity and convergence make sense in any vector space. The Euclidean length is then one norm among several. The later chapters state limits, continuity and convergence in normed spaces. On spaces of functions the truth of such a statement depends on the norm, and the last part of this section gives the first examples.

3.6.1 Norms

Definition 3.6.2 (Norm)**.** A *norm* on a real vector space V is a function $\mathbf{v} \mapsto \|\mathbf{v}\|$ from V to $\mathbb{R}$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$:

(N1) $\|\mathbf{v}\| \geq 0$, with $\|\mathbf{v}\| = 0$ only for $\mathbf{v} = \mathbf{0}$;

(N2) $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$;

(N3) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (the *triangle inequality*).

Definition 3.6.3 (Normed space)**.** A *normed space* $(V, \|\cdot\|)$ is a real vector space V together with a norm on V. When the norm is fixed by the context the space is written V.

The absolute value is a norm on $\mathbb{R}$: (N1) and (N2) are in Proposition 2.3.6 and (N3) is Lemma 2.3.8. The length $|\mathbf{v}|$ of Definition 3.5.2 is a norm on $\mathbb{R}^n$: (N1) and (N2) are parts (3) and (4) of Proposition 3.5.3 and (N3) is Theorem 3.5.5. Axiom (N2) with $c = 0$ gives $\|\mathbf{0}\| = 0$. Three non-examples on $\mathbb{R}^2$ show that no axiom follows from the other two. The function $(x, y) \mapsto |x|$ satisfies (N2) and (N3) and fails (N1) at $(0, 1)$. The function $\sqrt{|\mathbf{v}|}$ satisfies (N1) and (N3) because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. It fails (N2) because $\sqrt{|4\mathbf{v}|} = 2\sqrt{|\mathbf{v}|}$. The function $\left(\sqrt{|x|} + \sqrt{|y|}\right)^2$ satisfies (N1) and (N2) and fails (N3): it takes the value 1 at $(1, 0)$ and at $(0, 1)$ and the value 4 at their sum.

Proposition 3.6.4 (Triangle inequalities in a normed space)**.** *In a normed space, for all* $\mathbf{u}, \mathbf{v}, \mathbf{w}$*:*

1. $\|-\mathbf{v}\| = \|\mathbf{v}\|$;
2. $\|\mathbf{u} - \mathbf{w}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\|$;
3. $\big|\|\mathbf{u}\| - \|\mathbf{v}\|\big| \leq \|\mathbf{u} - \mathbf{v}\|$.

Proof. (1) is (N2) with $c = -1$. (2) is (N3) applied to $(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w})$. (3) By (N3), $\|\mathbf{u}\| = \|(\mathbf{u}-\mathbf{v})+\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$. Hence $\|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\|$. Exchanging $\mathbf{u}$ and $\mathbf{v}$ and using (1) gives $\|\mathbf{v}\| - \|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\|$. ∎

3.6.5 Norms on $\mathbb{R}^n$

Definition 3.6.6 (The 1-, 2- and ∞-norms)**.** For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|\mathbf{x}\|_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} = |\mathbf{x}|, \qquad \|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}. \tag{3.6.1}$$

For $\mathbf{x} = (3, -4)$ the three norms are 7, 5 and 4. On $\mathbb{R}$ all three are the absolute value. The 2-norm is the Euclidean length of Section 3.5. As a non-example, the smallest absolute value $\min\{|x_1|, \ldots, |x_n|\}$ is no norm for $n \geq 2$: it vanishes at the nonzero vector $\mathbf{e}_1$.

Proposition 3.6.7 (Comparison of the three norms)**.** *The* 1-, 2- *and* ∞*-norms are norms on* $\mathbb{R}^n$*. For every* $\mathbf{x} \in \mathbb{R}^n$*,*

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq n\,\|\mathbf{x}\|_\infty, \qquad \|\mathbf{x}\|_2 \leq \sqrt{n}\,\|\mathbf{x}\|_\infty, \qquad \|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2. \tag{3.6.2}$$

Proof. The 2-norm is the length and is a norm by the remarks after Definition 3.6.3. For the 1-norm, a sum of nonnegative terms is 0 only when every term is 0, the factor $|c|$ comes out of each term by Proposition 2.3.6, and (N3) follows from $|x_i + y_i| \leq |x_i| + |y_i|$ term by term. For the ∞-norm, (N1) and (N2) are clear. For (N3) every index i has $|x_i + y_i| \leq |x_i| + |y_i| \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$. The largest of the left sides obeys the same bound.

Let $|x_k|$ be the largest of the $|x_i|$. Then $x_k^2 \le \sum x_i^2 \le n\,x_k^2$. Taking nonnegative square roots (Proposition 2.2.5) gives the first and the fourth inequality. Expanding the square, $\left(\sum |x_i|\right)^2 = \sum x_i^2 + 2\sum_{i<j} |x_i||x_j| \ge \sum x_i^2$ gives the second. Each $|x_i| \le |x_k|$ gives the third. The last is Theorem 3.5.4 applied to $(|x_1|, \dots, |x_n|)$ and $(1, \dots, 1)$. ∎

The *closed unit ball* of a norm is the set $\{\mathbf{x} : \|\mathbf{x}\| \le 1\}$. In $\mathbb{R}^2$ the unit ball of the 1-norm is the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$, the unit ball of the 2-norm is the disc of radius 1, and the unit ball of the ∞-norm is the square $[-1, 1] \times [-1, 1]$ (Fig. 3.6.1). The inequalities $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$ of (3.6.2) say that the balls are nested in the reverse order: a vector with $\|\mathbf{x}\|_1 \le 1$ has $\|\mathbf{x}\|_2 \le 1$. The ball determines its norm. By (N2) the norm of $\mathbf{x} \neq \mathbf{0}$ is the number $t > 0$ with $\|\mathbf{x}/t\| = 1$.

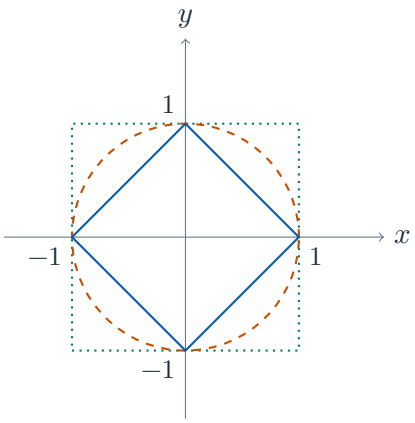


Figure 3.6.1: Unit balls in $\mathbb{R}^2$ of the 1-norm (square on the diagonals, solid), the 2-norm (disc, dashed) and the ∞-norm (square on the axes, dotted). Each ball lies inside the next.

3.6.8 Sup norm on functions

A function f on an interval I is *bounded* if some number M has $|f(x)| \le M$ for every $x \in I$. The bounded functions form a subspace $\mathcal{B}(I)$ of $\mathcal{F}(I)$. The zero function is bounded. If $|f| \le M$ and $|g| \le N$ on I, then $|f + cg| \le M + |c|N$ by Lemma 2.3.8.

Definition 3.6.9 (Sup norm)**.** The *sup norm* of a bounded function f on an interval I is

$$\|f\|_\infty = \sup\{|f(x)| : x \in I\}. \tag{3.6.3}$$

The supremum exists by Assumption 2.4.5 because the set is nonempty and bounded above. On a closed bounded interval $[a, b]$ every continuous function is bounded, and its sup norm is the maximum $\max\{|f(x)| : a \le x \le b\}$. For a continuous f on $[a, b]$ the Extreme Value Theorem (Theorem 5.4.6) proves both claims: f is bounded and takes a largest value M and a smallest value m. The largest value of $|f|$ is then the larger of M and $-m$. So the supremum is a maximum. That theorem is proved in Chapter 5 from completeness and uses nothing from this chapter. The forward reference is not circular. The subspace $C([a, b])$ of $\mathcal{B}([a, b])$ has the sup norm by restriction. The name and the subscript recall the ∞-norm of $\mathbb{R}^n$: a vector of $\mathbb{R}^n$ is a function on $\{1, \dots, n\}$ and its ∞-norm is the largest absolute value of that function.

Proposition 3.6.10 (Sup norm on bounded and continuous functions)**.** *The sup norm is a norm on $\mathcal{B}(I)$ and on $C([a, b])$.*

Proof. (N1) Each $|f(x)| \geq 0$. If $\|f\|_\infty = 0$, then $0 \leq |f(x)| \leq 0$ for every x. Hence f is the zero function. (N2) For $c \neq 0$ every x has $|cf(x)| = |c||f(x)| \leq |c|\,\|f\|_\infty$. Hence $\|cf\|_\infty \leq |c|\,\|f\|_\infty$. The same bound for cf and the scalar $1/c$ gives the reverse inequality. The case $c = 0$ is clear. (N3) Every x has $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|_\infty + \|g\|_\infty$. This number is an upper bound of the values $|f(x)+g(x)|$. The least upper bound is at most this number (Definition 2.4.3). ∎

On $[0,1]$ the identity $x - x^2 = \frac{1}{4} - (x - \frac{1}{2})^2$ shows that $0 \leq x - x^2 \leq \frac{1}{4}$ with equality at $x = \frac{1}{2}$. Hence $\|x - x^2\|_\infty = \frac{1}{4}$. Also $\|x^n\|_\infty = 1$ on $[0,1]$ for every n. As non-examples, the function x on $\mathbb{R}$ is unbounded and has no sup norm. On $C([0,1])$ the function $f \mapsto |f(0)|$ satisfies (N2) and (N3). It fails (N1) at the nonzero function x.

3.6.11 Distance

Definition 3.6.12 (Distance and balls)**.** In a normed space V the *distance* between $\mathbf{u}$ and $\mathbf{v}$ is $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. For $r > 0$ the *open ball* of radius r about $\mathbf{v}$ is $B_r(\mathbf{v}) = \{\mathbf{w} \in V : \|\mathbf{w}-\mathbf{v}\| < r\}$.

By (N1) and Proposition 3.6.4 the distance is nonnegative, vanishes only for $\mathbf{u} = \mathbf{v}$, is symmetric and satisfies $d(\mathbf{u},\mathbf{w}) \leq d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w})$. On $\mathbb{R}$ it is the distance $|u-v|$ of Chapter 2. In $\mathbb{R}^2$ the distance from $(0,0)$ to $(3,4)$ is 7, 5 or 4 in the 1-, 2- or ∞-norm. In $C([a,b])$ the distance $\|f-g\|_\infty$ is the largest vertical gap between the two graphs. The ball $B_\varepsilon(g)$ consists of the continuous functions whose graphs lie in the band of half-width ε about the graph of g (Fig. 3.6.2). As a non-example, the closed ball $\{\mathbf{w} : \|\mathbf{w}-\mathbf{v}\| \leq r\}$ is not the open ball $B_r(\mathbf{v})$: on $\mathbb{R}$ the first is $[v-r, v+r]$ and the second is $(v-r, v+r)$.

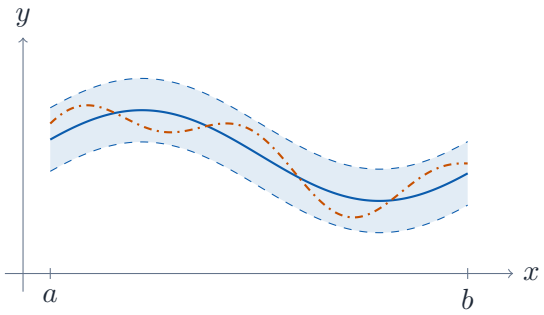


Figure 3.6.2: Ball $B_\varepsilon(g)$ of the sup norm on $C([a,b])$. The graph of g is solid and the band between $g-\varepsilon$ and $g+\varepsilon$ is shaded with dashed edges. A function f with $\|f-g\|_\infty < \varepsilon$ (dash-dotted) has its graph inside the band.

3.6.13 Equivalent norms

Definition 3.6.14 (Equivalent norms)**.** Two norms $\|\cdot\|$ and $\|\cdot\|'$ on V are *equivalent* if there are numbers $m > 0$ and $M > 0$ with

$$m\,\|\mathbf{v}\| \leq \|\mathbf{v}\|' \leq M\,\|\mathbf{v}\| \qquad \text{for every } \mathbf{v} \in V. \tag{3.6.4}$$

Equivalent norms make the same statements about smallness true. If $\|\mathbf{v}\|' \leq M\|\mathbf{v}\|$ as in (3.6.4), every ball $B_r(\mathbf{v})$ of the first norm lies inside the ball of radius Mr of the second. The reverse inequality gives the reverse inclusion. An ε–δ statement proved for one norm is true for the other after ε or δ is multiplied by a constant. By Proposition 3.6.7 and (3.6.2) the 1-, 2- and ∞-norms on $\mathbb{R}^n$ are equivalent: (3.6.2) gives $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$, which is (3.6.4) with $m = 1$ and $M = n$. The next theorem says that every norm on $\mathbb{R}^n$ is equivalent to these. Hence any two norms on $\mathbb{R}^n$ are equivalent.

Theorem 3.6.15 (Equivalence of norms)**.** *On a finite-dimensional vector space any two norms are equivalent.*

Proof. **1.** Upper bound on $\mathbb{R}^n$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and put $C = \|\mathbf{e}_1\| + \cdots + \|\mathbf{e}_n\|$. By (N3) and (N2),

$$\|\mathbf{x}\| = \Big\|\sum_{i=1}^{n} x_i \mathbf{e}_i\Big\| \le \sum_{i=1}^{n} |x_i|\,\|\mathbf{e}_i\| \le C\,\|\mathbf{x}\|_\infty. \tag{3.6.5}$$

With Proposition 3.6.4(3), (3.6.5) gives $\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \le C\,\|\mathbf{x}-\mathbf{y}\|_\infty$.
2. Lower bound on $\mathbb{R}^n$. Let $S = \{\mathbf{x} : \|\mathbf{x}\|_\infty = 1\}$ and let m be the infimum of the numbers $\|\mathbf{x}\|$ with $\mathbf{x} \in S$. It exists by Proposition 2.4.6 because the set is nonempty and bounded below by 0. We show $m > 0$. Call a box $Q = [a_1, a_1+\ell] \times \cdots \times [a_n, a_n+\ell]$ *bad* if for every $\varepsilon > 0$ some $\mathbf{x} \in Q \cap S$ has $\|\mathbf{x}\| < \varepsilon$. Suppose $m = 0$. Then the box $Q_0 = [-1,1]^n$ containing S is bad. Halving each side of a box Q cuts it into 2^n boxes of side $\ell/2$ whose union is Q. If none of them were bad, each would have a number $\varepsilon_j > 0$ with $\|\mathbf{x}\| \ge \varepsilon_j$ on its part of S. The smallest ε_j would then show that Q is not bad. Hence a bad box contains a bad box of half its side. Induction gives bad boxes $Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots$ with sides 2^{1-k}. Their ith coordinate intervals are nested, and Theorem 2.4.17 gives a number x_i^* in all of them. The point $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ lies in every Q_k.

For each k choose $\mathbf{y}_k \in Q_k \cap S$ with $\|\mathbf{y}_k\| < 2^{-k}$. Both $\mathbf{x}^*$ and $\mathbf{y}_k$ lie in Q_k. Hence $\|\mathbf{x}^* - \mathbf{y}_k\|_\infty \le 2^{1-k}$. By step 1 and Proposition 3.6.4(3) for the ∞-norm,

$$\|\mathbf{x}^*\| < (1+2C)\,2^{-k}, \qquad \|\mathbf{x}^*\|_\infty \ge 1 - 2^{1-k} \qquad \text{for every } k.$$

Since $2^k \ge 1+k$ (Theorem 2.3.11), the Archimedean property (Theorem 2.4.10) makes 2^{-k} smaller than any given positive number for some k. The first inequality then gives $\|\mathbf{x}^*\| = 0$ and the second gives $\|\mathbf{x}^*\|_\infty \ge 1$. A nonzero vector of norm 0 contradicts (N1). Hence $m > 0$. For $\mathbf{x} \ne \mathbf{0}$ the vector $\mathbf{x}/\|\mathbf{x}\|_\infty$ lies in S. Axiom (N2) gives $\|\mathbf{x}\| \ge m\,\|\mathbf{x}\|_\infty$. Every norm on $\mathbb{R}^n$ satisfies $m\,\|\mathbf{x}\|_\infty \le \|\mathbf{x}\| \le C\,\|\mathbf{x}\|_\infty$.
3. Finite-dimensional spaces. If $V = \{\mathbf{0}\}$ there is nothing to prove. Otherwise let $\dim V = n$ and let $\Phi\colon V \to \mathbb{R}^n$ send each vector to its coordinate vector in a fixed basis. The map Φ is a linear bijection by (3.2.3) and Theorem 3.2.23. For a norm $\|\cdot\|$ on V the function $\mathbf{x} \mapsto \|\Phi^{-1}\mathbf{x}\|$ is a norm on $\mathbb{R}^n$ because Φ^{-1} is linear and injective. Step 2 gives $m, C > 0$ with $m\|\Phi\mathbf{v}\|_\infty \le \|\mathbf{v}\| \le C\|\Phi\mathbf{v}\|_\infty$. A second norm $\|\cdot\|'$ gives constants m' and C' in the same way. Then $\|\mathbf{v}\|' \le C'\|\Phi\mathbf{v}\|_\infty \le (C'/m)\|\mathbf{v}\|$ and $\|\mathbf{v}\|' \ge m'\|\Phi\mathbf{v}\|_\infty \ge (m'/C)\|\mathbf{v}\|$. ∎

Step 2 is a compactness argument that uses nested intervals. The Bolzano–Weierstrass theorem of Chapter 5 (Theorem 5.1.25) states the same idea for sequences. The proof above needs only the completeness axiom in the form of Theorem 2.4.17.

Caution

Theorem 3.6.15 fails in infinite dimensions. On the space $\mathcal{P}$ of all polynomials compare the sup norm on $[0,1]$ with the coefficient norm $\|a_0 + a_1 x + \cdots + a_n x^n\|_c = |a_0| + \cdots + |a_n|$. Both are norms. The coefficients are unique by Example 3.2.19. A polynomial that vanishes on $[0,1]$ is 0 by Corollary 4.2.8. On $[0,1]$, $|x^k| \le 1$ gives $\|p\|_\infty \le \|p\|_c$. The reverse bound fails. The polynomial $p_n = (1-2x)^n$ has $\|p_n\|_\infty = 1$ because $|1-2x| \le 1$ on $[0,1]$ and $p_n(0) = 1$. By the binomial theorem (Theorem 4.6.14), proved in Chapter 4 by induction with no use of this chapter, its coefficients are $\binom{n}{k}(-2)^k$, with $\|p_n\|_c = \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$. No constant M has $3^n \le M$ for every n.

3.6.16 Continuity and bounded linear maps

Definition 3.6.17 (Continuity between normed spaces)**.** Let V and W be normed spaces and $F\colon V \to W$ a function. The function F is *continuous at* $\mathbf{v}_0 \in V$ if for every $\varepsilon > 0$ there is a $\delta > 0$ with

$$\|\mathbf{v} - \mathbf{v}_0\| < \delta \quad \Longrightarrow \quad \|F(\mathbf{v}) - F(\mathbf{v}_0)\| < \varepsilon. \tag{3.6.6}$$

It is *continuous* if it is continuous at every point of V.

For $V = W = \mathbb{R}$ with the absolute value this is the continuity of Definition 5.3.1 on the domain $\mathbb{R}$. The norm itself is continuous from V to $\mathbb{R}$: by Proposition 3.6.4(3) the choice $\delta = \varepsilon$ works at every point. By the remark after Definition 3.6.14, replacing a norm on V or on W by an equivalent one does not change which functions are continuous. As a non-example, the function on $\mathbb{R}$ equal to 0 for $x \le 0$ and to 1 for $x > 0$ is not continuous at 0: with $\varepsilon = \frac{1}{2}$ no δ works.

Definition 3.6.18 (Bounded linear map)**.** A linear map $T\colon V \to W$ between normed spaces is *bounded* if there is a number $M \ge 0$ with $\|T\mathbf{v}\| \le M\,\|\mathbf{v}\|$ for every $\mathbf{v} \in V$.

Definition 3.6.19 (Operator norm)**.** The *operator norm* of a bounded linear map $T\colon V \to W$ is

$$\|T\| = \sup\{\|T\mathbf{v}\| : \mathbf{v} \in V,\ \|\mathbf{v}\| \le 1\}. \tag{3.6.7}$$

The bounded linear maps from V to W form the set $\mathcal{L}(V, W)$.

Multiplication by a number a on a normed space is bounded with operator norm $|a|$ by (N2). Every linear map $\mathbb{R} \to \mathbb{R}$ is $v \mapsto av$ (Theorem 3.4.5) and so is bounded. A rotation of the plane preserves length (Theorem 3.5.10). In the 2-norm it has $\|R_{c,s}\mathbf{v}\|_2 = \|\mathbf{v}\|_2$ and operator norm 1. Every reflection $F_{c,s}$ has operator norm 1 as well. A non-example appears below: differentiation on the space of all polynomials with the sup norm on $[0, 1]$.

Proposition 3.6.20 (Properties of the operator norm)**.** *Let $T\colon V \to W$ and $S\colon W \to U$ be bounded linear maps.*

1. *$\|T\mathbf{v}\| \le \|T\|\,\|\mathbf{v}\|$ for every $\mathbf{v}$. The number $\|T\|$ is the smallest M allowed in Definition 3.6.18.*
2. *$\mathcal{L}(V, W)$ is a subspace of the space of all functions from V to W. The operator norm is a norm on it.*
3. *$S \circ T$ is bounded with $\|S \circ T\| \le \|S\|\,\|T\|$.*

Proof. (1) If $\|T\mathbf{v}\| \le M\|\mathbf{v}\|$ for all $\mathbf{v}$, then M is an upper bound of the set in (3.6.7). The supremum exists and $\|T\| \le M$. For $\mathbf{v} \neq \mathbf{0}$ the vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ has norm 1. Hence $\|T\mathbf{u}\| \le \|T\|$. Multiplying by $\|\mathbf{v}\|$ and using (N2) gives $\|T\mathbf{v}\| \le \|T\|\,\|\mathbf{v}\|$. The case $\mathbf{v} = \mathbf{0}$ is clear. Thus $\|T\|$ is allowed and lies below every allowed M.

(2) The zero map is bounded. For bounded T, T' and a scalar c, $\|(T + cT')\mathbf{v}\| \le (\|T\| + |c|\|T'\|)\|\mathbf{v}\|$. The subspace test (Theorem 3.2.13) applies. For the axioms: if $\|T\| = 0$ then $T\mathbf{v} = \mathbf{0}$ for every $\mathbf{v}$ by (1). For $c \neq 0$, (1) gives $\|cT\| \le |c|\|T\|$. The same bound for cT and the scalar $1/c$ gives the reverse. The bound with $c = 1$ and (1) give $\|T + T'\| \le \|T\| + \|T'\|$.

(3) By (1) twice, $\|S(T\mathbf{v})\| \le \|S\|\,\|T\mathbf{v}\| \le \|S\|\,\|T\|\,\|\mathbf{v}\|$. ∎

Theorem 3.6.21 (Bounded and continuous linear maps)**.** *For a linear map $T\colon V \to W$ between normed spaces the following are equivalent:*

1. *T is bounded;*

2. T is continuous;

3. T is continuous at $\mathbf{0}$.

Proof. (1)⇒(2). Let $\|T\mathbf{v}\| \le M\|\mathbf{v}\|$ for all $\mathbf{v}$, let $\mathbf{v}_0 \in V$ and $\varepsilon > 0$, and put $\delta = \varepsilon/(M+1)$. If $\|\mathbf{v} - \mathbf{v}_0\| < \delta$, linearity gives $\|T\mathbf{v} - T\mathbf{v}_0\| = \|T(\mathbf{v} - \mathbf{v}_0)\| \le M\delta < \varepsilon$.

(2)⇒(3) is a special case.

(3)⇒(1). Since $T\mathbf{0} = \mathbf{0}$, continuity at $\mathbf{0}$ with $\varepsilon = 1$ gives a $\delta > 0$ such that $\|\mathbf{w}\| < \delta$ implies $\|T\mathbf{w}\| < 1$. Let $\mathbf{v} \neq \mathbf{0}$ and put $\mathbf{w} = \dfrac{\delta}{2\|\mathbf{v}\|}\,\mathbf{v}$. Then $\|\mathbf{w}\| = \delta/2 < \delta$ and

$$\frac{\delta}{2\|\mathbf{v}\|}\,\|T\mathbf{v}\| = \|T\mathbf{w}\| < 1, \qquad \text{that is} \qquad \|T\mathbf{v}\| < \frac{2}{\delta}\,\|\mathbf{v}\|.$$

The bound $M = 2/\delta$ also holds at $\mathbf{v} = \mathbf{0}$. ∎

The proof uses the axioms of a norm, linearity and the order of $\mathbb{R}$. It uses no limit, no sequence and no theorem of a later chapter. Later chapters can apply it without circularity. The bound in Definition 3.6.18 is a bound on the unit ball. A nonzero linear map takes arbitrarily large values because $\|T(t\mathbf{v})\| = |t|\,\|T\mathbf{v}\|$.

Corollary 3.6.22 (Linear maps on finite-dimensional spaces)**.** *If V is finite-dimensional, every linear map from V to a normed space W is bounded and hence continuous, for every norm on V.*

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis and Φ the coordinate map of the proof of Theorem 3.6.15. For $\mathbf{v} = \sum x_i\mathbf{v}_i$, (N3) and (N2) give $\|T\mathbf{v}\| \le \sum|x_i|\,\|T\mathbf{v}_i\| \le K\|\Phi\mathbf{v}\|_\infty$ with $K = \sum\|T\mathbf{v}_i\|$. The function $\mathbf{v} \mapsto \|\Phi\mathbf{v}\|_\infty$ is a norm on V. By Theorem 3.6.15 some L has $\|\Phi\mathbf{v}\|_\infty \le L\|\mathbf{v}\|$. Hence $\|T\mathbf{v}\| \le KL\|\mathbf{v}\|$. Continuity follows by Theorem 3.6.21. ∎

3.6.23 Operator norms of evaluation and differentiation

Let p be a point of $[a, b]$. The evaluation $E_p f = f(p)$ of Definition 3.4.10, restricted to $C([a, b])$ with the sup norm, satisfies $|E_p f| \le \|f\|_\infty$. It is bounded with $\|E_p\| \le 1$. The constant function 1 has $\|1\|_\infty = 1$ and $E_p 1 = 1$. Hence $\|E_p\| = 1$ for every point p, the endpoint a included. By Theorem 3.6.21 the value at p depends continuously on f: two functions within ε of each other in the sup norm have values within ε at p.

The choice of space matters. On the space $\mathcal{P}$ of all polynomials with the sup norm on $[0, 1]$, evaluation at the point 2 outside $[0, 1]$ is unbounded. The polynomials x^n have $\|x^n\|_\infty = 1$ and the values 2^n at 2. No M exceeds every 2^n. By Theorem 3.6.21 the functional $p \mapsto p(2)$ is not continuous on this normed space.

Differentiation of polynomials (Section 3.3) gives the first example of an unbounded operator. Write $Tp = p'$ and give every polynomial space the sup norm on $[0, 1]$. On $\mathcal{P}_n$ the map T is bounded by Corollary 3.6.22. On all of $\mathcal{P}$ it is not. For $n \ge 1$ the polynomial x^n has $\|x^n\|_\infty = 1$ and $T(x^n) = nx^{n-1}$ has sup norm n, attained at $x = 1$. A bound $\|Tp\|_\infty \le M\|p\|_\infty$ would give $n \le M$ for every n, against Theorem 2.4.10. The polynomial x^n/n lies within $1/n$ of 0 in the sup norm and its derivative x^{n-1} has sup norm 1. Thus a polynomial uniformly close to 0 can have a derivative far from 0. By Theorem 3.6.21 the map $p \mapsto p'$ is discontinuous at 0. Continuity at a polynomial p_0 would give continuity at 0 because $(p + p_0)' - p_0' = p'$ and $\|(p + p_0) - p_0\|_\infty = \|p\|_\infty$. Thus the map is continuous at no point of $\mathcal{P}$. Chapter 7 meets the same phenomenon for differentiation on continuously differentiable functions and removes it by a change of norm on the domain.

Worked Example 3.6.1

Let $c = s = 1/\sqrt{2}$. Find the operator norm of the rotation $R_{c,s}$ on $\mathbb{R}^2$ with the 2-norm and with the ∞-norm.

In the 2-norm, $\|R_{c,s}\mathbf{v}\|_2 = \|\mathbf{v}\|_2$ by Theorem 3.5.10. Hence $\|R_{c,s}\| = 1$. In the ∞-norm, $R_{c,s}(x, y) = \frac{1}{\sqrt{2}}(x - y,\ x + y)$. Each component has absolute value at most $\frac{1}{\sqrt{2}}(|x| + |y|) \le \sqrt{2}\,\|(x, y)\|_\infty$. Hence $\|R_{c,s}\| \le \sqrt{2}$. The vector $(1, -1)$ has ∞-norm 1 and image $\frac{1}{\sqrt{2}}(2, 0) = (\sqrt{2}, 0)$ of ∞-norm $\sqrt{2}$. The operator norm is $\sqrt{2}$. The rotation turns the unit square of Fig. 3.6.1 through an eighth of a turn and moves its corners outside it. One linear map has different operator norms for different norms on its spaces.

Worked Example 3.6.2

Give $\mathcal{P}_2$ the sup norm on $[0, 1]$. Prove that the map $Tp = p'$ from $\mathcal{P}_2$ to $\mathcal{P}_1$ has operator norm 8.

A polynomial $\ell \in \mathcal{P}_1$ satisfies $\ell(x) = (1 - x)\,\ell(0) + x\,\ell(1)$ because both sides have degree at most 1 and agree at 0 and 1 (Corollary 4.2.8). For $0 \le x \le 1$ the weights $1 - x$ and x are nonnegative with sum 1. Hence $|\ell(x)| \le \max\{|\ell(0)|, |\ell(1)|\}$ and $\|\ell\|_\infty = \max\{|\ell(0)|, |\ell(1)|\}$. For $p = a_0 + a_1 x + a_2 x^2$ the values $p(0) = a_0$, $p(\frac{1}{2}) = a_0 + \frac{1}{2}a_1 + \frac{1}{4}a_2$ and $p(1) = a_0 + a_1 + a_2$ give

$$p'(0) = a_1 = -3p(0) + 4p(\tfrac{1}{2}) - p(1), \qquad p'(1) = a_1 + 2a_2 = p(0) - 4p(\tfrac{1}{2}) + 3p(1).$$

Each value of p has absolute value at most $\|p\|_\infty$ and $3 + 4 + 1 = 8$. Hence $\|p'\|_\infty = \max\{|p'(0)|, |p'(1)|\} \le 8\|p\|_\infty$ and $\|T\| \le 8$.

The polynomial $q(x) = 8x^2 - 8x + 1 = 1 - 8x(1 - x)$ attains the bound. On $[0, 1]$ the product $x(1 - x)$ lies in $[0, \frac{1}{4}]$. Hence $-1 \le q \le 1$. With $q(0) = 1$ this gives $\|q\|_\infty = 1$. Its derivative $q'(x) = 16x - 8$ has $q'(1) = 8$ and $\|q'\|_\infty = 8$. Hence $\|T\| = 8$. A theorem of A. A. Markov, not proved in this book, gives the operator norm $2n^2$ on $\mathcal{P}_n$. The cases $n = 1$ (Problem 3.6.8) and $n = 2$ agree with it.

Common misconception

The only linear map with bounded values is the zero map. Boundedness of a linear map is a bound on the unit ball, $\|T\mathbf{v}\| \le M$ for $\|\mathbf{v}\| \le 1$. The operator norm also depends on the norms of both spaces, as Worked Example 3.6.1 shows. A statement such as "this map has norm 1" is incomplete until the norms are named.

Study guide

Check the three norm axioms one at a time. Refute a candidate by one vector or one pair of vectors. Learn the three norms on $\mathbb{R}^n$ together with their unit balls and the chain $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$. The sup norm is the largest absolute value of a function. To compute an operator norm, prove an upper bound $\|T\mathbf{v}\| \le M\|\mathbf{v}\|$ and then find one vector that attains it. To show that a map is unbounded, find vectors of norm 1 with arbitrarily large images. In finite dimensions every norm is equivalent to every other and every linear map is continuous. In spaces of functions neither statement is automatic.

Problems

3.6.1 For $\mathbf{x} = (1, -2, 2)$ compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_\infty$, and check every inequality of Proposition 3.6.7 with $n = 3$. ⋆ Hint · Answer · Solution

3.6.2 Find the sup norm of $f(x) = x^2 - 2x$ on the interval $[-1, 2]$. ⋆ Hint · Answer · Solution

3.6.3 Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$. Prove that the operator norm of $\mathbf{x} \mapsto A\mathbf{x}$ is 4 when $\mathbb{R}^2$ has the ∞-norm and 3 when it has the 1-norm. ⋆⋆ Hint · Answer · Solution

3.6.4 Let $A = (a_{ij})$ be an $n \times n$ matrix. Prove that the operator norm of $\mathbf{x} \mapsto A\mathbf{x}$ on $(\mathbb{R}^n, \|\cdot\|_\infty)$ is the largest row sum $\max_i \sum_j |a_{ij}|$. ⋆⋆ Hint · Answer · Solution

3.6.5 Prove that $\|(x, y)\| = \max\{|x|, |y|, |x + y|\}$ is a norm on $\mathbb{R}^2$ and find the vertices of its unit ball. ⋆⋆ Hint · Answer · Solution

3.6.6 The parallelogram law $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ is true for the 2-norm (Problem 3.5.5). Show that it fails for the 1-norm and the ∞-norm on $\mathbb{R}^2$, and for the sup norm on $C([0, 1])$. Call a function $\langle \cdot, \cdot \rangle$ from pairs of vectors to $\mathbb{R}$ a *product* if it has properties (1)–(3) of Proposition 3.5.3: it is symmetric, linear in the first slot and positive at every nonzero vector. Conclude that none of these norms has the form $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ for such a product. ⋆⋆ Hint · Answer · Solution

3.6.7 On $C([0, 1])$ with the sup norm let $\lambda(f) = f(0) - 2f(1)$. Prove that λ is bounded and that $\|\lambda\| = 3$. ⋆⋆ Hint · Answer · Solution

3.6.8 On $\mathcal{P}_1$ compare the sup norm on $[0, 1]$ with the coefficient norm $\|a + bx\|_c = |a| + |b|$ of the caution after Theorem 3.6.15. Prove that $\|p\|_\infty \le \|p\|_c \le 3\|p\|_\infty$ and that the constant 3 cannot be lowered. Then give $\mathcal{P}_0$ the sup norm on $[0, 1]$ and find the operator norm of $Tp = p'$ from $\mathcal{P}_1$ to $\mathcal{P}_0$. ⋆⋆ Hint · Answer · Solution

3.6.9 Let $c = \frac{3}{5}$ and $s = \frac{4}{5}$. Find the operator norm of $R_{c,s}$ on $(\mathbb{R}^2, \|\cdot\|_\infty)$. ⋆⋆ Hint · Answer · Solution

3.6.10 Which of the following are norms on $C([0, 1])$? (a) $f \mapsto |f(0)| + \|f\|_\infty$; (b) $f \mapsto \max\{|f(0)|, |f(1)|\}$; (c) $f \mapsto \|f\|_\infty^2$; (d) $f \mapsto 2\|f\|_\infty$. ⋆⋆ Hint · Answer · Solution

3.6.11 Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms on V and let W be a normed space. Prove that a linear map $T\colon V \to W$ is bounded for $\|\cdot\|$ exactly when it is bounded for $\|\cdot\|'$. Then show that the identity map of $\mathcal{P}$ from the sup norm on $[0, 1]$ to the coefficient norm of the caution after Theorem 3.6.15 is unbounded. ⋆⋆ Hint · Answer · Solution

3.6.12 Give $\mathcal{P}$ the sup norm on $[0, 1]$ and let $\lambda(p) = p'(0)$. Prove that λ is unbounded. Then prove that every polynomial q lies within any given $\varepsilon > 0$ of some polynomial in $\ker \lambda$. The kernel of a discontinuous linear functional can come arbitrarily close to every vector. ⋆⋆⋆ Hint · Answer · Solution

Chapter 4

Elementary Functions and Algebra

Almost every function that calculus studies is built from a short list: polynomials, quotients of polynomials, powers and roots, exponentials and logarithms, and the trigonometric functions with their inverses. Chapter 1 defined a function as a rule with a stated domain and codomain (Definition 1.5.8) and Chapter 2 set out the real numbers on which these functions act. This chapter constructs the elementary functions from those two foundations and records the algebra that manipulates them.

Three of the constructions need more than algebra. A real power a^x with an irrational exponent has no meaning as repeated multiplication and is defined here as a supremum, by the completeness of $\mathbb{R}$ (Assumption 2.4.5). The number e is defined as a limit. The logarithm and the inverse trigonometric functions are inverses whose existence on the full expected range rests on the continuity theorems of Chapter 5. Radian measure is an arc length, and arc length is an integral defined in Chapter 8. Each such dependence is stated where it arises, together with the place where it is discharged. None of them is circular.

The chapter does not differentiate or integrate anything. Derivatives of the elementary functions are computed in Chapter 6 and their integrals in Chapter 8. Remarks headed *Later use* point forward to the places where each algebraic fact returns.

4.1 Functions and Their Graphs

Overview

Objectives: Find the natural domain of a function given by a formula; read and sketch a graph; predict the effect of a shift, a stretch or a reflection on a graph; decide whether a function is even, odd or neither; work with piecewise definitions; recognise a strictly monotone function as injective.
Prerequisites: Functions, injectivity and inverses (Definitions 1.5.8, 1.5.13 and 1.5.24); intervals and absolute value (Definition 2.1.8 and Section 2.3).
Key results: The transformation rules for graphs (Theorem 4.1.8); the even-odd decomposition (Theorem 4.1.13); a strictly monotone function is injective and has a strictly monotone inverse (Theorem 4.1.19).

Notation

$\operatorname{dom} f$	the domain of the function f
$\Gamma(f)$	the graph $\{(x, f(x)) : x \in \operatorname{dom} f\}$ of f
$\lfloor x \rfloor$	the floor of x, the largest integer not exceeding x

A formula such as $f(x) = \sqrt{4 - x^2}$ looks like a complete description of a function. Definition 1.5.8 requires a domain as well as a rule. The formula states only the rule. Calculus builds

new functions from old ones by shifting, scaling, adding and composing them. Each operation needs the domain and the graph of the result. This section fixes the conventions for both.

4.1.1 Natural domain

Definition 4.1.2 (Natural domain)**.** When a real function is given by a formula and no domain is stated, its domain is the *natural domain*: the set of all real x for which every operation in the formula is defined. The codomain is $\mathbb{R}$.

Three operations restrict the natural domain in this chapter. Division excludes the zeros of every denominator. An even root requires a nonnegative radicand. A logarithm (Section 4.4) requires a positive argument.

Example 4.1.3. The natural domain of $f(x) = \sqrt{4 - x^2}$ is the set where $4 - x^2 \geq 0$, the closed interval $[-2, 2]$. As a non-example of equality of functions, $g(x) = x$ and $h(x) = x^2/x$ agree at every $x \neq 0$ but are different functions: the natural domain of h is $\mathbb{R} \setminus \{0\}$ and that of g is $\mathbb{R}$.

Common misconception

Simplifying a formula can enlarge its natural domain. Cancelling x in x^2/x produces the function x, defined at 0 where the original quotient is not. A simplification is an identity only on the domain of the original formula. Record that domain first.

4.1.4 Graphs

Definition 4.1.5 (Graph)**.** The *graph* of a real function f is the set $\Gamma(f) = \{(x, f(x)) : x \in \operatorname{dom} f\}$ of points of the plane.

Because a function assigns exactly one output to each input, every vertical line $x = c$ meets the graph of a function at most once, at $(c, f(c))$ when $c \in \operatorname{dom} f$ and nowhere otherwise. A set of points meeting some vertical line twice is therefore not the graph of any function.

Example 4.1.6. The parabola $y = x^2$ is the graph of $f(x) = x^2$ on $\mathbb{R}$. As a non-example, the unit circle $x^2 + y^2 = 1$ is the graph of no function of x: the line $x = 0$ meets it at $(0, 1)$ and at $(0, -1)$. Its upper half is the graph of $x \mapsto \sqrt{1 - x^2}$ on $[-1, 1]$.

4.1.7 Transformations

Shifting, stretching and reflecting a graph correspond to simple changes in the formula. Theorem 4.1.8 states the correspondence and Fig. 4.1.1 shows it for the parabola.

Theorem 4.1.8 (Transformations of a graph)**.** *Let f have domain D and let $h, k \in \mathbb{R}$ and $c, b \neq 0$.*

1. *The graph of $g(x) = f(x - h) + k$, with domain $\{x : x - h \in D\}$, is the graph of f translated by h horizontally and by k vertically.*
2. *The graph of $g(x) = c\, f(x)$ is the graph of f with every y-coordinate multiplied by c. For $c = -1$ it is the reflection of the graph of f in the x-axis.*
3. *The graph of $g(x) = f(bx)$, with domain $\{x : bx \in D\}$, is the graph of f with every x-coordinate multiplied by $1/b$. For $b = -1$ it is the reflection of the graph of f in the y-axis.*

Proof. For the first statement, a point (x, y) lies on $\Gamma(g)$ exactly when $x - h \in D$ and $y - k = f(x - h)$, that is, exactly when $(x - h, y - k)$ lies on $\Gamma(f)$. Hence $\Gamma(g)$ consists of the points $(u+h, v+k)$ with $(u, v) \in \Gamma(f)$. For the second, $(x, y) \in \Gamma(g)$ exactly when $x \in D$ and $y/c = f(x)$.

Hence $\Gamma(g)$ consists of the points (u, cv) with $(u, v) \in \Gamma(f)$. For the third, $(x, y) \in \Gamma(g)$ exactly when $bx \in D$ and $y = f(bx)$, that is, when $(bx, y) \in \Gamma(f)$. Hence $\Gamma(g)$ consists of the points $(u/b, v)$ with $(u, v) \in \Gamma(f)$. ∎

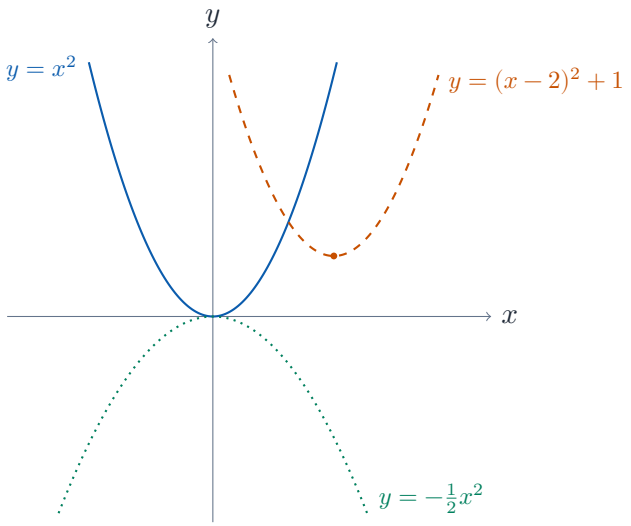


Figure 4.1.1: Parabola $y = x^2$ with two transforms. Translation by 2 to the right and 1 upwards moves the vertex to $(2, 1)$. Multiplication of every y-coordinate by $-\frac{1}{2}$ flattens the parabola and reflects it in the x-axis.

Example 4.1.9. By Theorem 4.1.8 the graph of $y = (x-2)^2+1$ is the parabola $y = x^2$ moved 2 units right and 1 unit up, with vertex $(2, 1)$. As a non-example of a translation, $y = x^2+2x$ is not obtained from $y = x^2$ by a vertical shift alone. Completing the square gives $x^2+2x = (x+1)^2-1$, a shift left by 1 and down by 1.

Common misconception

In $y = f(2x-6)$ the horizontal shift is 3, not 6. Factor the coefficient of x first: $f(2x-6) = f(2(x-3))$ is f compressed by the factor 2 and then translated 3 units to the right. A horizontal shift by h replaces x by $x-h$ wherever x occurs, with the sign opposite to the one that appears in the formula.

4.1.10 Even and odd functions

Definition 4.1.11 (Even and odd functions)**.** Let f have a domain D symmetric about 0: $-x \in D$ whenever $x \in D$. Then f is *even* if $f(-x) = f(x)$ for every $x \in D$ and *odd* if $f(-x) = -f(x)$ for every $x \in D$.

The graph of an even function is symmetric under reflection in the y-axis, and the graph of an odd function is symmetric under the half-turn about the origin. An odd function defined at 0 has $f(0) = -f(0)$ and therefore $f(0) = 0$.

Example 4.1.12. The functions x^2, x^4 and $|x|$ are even and x, x^3 and $1/x$ are odd. As a non-example, $f(x) = x^2 + x$ is neither: $f(1) = 2$ and $f(-1) = 0$. The value $f(-1)$ equals neither $f(1)$ nor $-f(1)$.

Theorem 4.1.13 (Even-odd decomposition)**.** *Every function f on a domain symmetric about 0 is the sum $f = E + O$ of an even function E and an odd function O in exactly one way, namely*

$$E(x) = \frac{f(x) + f(-x)}{2}, \qquad O(x) = \frac{f(x) - f(-x)}{2}. \tag{4.1.1}$$

Proof. The functions E and O of (4.1.1) add to f. Replacing x by $-x$ leaves E unchanged and changes the sign of O. Hence E is even and O is odd. For uniqueness suppose $f = E_1 + O_1$ is a second such decomposition. Then $E - E_1 = O_1 - O$ is both even and odd. A function g with $g(-x) = g(x)$ and $g(-x) = -g(x)$ satisfies $g(x) = -g(x)$ and is zero. Hence $E_1 = E$ and $O_1 = O$. ∎

Later use

On a symmetric interval $[-a, a]$ the integral of an odd function is zero and the integral of an even function is twice its integral over $[0, a]$ (Corollary 8.5.4). Splitting an integrand into its even and odd parts by Theorem 4.1.13 reduces a symmetric integral to its even part.

4.1.14 Piecewise functions

A function may be defined by different formulas on different parts of its domain. The absolute value is the first example:

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases} \tag{4.1.2}$$

The *floor function* assigns to x the largest integer n with $n \leq x$. By Lemma 2.4.11 exactly one integer $\lfloor x \rfloor$ satisfies $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, namely the largest integer not exceeding x.

Example 4.1.15. The function $f(x) = x^2$ for $x < 0$ and $f(x) = x + 1$ for $x \geq 0$ is piecewise defined, with $f(-2) = 4$ and $f(0) = 1$. A formula with cases is still one function. As a non-example, the cases $x \leq 1$ with value 0 and $x \geq 1$ with value 1 do not define a function: both apply at $x = 1$ and assign it two different outputs.

4.1.16 Monotone functions

Definition 4.1.17 (Monotone function)**.** Let f be defined on a set $S \subseteq \mathbb{R}$. Then f is *increasing* on S if $x < y$ in S implies $f(x) \leq f(y)$, and *strictly increasing* if $x < y$ implies $f(x) < f(y)$. Reversing the inequalities between the values defines *decreasing* and *strictly decreasing*. A function with any of these four properties is *monotone*.

Example 4.1.18. The function x^3 is strictly increasing on $\mathbb{R}$, because $y^3 - x^3 = (y - x)(x^2 + xy + y^2)$ and the second factor equals $(x + \frac{y}{2})^2 + \frac{3}{4}y^2 > 0$ unless $x = y = 0$. As a non-example, x^2 is neither increasing nor decreasing on $\mathbb{R}$: from $-1 < 0$ it gives $1 > 0$ and from $0 < 1$ it gives $0 < 1$. It is strictly increasing on $[0, \infty)$.

Theorem 4.1.19 (Strictly monotone functions are injective)**.** *A strictly increasing function $f : S \to \mathbb{R}$ is injective with a strictly increasing inverse $f^{-1} : f(S) \to S$. The same holds with "decreasing" in both places.*

Proof. Injectivity is Proposition 1.5.18. As a map onto its image, f has an inverse f^{-1} (Definition 1.5.24). Let $u < v$ in $f(S)$ and put $x = f^{-1}(u)$, $y = f^{-1}(v)$. If $y \leq x$, then $v = f(y) \leq f(x) = u$ by the monotonicity of f, contradicting $u < v$. Hence $x < y$. The decreasing case is the same with the inequalities between values reversed. ∎

Later use

Theorem 4.1.19 is the source of every inverse function in this chapter: the nth root, the logarithm and the three inverse trigonometric functions are each the inverse of a strictly monotone function restricted to a suitable interval. Chapter 6 differentiates such inverses (Theorem 6.3.9).

Study guide

Record the domain before simplifying a formula because cancellation can enlarge it. To transform a graph, rewrite the formula as $c\,f(b(x-h))+k$ and apply Theorem 4.1.8 from the inside outwards: scale horizontally by $1/b$, shift by h, scale vertically by c and shift by k. Test parity by substituting $-x$ and comparing with $f(x)$ and $-f(x)$. A function that is strictly monotone on an interval has an inverse on that interval that is monotone in the same sense.

Worked Example 4.1.1

Find the natural domain of $f(x)=\dfrac{\sqrt{x+3}}{x^2-4}$.
The square root requires $x+3\geq 0$, that is, $x\geq -3$. The denominator vanishes at $x=\pm 2$. Both points satisfy $x\geq -3$ and are therefore excluded. The natural domain is

$$[-3,-2)\cup(-2,2)\cup(2,\infty).$$

Worked Example 4.1.2

Describe the graph of $g(x)=3-2(x+1)^2$ as a transform of $y=x^2$ and find the range of g.
Write $g(x)=-2\,f(x-(-1))+3$ with $f(x)=x^2$. By Theorem 4.1.8 the graph of g is the parabola $y=x^2$ translated 1 unit to the left, stretched vertically by the factor 2 with a reflection in the x-axis, and translated 3 units up. The vertex moves from $(0,0)$ to $(-1,3)$ and the parabola opens downwards. Since $(x+1)^2\geq 0$ with equality at $x=-1$, the values satisfy $g(x)\leq 3$. Every $y\leq 3$ is attained at $x=-1+\sqrt{(3-y)/2}$. Hence the range is $(-\infty,3]$.

Problems

4.1.1 Find the natural domain of $f(x)=\dfrac{1}{\sqrt{x^2-9}}$. ⋆ Hint · Answer · Solution

4.1.2 Decide whether each function is even, odd or neither: $f(x)=\dfrac{x^3}{1+x^2}$, $g(x)=x^2+x$, $h(x)=|x|-x^4$. ⋆ Hint · Answer · Solution

4.1.3 Prove that the product of two odd functions is even and that the product of an even function and an odd function is odd. ⋆⋆ Hint · Answer · Solution

4.1.4 The function f has domain $[0,4]$ and its graph passes through $(2,7)$. Let $g(x)=f(2x-6)+1$. Find the domain of g and the point of the graph of g that corresponds to $(2,7)$. ⋆⋆ Hint · Answer · Solution

4.1.5 Write $f(x) = |x-1| + |x+1|$ without absolute values, as a piecewise function on the three intervals cut out by $x = -1$ and $x = 1$. ⋆⋆ Hint · Answer · Solution

4.1.6 Let $f(x) = \dfrac{2x+1}{x-3}$ on its natural domain. Show that f is injective, find its range and find f^{-1}. ⋆⋆ Hint · Answer · Solution

4.1.7 Prove that the composition of two increasing functions is increasing and that the composition of two decreasing functions is increasing. ⋆⋆ Hint · Answer · Solution

4.2 Polynomials and Rational Functions

Overview

Objectives: Divide one polynomial by another with remainder; use the remainder and factor theorems to locate roots and factors; list the candidate rational roots of an integer polynomial; factor a polynomial and complete the square; add, simplify and long-divide rational expressions; split a quotient with two distinct linear factors into partial fractions.
Prerequisites: Mathematical induction (Theorem 1.4.2); functions and their graphs (Section 4.1).
Key results: The division algorithm (Theorem 4.2.4); the remainder and factor theorems (Theorems 4.2.5 and 4.2.6); at most n roots for a polynomial of degree n (Corollary 4.2.7); the rational root theorem (Theorem 4.2.11); the quadratic formula (Theorem 4.2.15); two-factor partial fractions (Proposition 4.2.20).

Notation

$\deg p$ the degree of a nonzero polynomial p

Polynomials are the functions that addition and multiplication alone produce from x and the constants. Their roots, factors and quotients govern the behaviour of rational functions and the way a rational function is later integrated. This section develops the algebra of polynomials from the division algorithm.

4.2.1 Polynomials and the division algorithm

Definition 4.2.2 (Polynomial)**.** A *polynomial* is a function of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real coefficients $a_0, \dots, a_n$. If $a_n \neq 0$, the integer n is the *degree* $\deg p$ and a_n is the *leading coefficient*. The zero polynomial has no degree. A *root* of p is a real c with $p(c) = 0$.

Example 4.2.3. $p(x) = 3x^4 - x + 7$ is a polynomial of degree 4 with leading coefficient 3. Every constant $c \neq 0$ is a polynomial of degree 0. As a non-example, $1/x$ is not a polynomial because every polynomial is defined on all of $\mathbb{R}$ and $1/x$ is undefined at 0.

The degree of a product is the sum of the degrees because the product of the leading terms $a_n x^n$ and $b_m x^m$ is $a_n b_m x^{n+m}$ with $a_n b_m \neq 0$ while every other product of terms has lower degree. A sum is either the zero polynomial or of degree at most the larger of the two degrees.

Theorem 4.2.4 (Division algorithm)**.** *Let f and g be polynomials with g not the zero polynomial. There are unique polynomials q and r such that*

$$f = q\,g + r, \qquad \text{with } r = 0 \text{ or } \deg r < \deg g. \tag{4.2.1}$$

The polynomial q is the quotient *and r the* remainder.

Proof. Let $m = \deg g$ and let b be the leading coefficient of g. For existence we prove by induction on n (Theorem 1.4.2) the statement $P(n)$: every polynomial f that is zero or has degree at most n can be written as $qg + r$ with $r = 0$ or $\deg r < m$. When $n < m$ this is immediate with $q = 0$ and $r = f$. Suppose $P(n-1)$ with $n \geq m$ and let f have degree n with leading coefficient a. The polynomial

$$f_1 = f - \frac{a}{b}\,x^{n-m} g$$

has the leading term of f cancelled. Hence $f_1 = 0$ or $\deg f_1 \leq n-1$. By $P(n-1)$, $f_1 = q_1 g + r$ with $r = 0$ or $\deg r < m$. Then $f = (q_1 + \frac{a}{b}x^{n-m})g + r$. This representation proves $P(n)$.

For uniqueness suppose $qg + r = q'g + r'$ with both remainders zero or of degree less than m. Then $(q - q')g = r' - r$. If $q \neq q'$ the left side has degree at least m while the right side is zero or has degree less than m, a contradiction. Hence $q = q'$ and then $r = r'$. ∎

The proof is the procedure of polynomial long division: at each step the leading term of what remains is divided by the leading term of g, the multiple of g is subtracted and the process repeats until the degree of what remains falls below $\deg g$.

Theorem 4.2.5 (Remainder theorem)**.** *The remainder on dividing a polynomial f by $x - c$ is the constant $f(c)$.*

Proof. By Theorem 4.2.4, $f(x) = q(x)(x-c)+r$ with r zero or of degree 0. Hence r is a constant. Setting $x = c$ gives $f(c) = r$. ∎

Theorem 4.2.6 (Factor theorem)**.** *A polynomial f has $x - c$ as a factor, that is, $f(x) = (x - c)q(x)$ for a polynomial q, exactly when $f(c) = 0$.*

Proof. If $f(x) = (x - c)q(x)$, then $f(c) = 0$. Conversely, if $f(c) = 0$, the remainder in Theorem 4.2.5 is 0 and $f(x) = q(x)(x - c)$. ∎

Corollary 4.2.7 (Number of roots of a polynomial)**.** *A polynomial of degree $n \geq 0$ has at most n distinct roots.*

Proof. By induction on n. A polynomial of degree 0 is a nonzero constant and has no roots. Let f have degree $n \geq 1$ and suppose the statement for degree $n - 1$. If f has no root there is nothing to prove. Otherwise let c be a root. By Theorem 4.2.6, $f(x) = (x - c)q(x)$ with $\deg q = n - 1$. Any root $d \neq c$ of f satisfies $(d - c)q(d) = 0$ with $d - c \neq 0$ and hence $q(d) = 0$. By the induction hypothesis q has at most $n - 1$ roots. The roots of f are therefore c and at most $n - 1$ others. ∎

Corollary 4.2.8 (Identity of polynomials)**.** *If two polynomials f and g, each zero or of degree at most n, agree at $n + 1$ distinct points, then $f = g$ and their coefficients coincide.*

Proof. The difference $f - g$ is zero or has degree at most n and has $n + 1$ distinct roots. By Corollary 4.2.7 it has no degree and is therefore the zero polynomial. ∎

Since the proofs of Theorem 4.2.4 and Corollaries 4.2.7 and 4.2.8 use only a coefficient list $a_0, \ldots, a_n$ with $a_n \neq 0$, Corollary 4.2.8 applied to two such lists for one function shows that a polynomial function determines its coefficients and with them its degree.

Lemma 4.2.9 (Difference of powers)**.** *For all real a, b and every integer $n \geq 1$,*

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}). \tag{4.2.2}$$

Proof. Multiplying out, a times the bracket is $a^n + a^{n-1}b + \cdots + ab^{n-1}$ and b times the bracket is $a^{n-1}b + \cdots + ab^{n-1} + b^n$. Every term except a^n and b^n occurs in both products and cancels in their difference $a^n - b^n$. ∎

4.2.10 Rational roots and factoring

Theorem 4.2.11 (Rational root theorem)**.** *Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ have integer coefficients with $a_n \neq 0$ and $a_0 \neq 0$. If a fraction r/s in lowest terms is a root of p, then r divides a_0 and s divides a_n.*

Proof. Multiplying $p(r/s) = 0$ by s^n gives

$$a_n r^n + a_{n-1} r^{n-1} s + \cdots + a_1 r s^{n-1} + a_0 s^n = 0. \tag{4.2.3}$$

Every term of (4.2.3) except the last has the factor r. Hence r divides $a_0 s^n$ by Theorem 1.3.5. The integers r and s^n are coprime by Corollary 1.4.16 because r and s are (Definition 1.4.13). Euclid's lemma (Theorem 1.4.15) then gives $r \mid a_0$. Likewise s divides every term except the first and therefore divides $a_n r^n$. Since s and r^n are coprime by Corollary 1.4.16, Euclid's lemma gives $s \mid a_n$. ∎

Example 4.2.12. The candidate rational roots of $2x^3 + x^2 - 2x - 1$ are ± 1 and $\pm\frac{1}{2}$. Testing them gives roots 1, -1 and $-\frac{1}{2}$. As a non-example of the theorem's reach, $x^2 - 2$ has candidates ± 1 and ± 2. No candidate is a root. Since every rational number can be written in lowest terms (Corollary 1.4.17), $x^2 - 2$ has no rational root. Its real roots $\pm\sqrt{2}$ exist and are irrational (Theorem 2.1.3).

Factoring is the first step of partial fractions. Try the following moves in order, using the special products collected in Section 4.3:

1. **Greatest common factor.** $6x^3 - 9x = 3x(2x^2 - 3)$.
2. **Difference of squares**, from Eq. (4.3.4): $x^2 - 9 = (x-3)(x+3)$ and $x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$.
3. **Trinomials** $x^2 + bx + c = (x+p)(x+q)$ with $p+q = b$ and $pq = c$: $x^2 - 5x + 6 = (x-2)(x-3)$.
4. **Sum or difference of cubes**: $x^3 - 8 = (x-2)(x^2+2x+4)$.
5. **Grouping** (four terms): $x^3 + x^2 + x + 1 = x^2(x+1) + (x+1) = (x+1)(x^2+1)$.
6. **Rational roots** (Theorem 4.2.11) followed by division by the corresponding linear factor (Theorem 4.2.6).

4.2.13 Quadratics and completing the square

To complete the square in $x^2 + bx$, add and subtract the square of half the linear coefficient:

$$x^2 + bx = \left(x + \tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2. \tag{4.2.4}$$

For a leading coefficient $a \neq 1$, factor it out of the first two terms first.

Example 4.2.14. $x^2 + 6x + 13 = (x+3)^2 + 4$ and $2x^2 - 4x + 7 = 2(x^2 - 2x) + 7 = 2\big[(x-1)^2 - 1\big] + 7 = 2(x-1)^2 + 5$.

Theorem 4.2.15 (Quadratic formula)**.** *Let $a \neq 0$ and let $\Delta = b^2 - 4ac$ be the* discriminant *of $ax^2 + bx + c$. If $\Delta \geq 0$, the real roots are*

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \tag{4.2.5}$$

and $ax^2+bx+c = a(x-r_1)(x-r_2)$ *with* r_1, r_2 *these roots. If* $\Delta < 0$*, there is no real root and* ax^2+bx+c *has no factor of degree* 1 *with real coefficients.*

Proof. By (4.2.4),

$$ax^2+bx+c = a\Big[\Big(x+\frac{b}{2a}\Big)^2 - \frac{\Delta}{4a^2}\Big]. \tag{4.2.6}$$

If $\Delta < 0$ the bracket is at least $-\Delta/(4a^2) > 0$. Hence there is no real root and, by Theorem 4.2.6, no factor $x-c$. If $\Delta \geq 0$, the bracket is a difference of squares and factors by Eq. (4.3.4) as $(x+\frac{b}{2a}-\frac{\sqrt{\Delta}}{2a})(x+\frac{b}{2a}+\frac{\sqrt{\Delta}}{2a}) = (x-r_1)(x-r_2)$. ∎

A quadratic with negative discriminant is called *irreducible*.

Later use

A denominator that will *not* factor over the reals (negative discriminant) is handled by completing the square and then a substitution into an arctangent (Proposition 9.1.7) or a trigonometric substitution (Eq. (9.4.4)):

$$\int \frac{\mathrm{d}x}{x^2+6x+13} = \int \frac{\mathrm{d}x}{(x+3)^2+4} = \tfrac{1}{2}\arctan\Big(\tfrac{x+3}{2}\Big) + C.$$

4.2.16 Rational functions and long division

Definition 4.2.17 (Rational function)**.** A *rational function* is a quotient p/q of polynomials with q not the zero polynomial, on its natural domain $\{x : q(x) \neq 0\}$. It is *proper* if $p = 0$ or $\deg p < \deg q$, and *improper* otherwise.

Example 4.2.18. $\dfrac{x+1}{x^2-4}$ is a proper rational function with domain $\mathbb{R}\setminus\{\pm 2\}$. The function $\dfrac{x^2}{x+1}$ is improper. As a non-example, $\dfrac{\sqrt{x}}{x+1}$ is not a rational function because its numerator is not a polynomial.

Add rational expressions over a common denominator and cancel only *common factors*:

$$\frac{1}{x-1}+\frac{1}{x+1} = \frac{(x+1)+(x-1)}{(x-1)(x+1)} = \frac{2x}{x^2-1}.$$

Common misconception

You may cancel a *factor* shared by numerator and denominator, as in $\dfrac{(x-2)(x+3)}{(x-2)} = x+3$ for $x \neq 2$. You may *not* cancel a term: $\dfrac{x+3}{x} \neq 3$. The test is whether the thing you are cancelling multiplies the *whole* numerator and the *whole* denominator.

For an improper rational function, Theorem 4.2.4 writes $p = Qq + R$ and therefore

$$\frac{p}{q} = Q + \frac{R}{q},$$

a polynomial plus a proper rational function. For example

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

For large $|x|$ the proper part $1/(x+1)$ is small and the graph approaches the line $y = x - 1$, as Fig. 4.2.1 shows. The vertical line $x = -1$, where the denominator vanishes, is excluded from the domain.

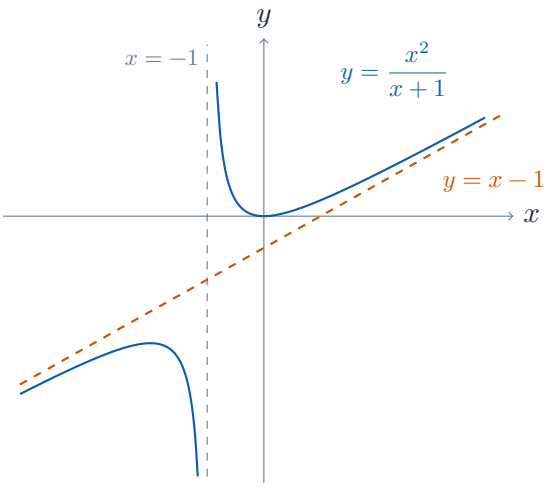


Figure 4.2.1: Graph of $x^2/(x+1)$ beside the line $y = x - 1$ given by long division. The two curves differ by $1/(x+1)$, small for large $|x|$.

Later use

Partial fractions requires a *proper* fraction. Long-divide an improper integrand first (Eq. (9.5.1)). The polynomial part integrates directly and the proper remainder gives the ln and arctan terms. The method has four steps: divide, factor, decompose, integrate.

4.2.19 Partial fractions (preview)

A proper rational function whose denominator is a product of distinct linear factors is a sum of simple fractions, one for each factor. The case of two factors already shows the method.

Proposition 4.2.20 (Partial fractions over two distinct linear factors)**.** *Let $a \neq b$ and let p be zero or a polynomial of degree at most 1. Then for $x \neq a, b$,*

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, \qquad A = \frac{p(a)}{a-b}, \quad B = \frac{p(b)}{b-a}. \tag{4.2.7}$$

Proof. The right side has numerator $A(x-b) + B(x-a)$ over $(x-a)(x-b)$. This numerator is zero or of degree at most 1. At $x = a$ it equals $A(a-b) = p(a)$ and at $x = b$ it equals $B(b-a) = p(b)$. Two polynomials of degree at most 1 that agree at two points are equal by Corollary 4.2.8. Hence the numerators coincide. ∎

The formula for A is the *cover-up rule*: cover the factor $x-a$ in the denominator and evaluate what remains at $x = a$.

Example 4.2.21. $\dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)} = \dfrac{1/2}{x-1} - \dfrac{1/2}{x+1}$, by Proposition 4.2.20 with $p = 1$, $a = 1$ and $b = -1$.

Later use

Partial fractions (Theorem 9.5.2) cannot begin until the step of *factoring the denominator* completely into linear and irreducible-quadratic pieces is done. A difference of squares in

a denominator, $\dfrac{1}{x^2-a^2} = \dfrac{1}{(x-a)(x+a)}$, is the archetype. Matching coefficients in a general decomposition rests on Corollary 4.2.8.

Study guide

To locate the roots of an integer polynomial, list the candidates r/s of Theorem 4.2.11, test them with Theorem 4.2.5, and divide out each root you find with Theorem 4.2.6 until a quadratic remains. Settle the quadratic by its discriminant (Theorem 4.2.15). A rational function is long-divided before anything else when it is improper. Cancel factors, never terms, and record the excluded points of the domain before cancelling. For a denominator with two distinct linear factors use the cover-up rule of Proposition 4.2.20.

Worked Example 4.2.1

Factor $p(x) = 2x^3 - 3x^2 - 11x + 6$ completely.
By Theorem 4.2.11 a rational root r/s has $r \mid 6$ and $s \mid 2$. Testing candidates gives $p(3) = 54 - 27 - 33 + 6 = 0$. Hence $x - 3$ is a factor by Theorem 4.2.6. Division gives $p(x) = (x-3)(2x^2+3x-2)$. The quadratic has discriminant $9 + 16 = 25$ and roots $\frac{-3 \pm 5}{4}$, that is $\frac{1}{2}$ and -2. By Theorem 4.2.15,

$$p(x) = (x-3)\cdot 2\Big(x - \tfrac{1}{2}\Big)(x+2) = (x-3)(2x-1)(x+2).$$

Worked Example 4.2.2

Split $\dfrac{3x+1}{x^2+x-2}$ into partial fractions.
The denominator factors as $(x-1)(x+2)$. By the cover-up rule of Proposition 4.2.20 with $p(x) = 3x+1$, $a = 1$ and $b = -2$,

$$A = \frac{p(1)}{1-(-2)} = \frac{4}{3}, \qquad B = \frac{p(-2)}{-2-1} = \frac{-5}{-3} = \frac{5}{3}, \qquad \frac{3x+1}{x^2+x-2} = \frac{4/3}{x-1} + \frac{5/3}{x+2}.$$

As a check, $\frac{4}{3}(x+2) + \frac{5}{3}(x-1) = \frac{9x+3}{3} = 3x+1$.

Problems

4.2.1 Divide $x^3 - 2x^2 + 4$ by $x - 3$ and confirm the remainder with Theorem 4.2.5.
⋆ Hint · Answer · Solution

4.2.2 Factor $x^3 - 6x^2 + 11x - 6$ completely. ⋆⋆ Hint · Answer · Solution

4.2.3 Complete the square in $3x^2 - 12x + 7$ and state the least value of the expression.
⋆ Hint · Answer · Solution

4.2.4 Find k such that $x - 2$ is a factor of $x^3 + kx^2 - 4x + 4$, and factor the resulting polynomial. ⋆⋆ Hint · Answer · Solution

4.2.5 Split $\dfrac{5}{x^2+x-6}$ into partial fractions. ⋆⋆ Hint · Answer · Solution

4.2.6 Show that $x^3 - 3x + 1$ has no rational root. ⋆⋆ Hint · Answer · Solution

4.2.7 Verify that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$ and show that each quadratic factor is irreducible. ⋆⋆ Hint · Answer · Solution

4.2.8 Use Theorem 4.2.11 to prove that $\sqrt{2}$ is irrational. ⋆⋆⋆ Hint · Answer · Solution

4.3 Algebra Toolkit

Overview

Objectives: Apply the laws of exponents to integer, fractional and negative powers; expand and factor with the special products; rationalise a numerator or denominator; solve equations with radicals and check for extraneous roots; recognise and avoid the common algebraic errors.
Prerequisites: Polynomials and factoring (Section 4.2); inequalities (Section 2.3).
Key results: The special products Eqs. (4.3.3) and (4.3.4); the laws of exponents, proved in Section 4.4; $\sqrt{x^2} = |x|$.

Most difficulties students meet in limits, derivatives, integrals and series are old difficulties in a new setting: a sign slip in a factoring step, a forgotten exponent law or an illegal cancellation. Once the algebra is correct the calculus is usually the easier part. This section collects the algebra that the rest of the book assumes without comment and records for each fact where it returns later.

4.3.1 Laws of exponents

For positive bases x, y and real exponents a, b,

$$\begin{aligned} x^a x^b &= x^{a+b}, \qquad \frac{x^a}{x^b} = x^{a-b}, \quad (x^a)^b = x^{ab}, \\ (xy)^a &= x^a y^a, \quad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, \qquad x^0 = 1. \end{aligned} \tag{4.3.1}$$

For integer exponents the same laws hold for every nonzero real base. Negative and fractional exponents come from definitions:

$$x^{-a} = \frac{1}{x^a}, \qquad x^{1/n} = \sqrt[n]{x}, \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m. \tag{4.3.2}$$

The laws (4.3.1) are proved in Section 4.4: for integer exponents in Theorem 4.4.4, for rational exponents in Theorem 4.4.10 and for real exponents in Theorem 4.4.16.

Example 4.3.2. For $x > 0$, $\dfrac{x^3\sqrt{x}}{x^{-1}} = x^{3+1/2+1} = x^{9/2}$. As a non-example, $(-8)^{1/3}$ and $(-8)^{2/6}$ name the same exponent but the second, read as $\sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$, differs from the real cube root -2. The fractional-exponent laws require a positive base.

Roots and reciprocals as powers

Rules for powers apply visibly only to an expression written as a power. Turn every root and every reciprocal into an exponent before simplifying:

$$\frac{1}{\sqrt{x}} = x^{-1/2}, \qquad \sqrt[3]{x^2} = x^{2/3}, \qquad \frac{1}{x^4} = x^{-4}.$$

Later use

The power rule for derivatives (Chapter 6) and for antiderivatives, $\int x^r \,\mathrm{d}x = \dfrac{x^{r+1}}{r+1} + C$ for $r \neq -1$ (Chapter 8), applies only to a function written as a power: $\int x^{-1/2} \,\mathrm{d}x = 2x^{1/2} + C$ is immediate once $1/\sqrt{x}$ is rewritten. The antiderivative of the one exception x^{-1} is the

natural logarithm $\ln|x|$ (Subsection 4.4.25). Arc length (Eq. (8.6.1), with its $\sqrt{1+(f')^2}$) and surface area require fluency in rewriting radicals in exponent form.

Common misconception

$x^a + x^b$ does *not* simplify to x^{a+b} because the product law applies only to *multiplication*. Likewise $(x+y)^2 \neq x^2+y^2$ and $\sqrt{x^2+y^2} \neq x+y$. Roots and powers do not distribute over sums.

4.3.3 Special products

FOIL (First, Outer, Inner, Last) is the distributive law applied to two binomials:

$$(a+b)(c+d) = \underbrace{ac}_{\text{First}} + \underbrace{ad}_{\text{Outer}} + \underbrace{bc}_{\text{Inner}} + \underbrace{bd}_{\text{Last}}.$$

Memorise the three special products for reading in *both* directions:

$$(a+b)^2 = a^2 + 2ab + b^2, \tag{4.3.3}$$
$$(a-b)^2 = a^2 - 2ab + b^2,$$
$$(a+b)(a-b) = a^2 - b^2. \tag{4.3.4}$$

Each follows from FOIL and the commutativity of multiplication. For cubes,

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3, \qquad a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2). \tag{4.3.5}$$

The general expansion of $(a+b)^n$ is the binomial theorem (Theorem 4.6.14) and the general factorisation of $a^n - b^n$ is Lemma 4.2.9.

Example 4.3.4. Read left to right, Eq. (4.3.3) expands $(x+3)^2 = x^2+6x+9$. Read right to left, it recognises $4x^2 - 12x + 9$ as $(2x-3)^2$. As a non-example, x^2+9 is not a difference of squares and has no real factorisation: its discriminant is -36 (Theorem 4.2.15).

4.3.5 Rationalising

Because $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$ by Eq. (4.3.4), the *conjugate* $\sqrt{a}+\sqrt{b}$ clears the square roots from $\sqrt{a}-\sqrt{b}$. Multiplying numerator and denominator by a conjugate *rationalises* a quotient:

$$\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2}.$$

Later use

Rationalising a *numerator* is the standard move for a limit of the form $\frac{0}{0}$ that involves a root (Chapter 5): the quotient $\dfrac{\sqrt{x+4}-2}{x}$ becomes $\dfrac{1}{\sqrt{x+4}+2}$ and can then be evaluated at $x=0$. The same move computes the derivative of $\sqrt{x}$ from the definition (Definition 6.1.2).

4.3.6 Common algebraic errors

Each of the following errors appears in a large share of written solutions. Each is an instance of applying a law outside its hypotheses.

Common misconception

$\sqrt{x^2} = x$ is false for negative x: $\sqrt{(-3)^2} = \sqrt{9} = 3$. The square root denotes the *nonnegative* root. The correct identity is

$$\sqrt{x^2} = |x| \qquad \text{for every real } x.$$

Similarly $-x^2$ means $-(x^2)$ and differs from $(-x)^2 = x^2$.

Common misconception

Fractions do not split over a sum in the denominator: $\dfrac{1}{a+b} \neq \dfrac{1}{a} + \dfrac{1}{b}$. Take $a = b = 1$: the left side is $\frac{1}{2}$ and the right side is 2. A sum in the *numerator* does split, $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$.

Common misconception

Dividing both sides of an equation by an expression that can be zero loses solutions. From $x^2 = 3x$ dividing by x gives $x = 3$ and loses $x = 0$. Move everything to one side and factor instead: $x(x-3) = 0$ gives $x = 0$ or $x = 3$.

Common misconception

Squaring both sides of an equation can create solutions. The equation $\sqrt{x+2} = x$ squares to $x + 2 = x^2$, with roots 2 and -1, yet $x = -1$ fails the original equation because $\sqrt{1} = 1 \neq -1$. Every candidate obtained after squaring must be substituted into the original equation.

Caution

Multiplying or dividing an inequality by a negative number reverses it (Section 2.3). When the sign of a multiplier such as $x-1$ is unknown, split into the cases $x-1 > 0$ and $x-1 < 0$.

Later use

The algebra of this chapter returns throughout the book. The table records where.

Method	Assumed background
Substitution (u-sub)	Chain rule read backwards; composition $f(g(x))$; differentials $\mathrm{d}u = g'(x)\,\mathrm{d}x$.
Integration by parts	Product rule read backwards; the derivative and antiderivative tables.
Trigonometric integrals	Pythagorean and double-angle identities (Section 4.5).
Trigonometric substitution	Completing the square; the identities $1-\sin^2 = \cos^2$, $1+\tan^2 = \sec^2$; inverse-trig ranges.
Partial fractions	Factoring and polynomial long division (Section 4.2); matching coefficients (Corollary 4.2.8).
Improper integrals	Limits at ∞ and at a bad point; ln and e^x growth rates.
Arc length, surface area, work	Radicals and exponent laws; algebra of $\sqrt{1+(f')^2}$.
Sequences and series	Exponent and log laws; factorials; geometric-sum algebra (Section 4.6).
Parametric and polar	Trigonometry; the chain rule in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$.

Study guide

Convert roots and reciprocals into powers before applying an exponent law, and keep the base positive whenever the exponent is fractional. Read the special products Eqs. (4.3.3) and (4.3.4) in both directions: forwards to expand and backwards to factor. Rationalise with a conjugate. Before dividing by an expression, ask whether it can be zero. After squaring an equation, substitute every candidate back into the original. Write $\sqrt{x^2} = |x|$.

Worked Example 4.3.1

Simplify $\dfrac{(8x^6y^{-3})^{2/3}}{x^2y^{-1}}$ for $x, y > 0$.

By the laws of exponents, $(8x^6y^{-3})^{2/3} = 8^{2/3}x^4y^{-2} = 4x^4y^{-2}$, since $8^{2/3} = (\sqrt[3]{8})^2 = 4$. Dividing,

$$\frac{4x^4y^{-2}}{x^2y^{-1}} = 4x^{4-2}y^{-2+1} = 4x^2y^{-1} = \frac{4x^2}{y}.$$

Worked Example 4.3.2

Rationalise the numerator of $\dfrac{\sqrt{x+4}-2}{x}$ for $x \geq -4$, $x \neq 0$.

Multiply numerator and denominator by the positive conjugate $\sqrt{x+4}+2$. By Eq. (4.3.4) the numerator becomes $(x+4)-4=x$, and

$$\frac{\sqrt{x+4}-2}{x}=\frac{x}{x(\sqrt{x+4}+2)}=\frac{1}{\sqrt{x+4}+2},$$

where cancelling the factor x is legal because $x\neq 0$.

Problems

4.3.1 Write each expression as a single power of x (for $x>0$): $\dfrac{1}{\sqrt[3]{x^2}}$, $x\sqrt{x}$, $\dfrac{(\sqrt{x})^5}{x^3}$. ⋆ Hint · Answer · Solution

4.3.2 Expand $(2x-3)^2$ and $(x^2+1)(x^2-1)$, and factor $8x^3-27$. ⋆ Hint · Answer · Solution

4.3.3 Rationalise the denominator of $\dfrac{1}{\sqrt{5}-\sqrt{3}}$. ⋆⋆ Hint · Answer · Solution

4.3.4 Solve $\sqrt{2x+3}=x$. ⋆⋆ Hint · Answer · Solution

4.3.5 Find every real x with $\dfrac{x}{x-1}>2$. ⋆⋆ Hint · Answer · Solution

4.3.6 Prove that $\sqrt{ab}\le\dfrac{a+b}{2}$ for all $a,b\ge 0$, with equality exactly when $a=b$. ⋆⋆ Hint · Answer · Solution

4.3.7 Prove that $\left|\sqrt{a}-\sqrt{b}\right|\le\sqrt{|a-b|}$ for all $a,b\ge 0$. ⋆⋆⋆ Hint · Answer · Solution

4.4 Exponential and Logarithmic Functions

Overview

Objectives: Define a^x for integer, then rational, then real x and prove the laws of exponents at each stage; define e and the natural exponential; define the logarithm as the inverse of an exponential and derive the log laws; solve exponential and logarithmic equations; explain why $\ln|x|$, and not $\ln x$, is defined on both sides of 0.
Prerequisites: Completeness and suprema (Assumption 2.4.5 and Definition 2.4.3); the Archimedean property and the density of $\mathbb{Q}$ (Theorems 2.4.10 and 2.4.12); Bernoulli's inequality (Theorem 2.3.11); inverses of strictly monotone functions (Theorem 4.1.19).
Key results: Existence of nth roots (Theorem 4.4.6); the laws of exponents (Theorems 4.4.4, 4.4.10 and 4.4.16); the number e (Definition 4.4.19); the logarithm (Definition 4.4.22) and the log laws Eqs. (4.4.7) and (4.4.8).

Notation

$\sqrt[n]{a} = a^{1/n}$	the positive nth root of $a > 0$
a^x	the power with base $a > 0$ and real exponent x
e, $\exp x = e^x$	Euler's number and the natural exponential
$\log_a x$	the logarithm of $x > 0$ to the base $a > 0$, $a \neq 1$
$\ln x$	the natural logarithm $\log_e x$

Repeated multiplication gives $2^3 = 8$ a meaning that reciprocals and roots extend to $2^{-3} = \frac{1}{8}$ and $2^{3/2} = \sqrt{8}$. The expression $2^{\sqrt{2}}$ has no such reading: no number of multiplications or root extractions produces an irrational exponent. Calculus needs a^x for every real x because a function such as 2^x must be defined on an interval before its continuity or derivative can be discussed. This section builds a^x in three stages. The completeness axiom enters at the last.

4.4.1 Integer exponents

Definition 4.4.2 (Integer power)**.** For real a define $a^0 = 1$ and $a^{n+1} = a^n \cdot a$ for integers $n \geq 0$. For $a \neq 0$ and an integer $n \geq 1$ define $a^{-n} = 1/a^n$.

Example 4.4.3. $2^3 = 2^2 \cdot 2 = (2^1 \cdot 2) \cdot 2 = 8$ and $2^{-3} = \frac{1}{8}$. As a non-example, 0^{-1} is undefined since it would be $1/0^1 = 1/0$.

Theorem 4.4.4 (Laws for integer exponents)**.** *For nonzero reals a, b and integers m, n,*

$$a^m a^n = a^{m+n}, \qquad (a^m)^n = a^{mn}, \qquad (ab)^n = a^n b^n. \tag{4.4.1}$$

If $a > 1$ and $m < n$, then $a^m < a^n$.

Proof. For $m, n \geq 0$ each law follows by induction on n (Theorem 1.4.2). The case $n = 0$ reads $a^m = a^m$, $1 = 1$ and $1 = 1$. If $a^m a^n = a^{m+n}$, then $a^m a^{n+1} = a^m a^n a = a^{m+n} a = a^{m+n+1}$. If $(a^m)^n = a^{mn}$, then $(a^m)^{n+1} = a^{mn} a^m = a^{mn+m}$ by the first law. If $(ab)^n = a^n b^n$, then $(ab)^{n+1} = a^n b^n ab = a^{n+1} b^{n+1}$.

Negative exponents reduce to these cases. For $m, n \geq 0$, the first law gives $a^m = a^{m-n} a^n$ when $m \geq n$ and $a^n = a^{n-m} a^m$ when $m < n$, and in both cases $a^m a^{-n} = a^m / a^n = a^{m-n}$. Also $a^{-m} a^{-n} = 1/(a^m a^n) = a^{-(m+n)}$. Together these give the first law for all integers. The second

and third laws for negative exponents follow by taking reciprocals in the nonnegative case, for instance $(a^m)^{-n} = 1/(a^m)^n = 1/a^{mn} = a^{-mn}$.

For the inequality, $a^n - a^m = a^m(a^{n-m} - 1)$. Here $a^m > 0$, and $a^k > 1$ for every integer $k \geq 1$ by induction because $a^{k+1} = a^k a > a^k$. ∎

4.4.5 Roots

Completeness enters first through the existence of nth roots.

Theorem 4.4.6 (Existence of nth roots)**.** *For every $a > 0$ and every integer $n \geq 1$ there is exactly one $b > 0$ with $b^n = a$. It is written $\sqrt[n]{a}$ or $a^{1/n}$.*

Proof. The map $t \mapsto t^n$ is strictly increasing on $[0, \infty)$: for $0 \leq s < t$, Lemma 4.2.9 gives $t^n - s^n = (t - s)\,P$ where P is a sum of nonnegative terms including $t^{n-1} > 0$. This proves uniqueness and gives two estimates. For $c \geq 0$ and $0 < h \leq 1$ every term of P for the pair c, $c + h$ is at most $(c+1)^{n-1}$. For $0 < h < c$ every term for the pair $c - h$, c is at most c^{n-1}. Hence

$$(c + h)^n - c^n \leq h\, n(c + 1)^{n-1}, \qquad c^n - (c - h)^n \leq h\, nc^{n-1}. \tag{4.4.2}$$

For existence let $S = \{t \geq 0 : t^n \leq a\}$. It contains $\min(1, a)$, because $\min(1, a)^n \leq \min(1, a) \leq a$. It is bounded above by $\max(1, a)$ because $t > \max(1, a)$ gives $t^n \geq t > a$. By completeness (Assumption 2.4.5) it has a supremum $b = \sup S$ (Definition 2.4.3), and $b \geq \min(1, a) > 0$.

Suppose $b^n < a$. Choose h with $0 < h \leq 1$ and $h\, n(b + 1)^{n-1} < a - b^n$. By Eq. (4.4.2), $(b + h)^n < a$. Then $b + h \in S$ contradicts $b = \sup S$. Suppose $b^n > a$. Choose h with $0 < h < b$ and $h\, nb^{n-1} < b^n - a$. By Eq. (4.4.2), $(b - h)^n > a$. Every $t \in S$ therefore satisfies $t^n \leq a < (b - h)^n$ and hence $t < b - h$. Then $b - h$ is an upper bound of S smaller than b, again a contradiction. Hence $b^n = a$. ∎

Example 4.4.7. $\sqrt[3]{8} = 2$ and $\sqrt[4]{81} = 3$ because $2^3 = 8$ and $3^4 = 81$. As a non-example, -2 is a real number whose fourth power is 16, yet $\sqrt[4]{16} = 2$: the root symbol always denotes the positive root of Theorem 4.4.6.

For an odd integer n the power t^n is negative for $t < 0$ and $(-t)^n = -t^n$. A negative number a therefore has the real nth root $-\sqrt[n]{-a}$. It is the only one because $t \mapsto t^n$ is strictly increasing on $\mathbb{R}$ for odd n. Let $s < t$. For $0 \leq s$ this is the monotonicity on $[0, \infty)$ in the proof of Theorem 4.4.6. For $s < 0 \leq t$ it follows from $s^n < 0 \leq t^n$. For $s < t < 0$ the same monotonicity applied to $0 < -t < -s$ gives $-t^n = (-t)^n < (-s)^n = -s^n$. The odd root $\sqrt[n]{a}$, also written $a^{1/n}$, is thus defined for every real a. The real cube root $x^{1/3}$ of Example 6.1.17 is this function. Powers with a non-integer exponent and a negative base are not defined beyond these odd roots.

4.4.8 Rational exponents

Definition 4.4.9 (Rational power)**.** For $a > 0$ and a rational $r = m/n$ with m an integer and $n \geq 1$ an integer, define $a^r = \left(a^{1/n}\right)^m$.

The same rational number has many representations, $\frac{2}{3} = \frac{4}{6}$, and the definition is legitimate only if every representation gives the same value. Let $m/n = p/q$ and put $c = a^{1/(nq)}$. Then $(c^q)^n = c^{nq} = a$ and $c^q = a^{1/n}$ by the uniqueness in Theorem 4.4.6. Similarly $c^n = a^{1/q}$. Hence $(a^{1/n})^m = c^{qm}$ and $(a^{1/q})^p = c^{np}$. The two agree because $qm = np$. With $n = 1$ the definition returns the integer power.

Theorem 4.4.10 (Laws for rational exponents)**.** *For $a, b > 0$ and rationals r, s,*

$$a^r a^s = a^{r+s}, \qquad (a^r)^s = a^{rs}, \qquad (ab)^r = a^r b^r. \tag{4.4.3}$$

If $a > 1$ and $r < s$, then $a^r < a^s$.

Proof. Write $r = m/N$ and $s = k/N$ over a common denominator $N \geq 1$ and put $c = a^{1/N}$. Then $a^r a^s = c^m c^k = c^{m+k} = a^{r+s}$ by Theorem 4.4.4. For the product law, $(a^{1/N} b^{1/N})^N = ab$ and $(ab)^{1/N} = a^{1/N} b^{1/N}$ by uniqueness of roots. Raising both sides to the integer power m gives $(ab)^r = a^r b^r$. For the power law write $r = m/n$ and $s = p/q$ and let $c = a^{1/(nq)}$. Then $a^r = c^{mq} = (c^m)^q$. Hence $(a^r)^{1/q} = c^m$ and $(a^r)^s = c^{mp} = a^{mp/(nq)} = a^{rs}$.

For the inequality, $c = a^{1/N} > 1$ because $c \leq 1$ would give $a = c^N \leq 1$. With $k > m$, Theorem 4.4.4 gives $c^m < c^k$. ∎

Lemma 4.4.11 (Roots of a base above one)**.** *If $a > 1$ and $n \geq 1$ is an integer, then $1 < a^{1/n} \leq 1 + \dfrac{a-1}{n}$.*

Proof. The lower bound is the case $r = 0 < s = 1/n$ of Theorem 4.4.10. Bernoulli's inequality (Theorem 2.3.11) with the positive number $(a-1)/n$ gives $\left(1+\frac{a-1}{n}\right)^n \geq 1+(a-1) = a = \left(a^{1/n}\right)^n$. Because $t \mapsto t^n$ is strictly increasing on $[0, \infty)$, the upper bound follows. ∎

4.4.12 Real exponents

For $a > 1$ the powers a^r increase with the rational exponent r. The value at an irrational x should therefore be the least number above every a^r with $r < x$. Completeness supplies that number.

Definition 4.4.13 (Real power)**.** Let x be real. For $a > 1$ define

$$a^x = \sup\{a^r : r \in \mathbb{Q},\ r < x\}. \tag{4.4.4}$$

Define $1^x = 1$. For $0 < a < 1$ define $a^x = 1/(1/a)^x$.

The set in (4.4.4) is nonempty because some integer lies below x (Theorem 2.4.10). It is bounded above by a^N for any integer $N > x$ (Theorem 4.4.10). The supremum therefore exists by Assumption 2.4.5.

Example 4.4.14. The number $2^{\sqrt{2}}$ is the supremum of 2^1, $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, ... and of every other 2^r with rational $r < \sqrt{2}$. Since $1.4 < \sqrt{2} < 1.5$, it lies between $2^{1.4} \approx 2.639$ and $2^{1.5} \approx 2.828$ and equals $2.6651\ldots$. As a non-example, the definition gives no meaning to $(-2)^{\sqrt{2}}$: real powers with irrational exponents are defined only for positive bases.

A definition by supremum must agree with the earlier definition whenever both apply.

Theorem 4.4.15 (Consistency with rational powers)**.** *If $a > 0$ and x is rational, then the power a^x of Definition 4.4.13 equals the rational power of Definition 4.4.9.*

Proof. Let $a > 1$, write a^x for the rational power and σ for the supremum. Each rational $r < x$ has $a^r < a^x$ by Theorem 4.4.10. Hence $\sigma \leq a^x$. For each integer $n \geq 1$, the rational $x - \frac{1}{n}$ is less than x and, by Theorem 4.4.10 and Lemma 4.4.11,

$$\sigma \geq a^{x-1/n} = \frac{a^x}{a^{1/n}} \geq \frac{a^x}{1 + (a-1)/n}.$$

If $\sigma < a^x$, this gives $n(a^x/\sigma - 1) \leq a - 1$ for every n, with $a^x/\sigma - 1 > 0$, contradicting Theorem 2.4.10. Hence $\sigma = a^x$. For $0 < a < 1$, the product law of Theorem 4.4.10 gives $a^x(1/a)^x = 1^x = 1$ in agreement with the defining relation of Definition 4.4.13. ∎

Theorem 4.4.16 (Laws of exponents)**.** *Let $a, b > 0$ and let x, y be real. Then*

1. *$a^x > 0$ and $a^0 = 1$;*
2. *$a^{x+y} = a^x a^y$ and $a^{-x} = 1/a^x$;*
3. *$(ab)^x = a^x b^x$;*
4. *$x \mapsto a^x$ is strictly increasing if $a > 1$ and strictly decreasing if $0 < a < 1$;*
5. *$(a^x)^y = a^{xy}$.*

Proof. First let $a, b > 1$. For (1), $a^x \geq a^r > 0$ for any rational $r < x$, and $a^0 = 1$ by Theorem 4.4.15. For (4), let $x < y$. By Theorem 2.4.12 there are rationals $q_1 < q_2$ with $x < q_1 < q_2 < y$. Every rational $r < x$ satisfies $r < q_1$ and hence $a^r < a^{q_1}$. It follows that $a^x \leq a^{q_1}$. Also a^{q_2} belongs to the set whose supremum is a^y. Hence $a^x \leq a^{q_1} < a^{q_2} \leq a^y$.

For (2), let $r < x$ and $s < y$ be rational. Then $r + s < x + y$ and $a^r a^s = a^{r+s} \leq a^{x+y}$. Fixing s and taking the supremum over r gives $a^x a^s \leq a^{x+y}$. Taking the supremum over s then gives $a^x a^y \leq a^{x+y}$. Conversely, let $t < x + y$ be rational. By Theorem 2.4.12 there is a rational r with $t - y < r < x$. The rational $s = t - r$ is less than y and $a^t = a^r a^s \leq a^x a^y$. Taking the supremum over t gives $a^{x+y} \leq a^x a^y$. With $y = -x$ this gives $a^x a^{-x} = a^0 = 1$.

For (3), each rational $r < x$ gives $(ab)^r = a^r b^r \leq a^x b^x$. Hence $(ab)^x \leq a^x b^x$. Conversely, for rationals $r, s < x$ let $m = \max(r, s)$. Then $a^r b^s \leq a^m b^m = (ab)^m \leq (ab)^x$, and taking suprema over r and then s gives $a^x b^x \leq (ab)^x$.

The cases with a base in $(0, 1]$ reduce to these by Definition 4.4.13. For $0 < a < 1$, $a^{x+y} = 1/(1/a)^{x+y} = 1/((1/a)^x (1/a)^y) = a^x a^y$, and $a^x = 1/(1/a)^x$ is positive and strictly decreasing as the reciprocal of a positive increasing function. For (3) a base equal to 1 gives both sides equal by $1^x = 1$. The remaining cases follow from the case $a, b > 1$ and $c^x = 1/(1/c)^x$ for $0 < c < 1$.
(i) $a, b < 1$. Then $1/a, 1/b > 1$ and $(1/(ab))^x = (1/a)^x (1/b)^x$. Taking reciprocals gives $(ab)^x = a^x b^x$.
(ii) $b < 1 < a$, $ab > 1$. The factorisation $a = (ab) \cdot (1/b)$ into two numbers greater than 1 gives $a^x = (ab)^x (1/b)^x = (ab)^x / b^x$.
(iii) $b < 1 < a$, $ab < 1$. The factorisation $1/b = (1/(ab)) \cdot a$ into two numbers greater than 1 gives $1/b^x = (1/b)^x = (1/(ab))^x a^x = a^x/(ab)^x$.
(iv) $b < 1 < a$, $ab = 1$. Here $b = 1/a$ and $a^x b^x = a^x/a^x = 1 = 1^x$.
The case $a < 1 < b$ is the same with a and b exchanged.

For (5) with $y = p/q$ rational, (2) gives $(a^{x/q})^q = a^x$ and $(a^x)^{1/q} = a^{x/q}$ by uniqueness of roots. Raising to the integer power p gives $(a^x)^{p/q} = a^{px/q}$ by (2) again. For irrational y the two sides of (5) are continuous functions of y that agree at every rational y. Theorem 5.4.12 of Chapter 5 concludes from this that they are equal. Its proof of continuity uses only parts (1), (2) and (4) and never assumes part (5). ∎

Theorem 4.4.17 (Exponentials are bijections)**.** *For $a > 0$ with $a \neq 1$, the function $x \mapsto a^x$ is a bijection from $\mathbb{R}$ onto $(0, \infty)$.*

Injectivity follows from Theorem 4.4.16(4) and Theorem 4.1.19. The values are positive by Theorem 4.4.16(1). Surjectivity needs the continuity of $x \mapsto a^x$ and the intermediate value theorem. Part (2) of Theorem 5.4.12 proves the theorem in full once both are available.

4.4.18 Euler's number and the natural exponential

Of the exponential functions given by the bases $a > 0$, one makes the calculus of exponentials simplest. Its base is defined by a limit.

Definition 4.4.19 (Euler's number and the natural exponential)**.** Euler's number is

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{4.4.5}$$

The *natural exponential* is the function $\exp x = e^x$ on $\mathbb{R}$, defined by Definition 4.4.13.

That the limit exists is proved in Theorem 5.1.20: the sequence $\left(1+\frac{1}{n}\right)^n$ is increasing, bounded above by 3 (Problem 4.6.7 proves the bound) and therefore converges by the monotone convergence theorem (Theorem 5.1.18). Its terms are at least 2 by Bernoulli's inequality (Problem 4.4.7). Hence $e \geq 2 > 1$ and exp is strictly increasing. Numerically $e = 2.718281828\ldots$ and the convergence is slow: the term with $n = 1000$ is $2.7169\ldots$. The same limit with x/n in place of $1/n$ equals e^x for every real x (Theorem 5.4.15). Chapter 6 differentiates e^x from (5.4.1).

Example 4.4.20. By Theorem 4.4.16, $e^2e^{-1} = e^1 = e$ and $(e^{1/2})^2 = e$. As a non-example, $(1+1/n)^n$ does not approach 1 even though $1+1/n$ does: a limit of a power whose exponent grows cannot be computed by taking the limit of the base alone.

Later use

The base e is singled out by calculus because e^x is its own derivative (Chapter 6). Every other exponential is a rescaling of it by $a^x = e^{x\ln a}$ (Theorem 4.4.24). In series, every exponential enters the Taylor series of e^u through this identity (Section 11.2).

4.4.21 Logarithms

Definition 4.4.22 (Logarithm)**.** Let $a > 0$ with $a \neq 1$. The *logarithm to the base* a is the inverse (Definition 1.5.24) of the bijection $x \mapsto a^x$ of Theorem 4.4.17:

$$\log_a : (0,\infty) \to \mathbb{R}, \qquad y = \log_a x \iff a^y = x. \tag{4.4.6}$$

The *natural logarithm* is $\ln x = \log_e x$.

The definition uses the surjectivity in Theorem 4.4.17, proved in full by Theorem 5.4.12. Because $\log_a$ and $a^{(\cdot)}$ are inverse to each other, $a^{\log_a x} = x$ for $x > 0$ and $\log_a(a^x) = x$ for every real x. For $a > 1$ the logarithm is strictly increasing by Theorem 4.1.19.

Example 4.4.23. $\log_2 8 = 3$ because $2^3 = 8$, $\log_{10} 0.01 = -2$ and $\ln e^5 = 5$. As a non-example, $\log_2(-4)$ and $\ln 0$ are undefined because every power a^y is positive and no exponent produces -4 or 0.

Theorem 4.4.24 (Laws of logarithms)**.** *Let $a > 0$ with $a \neq 1$, let $x, y > 0$ and let r be real. Then*

$$\log_a(xy) = \log_a x + \log_a y, \tag{4.4.7}$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y,$$
$$\log_a(x^r) = r\log_a x, \tag{4.4.8}$$
$$\log_a 1 = 0, \qquad \log_a a = 1.$$

The change-of-base formulas are $\log_a x = \dfrac{\ln x}{\ln a}$ *and* $a^x = e^{x \ln a}$.

Proof. Put $u = \log_a x$ and $v = \log_a y$ with $a^u = x$ and $a^v = y$. By Theorem 4.4.16, $a^{u+v} = xy$ and $a^{u-v} = x/y$. These equations say $\log_a(xy) = u+v$ and $\log_a(x/y) = u-v$. By Theorem 4.4.16(5), $a^{ru} = (a^u)^r = x^r$ and hence $\log_a(x^r) = ru$. The values at 1 and a restate $a^0 = 1$ and $a^1 = a$. Applying ln to $x = a^u$ gives $\ln x = u \ln a$ by Eq. (4.4.8) with $\ln a \neq 0$ because $a \neq 1$. Finally $e^{x \ln a} = (e^{\ln a})^x = a^x$ by Theorem 4.4.16(5). ∎

Two moves solve almost every exponential or logarithmic equation. With the unknown in the *exponent*, take ln of both sides and use Eq. (4.4.8). With the unknown *inside* a logarithm, exponentiate both sides and then check that the solution lies in the domain of every logarithm in the original equation. Worked Example 4.4.2 applies both moves.

Common misconception

There is no law for the log of a *sum*: $\ln(x+y) \neq \ln x + \ln y$. The product law (4.4.7) applies to the different expression $\ln(xy)$. Similarly $\dfrac{\ln x}{\ln y} \neq \ln \dfrac{x}{y}$ and $(\ln x)^2 \neq \ln(x^2)$.

Later use

Logarithmic differentiation uses (4.4.7) to (4.4.8) to turn a product, quotient, or power such as $y = \dfrac{x^2\sqrt{x+1}}{(3x-2)^5}$ into a sum of logs before differentiating (Chapter 6). In integration, the log laws *combine* a sum of logarithms into a single answer.

Figure 4.4.1 shows e^x and $\ln x$, each the reflection of the other in the line $y = x$ as the graphs of a function and its inverse always are.

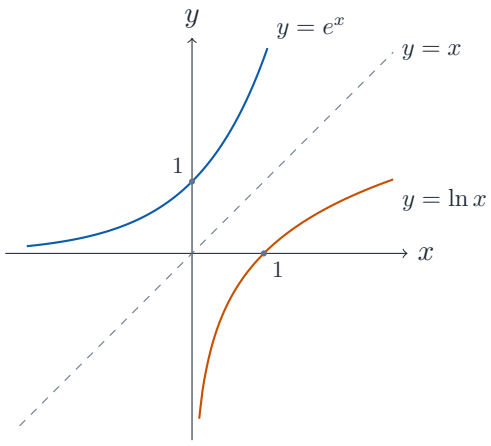


Figure 4.4.1: Mutually inverse functions e^x and $\ln x$ as reflections across the line $y = x$. The exponential satisfies $e^x > 0$ always (range $(0, \infty)$, domain all reals) and $\ln x$ is defined only for $x > 0$ (domain $(0, \infty)$, range all reals).

4.4.25 $\ln|x|$ versus e^x

These two functions behave oppositely in the one respect that matters for integration: the sign of the input.

	e^x	$\ln x$	$\ln\lvert x\rvert$
Domain	all real x	$x > 0$	$x \neq 0$
Range	$(0, \infty)$	all reals	all reals
Symmetry	none	none	even

The exponential needs no absolute value because e^x is defined and positive for *every* real x. The logarithm requires a *positive* argument. The function $\ln|x|$ extends it to the negative axis by reflection in the y-axis: $\ln|x| = \ln(-x)$ for $x < 0$. It is an even function on $\mathbb{R} \setminus \{0\}$ (Definition 4.1.11). Figure 4.4.2 compares $\ln x$ with $\ln|x|$ and its mirror branch on $x < 0$.

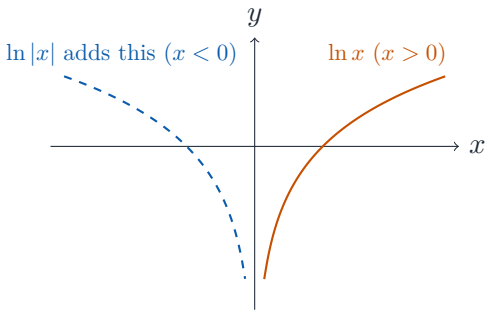


Figure 4.4.2: $\ln x$ (right branch only) versus the even function $\ln|x|$ with its added mirror branch for $x < 0$. The extra branch extends the logarithm to the negative axis by reflection in the y-axis.

Later use

Although $\dfrac{1}{x}$ is defined for *all* $x \neq 0$, the function $\ln x$ does not exist for $x < 0$ and cannot be an antiderivative there. For $x > 0$, $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln x = \dfrac{1}{x}$ (Chapter 6). For $x < 0$, $|x| = -x$ and the chain rule (Theorem 6.3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

Hence $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$ on any interval avoiding 0 (Chapter 8). Every substitution that reduces to $\displaystyle\int \frac{1}{u}\,\mathrm{d}u$ produces $\ln|u| + C$. The bars account for

$$\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C, \qquad \int \frac{\mathrm{d}x}{x-a} = \ln|x-a| + C,$$

and for the sums of $\ln|\cdot|$ terms in partial-fraction answers (Eq. (9.5.7)). The exponential needs no such care: $\int e^u\,\mathrm{d}u = e^u + C$ everywhere.

Common misconception

$\ln|x|$ differs from $|\ln x|$. The bars in $\ln|x|$ belong around the *argument* and extend the domain to $x < 0$. The function $|\ln x|$ is defined only for $x > 0$ and reflects the negative values of $\ln x$, those on $(0, 1)$, above the axis.

Study guide

The power a^x is built in stages: repeated multiplication for integer x, roots for rational x (Theorem 4.4.6), and a supremum for real x (Definition 4.4.13). The laws of exponents hold at every stage and require a positive base for a non-integer exponent. The logarithm is defined as the inverse of the exponential. Hence every log law is an exponent law read backwards. To solve an equation, take ln when the unknown is in an exponent and exponentiate when it is inside a logarithm. Then check the domain. Use $\ln|x|$ when the argument may be negative.

Worked Example 4.4.1

Evaluate $8^{2/3}$ and $16^{-3/4}$, and decide which is larger, $2^{\sqrt{3}}$ or $3^{\sqrt{2}}$.
By Definition 4.4.9, $8^{2/3} = (\sqrt[3]{8})^2 = 4$ and $16^{-3/4} = (\sqrt[4]{16})^{-3} = 2^{-3} = \frac{1}{8}$. For the comparison take logarithms. Since ln is strictly increasing, $2^{\sqrt{3}} < 3^{\sqrt{2}}$ exactly when $\sqrt{3}\ln 2 < \sqrt{2}\ln 3$, that is, when $\sqrt{3}\cdot 0.6931\ldots = 1.2005\ldots < \sqrt{2}\cdot 1.0986\ldots = 1.5536\ldots$.
The inequality is true and $3^{\sqrt{2}}$ is the larger.

Worked Example 4.4.2

Solve $3\cdot 2^x = 48$ and $\ln(x-1) = 2$.
The first equation gives $2^x = 16$. Taking ln and using Eq. (4.4.8), $x\ln 2 = \ln 16 = 4\ln 2$ and hence $x = 4$. In the second the unknown is inside a logarithm. Exponentiating gives $x - 1 = e^2$. The value $x = 1 + e^2$ satisfies the domain condition $x - 1 > 0$.

Problems

4.4.1 Evaluate $27^{-2/3}$, $16^{3/4}$, $\log_2 \frac{1}{32}$ and $\log_9 27$. ⋆ Hint · Answer · Solution

4.4.2 Write $2\ln x - \ln(x+1) + \frac{1}{2}\ln(x^2+1)$ as a single logarithm, for $x > 0$.
⋆ Hint · Answer · Solution

4.4.3 Solve $5^{x+1} = 3^{2x}$. ⋆⋆ Hint · Answer · Solution

4.4.4 Find the natural domain of $f(x) = \ln(x^2 - 4x + 3)$. ⋆⋆ Hint · Answer · Solution

4.4.5 Solve $\log_2 x + \log_2(x-2) = 3$. ⋆⋆ Hint · Answer · Solution

4.4.6 Prove that $\log_a b \cdot \log_b a = 1$ for $a, b > 0$ with $a, b \neq 1$. ⋆⋆ Hint · Answer · Solution

4.4.7 Use Bernoulli's inequality to show that $\left(1 + \frac{1}{n}\right)^n \geq 2$ for every integer $n \geq 1$.
⋆ Hint · Answer · Solution

4.4.8 Prove that $\log_2 3$ is irrational. ⋆⋆⋆ Hint · Answer · Solution

4.5 Trigonometric Functions

Overview

Objectives: Measure angles in radians by arc length; define the six trigonometric functions on the unit circle; derive the Pythagorean, angle-sum, double-angle and half-angle identities; read the graphs and transformations of sine and cosine; define the inverse trigonometric functions by restricting the domain and evaluate them on the correct branch.
Prerequisites: Graphs and transformations (Theorem 4.1.8); inverses of strictly monotone functions (Theorem 4.1.19); the special products (Section 4.3); rotations, reflections and lengths in the plane (Definitions 3.3.21 and 3.3.22 and Theorem 3.5.10).
Key results: Radian measure (Definition 4.5.2); the angle-sum formulas (Theorem 4.5.9); the double-angle and power-reduction formulas (Eqs. (4.5.4) and (4.5.5)); arcsin, arccos, arctan and their ranges (Definition 4.5.14).

Notation

$P(\theta)$	the point of the unit circle at arc length θ from $(1,0)$
$\sin, \cos, \tan$	sine, cosine, tangent; $\sec, \csc, \cot$ their reciprocals
$\arcsin, \arccos, \arctan$	the inverse trigonometric functions

Periodic phenomena such as oscillations, rotations and waves are described by the trigonometric functions. School trigonometry defines them for the acute angles of a right triangle and measures angles in degrees. Calculus needs them for every real input and needs a unit of angle in which the simplest limits and derivatives involve no conversion factor. Both needs are met by measuring angles by arc length on the unit circle.

4.5.1 Radian measure

Definition 4.5.2 (Radian measure)**.** Let $\theta \geq 0$. Starting at $(1,0)$, travel anticlockwise along the unit circle $x^2+y^2=1$ a distance θ, and call the endpoint $P(\theta)$. For $\theta < 0$ travel the distance $|\theta|$ clockwise. The angle between the positive x-axis and the ray from the origin through $P(\theta)$ has *radian measure* θ. The number π is half the circumference of the unit circle.

The length of a curve is defined in Definition A.1.1. Proposition A.1.12 proves that the upper semicircle has a finite length, the number π of Definition 4.5.2, and Theorem A.1.14(4) gives the whole circle the length 2π.

A full turn is 2π radians and corresponds to 360°. Hence $180^\circ = \pi$ radians, $1^\circ = \pi/180$ radians and $P(\theta+2\pi) = P(\theta)$ (Theorem A.1.14(1)). On a circle of radius ρ an angle of θ radians cuts off an arc of length $\rho\theta$ (Lemma A.1.4).

Logical order of radian measure

Arc length is defined by an integral in Chapter 8 (Definition 8.6.2). The arc of the unit circle from $(1,0)$ to the point $(\sqrt{1-s^2}, s)$, $0 \leq s < 1$, is the graph of $x = \sqrt{1-y^2}$ over $[0,s]$ with length

$$\theta = \int_0^s \frac{\mathrm{d}y}{\sqrt{1-y^2}}, \tag{4.5.1}$$

the integral of an algebraic function of y. In the first quadrant Definition 4.5.2 says $P(\theta) = (\sqrt{1-s^2}, s)$ with s determined by Eq. (4.5.1). Theorem A.1.14 extends this to every real

θ through reflections in the axes and the period 2π. It proves that P is well defined and continuous, that P takes each point of the circle exactly once for $0 \le \theta < 2\pi$ and that the arc from $(1,0)$ to $P(\theta)$ travelled anticlockwise has length θ. Theorem 8.6.6 shows that each $\theta \in [0, \pi/2)$ arises from exactly one s, that the supremum of the integral is $\pi/2$ (Proposition A.1.12) and that $\sin\theta < \theta < \tan\theta$ for $0 < \theta < \pi/2$. The integral, the Fundamental Theorem and Definition 8.6.2 use no trigonometric function. Hence the definition is completed in Chapter 8 without circularity. The logical order is: the definitions of this section; continuity and $\lim_{x\to 0}(\sin x)/x = 1$ in Chapter 5, from the inequality just stated; derivatives in Chapter 6; the integral and arc length of Chapter 8 as the source of Eq. (4.5.1) and its consequences.

The section uses two further properties of arc length. Rotations about the origin and reflections in a line through the origin (Definitions 3.3.21 and 3.3.22) map the unit circle to itself and preserve distance (Theorem 3.5.10). A rotation preserves orientation and a reflection reverses it (Corollary 3.5.16). Appendix A.1 deduces the statements about curves: both maps preserve the length of every curve. A rotation preserves the direction of travel round the circle and a reflection reverses it.

Example 4.5.3. In radians $90^\circ = \pi/2$ and $45^\circ = \pi/4$. The points $P(\pi/2) = (0,1)$ and $P(\pi) = (-1,0)$ lie a quarter turn and a half turn from $(1,0)$. As a non-example, an angle of 1 radian is not 1°: it is $180/\pi \approx 57.3$ degrees.

4.5.4 Unit-circle definitions

Definition 4.5.5 (Trigonometric functions)**.** For real θ, $\cos\theta$ and $\sin\theta$ are the coordinates of $P(\theta) = (\cos\theta, \sin\theta)$ (Fig. 4.5.1), the point of Definition 4.5.2 that Theorem A.1.14 constructs for every real θ. Where the denominators are nonzero,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}. \tag{4.5.2}$$

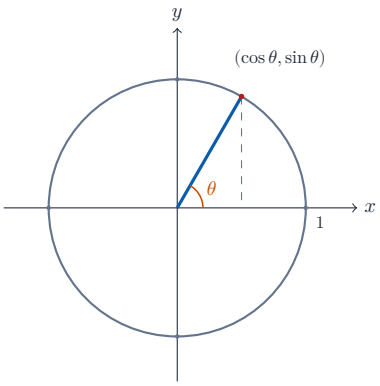


Figure 4.5.1: Unit circle with $\cos\theta$ and $\sin\theta$ as the coordinates of the point at angle θ. The quadrant determines the signs of cos and sin and is what the inverse-trig ranges specify.

For an acute angle the definition agrees with the right-triangle ratios: the dashed segment in Fig. 4.5.1 completes a right triangle with hypotenuse 1, adjacent side $\cos\theta$ and opposite side $\sin\theta$.

Example 4.5.6. $\cos 0 = 1$, $\sin 0 = 0$, $\cos\pi = -1$ and $\tan\pi = 0$. As a non-example, $\tan(\pi/2)$ is undefined since $\cos(\pi/2) = 0$. The natural domain of tan excludes every odd multiple of $\pi/2$.

Theorem 4.5.7 (Basic trigonometric properties)**.** *For every real θ:*

1. $\sin^2\theta + \cos^2\theta = 1$;
2. $\cos(\theta + 2\pi) = \cos\theta$ *and* $\sin(\theta + 2\pi) = \sin\theta$;
3. $\cos(-\theta) = \cos\theta$ *and* $\sin(-\theta) = -\sin\theta$;
4. $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ *and* $\sin(\frac{\pi}{2} - \theta) = \cos\theta$;
5. $\sin\theta > 0$ *for* $0 < \theta < \pi$ *and* $\cos\theta > 0$ *for* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Dividing (1) by $\cos^2\theta$ or by $\sin^2\theta$ gives $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$ wherever the denominators are nonzero.

Proof. Statement (1) says that $P(\theta)$ lies on the unit circle and (2) that a full turn has length 2π. For (3) the reflection $F_{1,0}$ in the x-axis (Definition 3.3.22) fixes $(1,0)$, reverses the direction of travel and preserves arc length. It maps $P(\theta)$ to $P(-\theta)$ while changing the sign of the second coordinate. For (4) the reflection $F_{c,c}$ with $c = 1/\sqrt{2}$ in the line $y = x$ swaps the coordinates, maps $(1,0)$ to $(0,1) = P(\frac{\pi}{2})$ and reverses the direction of travel. It maps $P(\theta)$ to $P(\frac{\pi}{2} - \theta)$. Theorem A.1.14 and Lemma A.1.15 prove (2), (3) and (4) from the construction of P. For (5), P maps $(0,\pi)$ into $\{y > 0\}$ by Theorem A.1.14(3). By (4), $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ with $0 < \frac{\pi}{2} - \theta < \pi$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. ∎

4.5.8 Angle-sum and multiple-angle identities

Theorem 4.5.9 (Angle-sum formulas)**.** *For all real α, β,*

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta, \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \end{aligned} \tag{4.5.3}$$

Proof. The squared distance between $P(\alpha)$ and $P(\beta)$ is

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

by Eq. (4.3.3) and Theorem 4.5.7(1). The rotation about the origin through the angle $-\beta$ preserves distance and maps $P(\beta)$ to $P(0) = (1,0)$ and $P(\alpha)$ to $P(\alpha - \beta)$. This rotation is $R_{c,s}$ of Definition 3.3.21 with $(c,s) = (\cos\beta, -\sin\beta) = P(-\beta)$ (Lemma A.1.15). Corollary A.1.20 with $\varphi = -\beta$ proves both statements about it. The squared distance between those images is $(\cos(\alpha - \beta) - 1)^2 + \sin^2(\alpha - \beta) = 2 - 2\cos(\alpha - \beta)$. Equating the two expressions proves the first formula of (4.5.3). Replacing β by $-\beta$ and using Theorem 4.5.7(3) gives the second. By Theorem 4.5.7(4) and the first formula,

$$\sin(\alpha + \beta) = \cos((\tfrac{\pi}{2} - \alpha) - \beta) = \cos(\tfrac{\pi}{2} - \alpha)\cos\beta + \sin(\tfrac{\pi}{2} - \alpha)\sin\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

and replacing β by $-\beta$ gives the fourth. ∎

Taking $\beta = \alpha = \theta$ in the sum formulas (4.5.3) and using Theorem 4.5.7(1) gives the double-angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \tag{4.5.4}$$

Solving the last two forms for the squares gives the half-angle (power-reduction) formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}. \tag{4.5.5}$$

Example 4.5.10. The number $s = \sin\frac{\pi}{6}$ is positive by Theorem 4.5.7(5). By Theorem 4.5.7(4), $s = \cos(\frac{\pi}{2} - \frac{\pi}{6}) = \cos\frac{\pi}{3} = \cos(2 \cdot \frac{\pi}{6}) = 1 - 2s^2$ by Eq. (4.5.4). Hence $2s^2 + s - 1 = (2s - 1)(s + 1) = 0$ and $s = \frac{1}{2}$. Then $\cos\frac{\pi}{6} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$. At $\frac{\pi}{4}$ the point lies on the line $y = x$ by Theorem 4.5.7(4). There $2\cos^2\frac{\pi}{4} = 1$ gives $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. As a non-example, $\sin 2\theta \neq 2\sin\theta$: at $\theta = \frac{\pi}{2}$ the left side is 0 and the right side is 2.

These values give the first-quadrant reference table to memorise:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Values in other quadrants follow from the symmetries of Theorem 4.5.7 and from $P(\theta + \pi) = -P(\theta)$, the half-turn $\cos(\theta + \pi) = -\cos\theta$, $\sin(\theta + \pi) = -\sin\theta$ given by Theorem 4.5.9.

Origin of the trigonometric functions

The cosine and the sine are the entries of a rotation. The rotation $R_{c,s}$ of Definition 3.3.21 sends $\mathbf{e}_1 = (1, 0)$ to the point (c, s) of the unit circle. With $(c, s) = P(\theta)$ it is the rotation through the angle θ. By Corollary A.1.20 it sends $P(\varphi)$ to $P(\varphi + \theta)$. Composing two rotations adds their angles. Proposition 3.3.23(1) computes the product $R_{c,s}R_{c',s'}$ entry by entry. With $(c, s) = P(\alpha)$ and $(c', s') = P(\beta)$ its entries are the angle-sum formulas of Theorem 4.5.9. The functions cos and sin are the coordinates of the first column of the rotation through θ.

The same computation is shorter in the complex numbers. Write the point (x, y) as $x + iy$ and multiply with the rule $i^2 = -1$. For $c^2 + s^2 = 1$,

$$(c + is)(x + iy) = (cx - sy) + i(sx + cy),$$

the complex form of $R_{c,s}(x, y)$. Rotation through θ is multiplication by the unit complex number $\cos\theta + i\sin\theta$. The product of $\cos\alpha + i\sin\alpha$ and $\cos\beta + i\sin\beta$ is $\cos(\alpha+\beta) + i\sin(\alpha+\beta)$ by Theorem 4.5.9.

Complex multiplication rotates the plane only. Geometric algebra extends the construction to every dimension. Vectors are multiplied with the rules $\mathbf{e}_j\mathbf{e}_j = 1$ and $\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j$ for $j \neq k$. The product $\mathbf{e}_1\mathbf{e}_2$ is the unit bivector of the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, with

$$(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1.$$

The bivector has the role of i. The rotor $R = \exp(-\theta\mathbf{e}_1\mathbf{e}_2/2) = \cos\frac{\theta}{2} - \mathbf{e}_1\mathbf{e}_2\sin\frac{\theta}{2}$ sends a vector $\mathbf{v}$ of that plane to $R\mathbf{v}R^{-1}$, its rotation through θ. In $\mathbb{R}^3$ or $\mathbb{R}^n$ the same formula rotates the part of $\mathbf{v}$ in the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ and fixes the part orthogonal to it. The unit bivector of any other plane gives the rotations of that plane in the same way. Rotation and the trigonometric functions generalise beyond $\mathbb{R}^2$ along this route.

The rotor is the exponential series of Chapter 11 evaluated at the bivector $-\theta\mathbf{e}_1\mathbf{e}_2/2$ in place of a number. Since the bivector squares to -1, the even terms of the series give the cosine series and the odd terms the sine series. With i in place of the bivector the same computation is Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. This book stays in one real variable.

Complex numbers, bivectors and rotors appear in this remark only. No later result depends on them.

Later use

Trigonometric *integrals* (splitting off a $\sec^2$ or converting powers) and trigonometric *substitution* (turning $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ into single trigonometric functions) both rest on the Pythagorean identities, restated as Eq. (9.3.1). Power reduction Eq. (4.5.5), restated as Eq. (9.3.2), is the standard way to integrate even powers of sine and cosine. The product-to-sum forms Eq. (9.3.3) follow from Theorem 4.5.9. The parity of Theorem 4.5.7(3) is the property that Corollary 8.5.4 uses on a symmetric interval. Calculus uses radians throughout because the derivative of $\sin x$ is $\cos x$ only when x is measured in radians (Chapter 6).

4.5.11 Graphs

By Theorem 4.5.7, sine and cosine are 2π-periodic with values in $[-1,1]$. No $p \in (0,2\pi)$ is a period of either. A point of the unit circle with second coordinate 1 is $(0,1)$ and one with first coordinate 1 is $(1,0)$. Hence $\sin(\frac{\pi}{2}+p)=1$ would give $P(\frac{\pi}{2}+p)=P(\frac{\pi}{2})$ and $\cos p = 1$ would give $P(p)=P(0)$, against Theorem A.1.14(1) and (3). Cosine is even and sine is odd. The identity $\sin\theta = \cos(\theta-\frac{\pi}{2})$ of Theorem 4.5.7(4) shows by Theorem 4.1.8 that the graph of sine is the graph of cosine translated $\frac{\pi}{2}$ to the right (Fig. 4.5.2). The tangent is π-periodic since $\tan(\theta+\pi)=\dfrac{-\sin\theta}{-\cos\theta}=\tan\theta$.

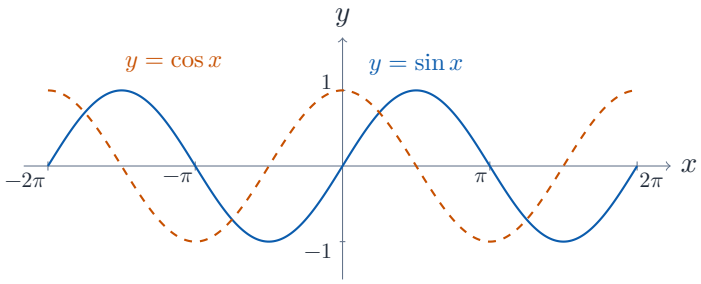


Figure 4.5.2: Graphs of $\sin x$ and $\cos x$ over two periods. Each oscillates between -1 and 1 with period 2π. The sine curve is the cosine curve translated $\pi/2$ to the right.

For $A \neq 0$ and $B > 0$, the graph of $y = A\sin(B(x-C))+D$ is obtained from the sine curve by Theorem 4.1.8. It has *amplitude* $|A|$, *period* $2\pi/B$, horizontal shift C and range $[D-|A|, D+|A|]$. The range uses that sine takes every value in $[-1,1]$ (Corollary 5.4.13).

4.5.12 Inverse trigonometric functions

A periodic function takes each value infinitely often and has no inverse. To invert one, first restrict its domain to an interval on which it is strictly monotone. The chosen interval fixes the *range* of the inverse and with it the quadrant in which an answer lies.

Theorem 4.5.13 (Monotonicity of the trigonometric functions)**.** *Sine is strictly increasing on* $[-\frac{\pi}{2},\frac{\pi}{2}]$*, cosine is strictly decreasing on* $[0,\pi]$*, and tangent is strictly increasing on* $(-\frac{\pi}{2},\frac{\pi}{2})$*.*

Proof. Let $a<b$ and put $u=\frac{a+b}{2}$, $v=\frac{b-a}{2}>0$. By Theorem 4.5.9, $\sin b-\sin a=\sin(u+v)-\sin(u-v)=2\cos u\sin v$. If $-\frac{\pi}{2}\le a<b\le\frac{\pi}{2}$, then $-\frac{\pi}{2}<u<\frac{\pi}{2}$ and $0<v\le\frac{\pi}{2}$. Hence both

factors are positive by Theorem 4.5.7(5). For cosine, $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ and $\theta \mapsto \frac{\pi}{2} - \theta$ maps $[0, \pi]$ onto $[-\frac{\pi}{2}, \frac{\pi}{2}]$ reversing order. For tangent with $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$,

$$\tan b - \tan a = \frac{\sin b \cos a - \cos b \sin a}{\cos a \cos b} = \frac{\sin(b-a)}{\cos a \cos b},$$

where $0 < b - a < \pi$ makes the numerator positive and both cosines are positive. ∎

By Theorem 4.1.19 each restricted function has a strictly monotone inverse on its image. The images are $[-1, 1]$, $[-1, 1]$ and $\mathbb{R}$. That every value in these sets is attained uses the continuity of the trigonometric functions and the intermediate value theorem (Theorem 5.4.2), and Corollary 5.4.13 gives the proof.

Definition 4.5.14 (Inverse trigonometric functions)**.** The functions arcsin, arccos and arctan are the inverses (Definition 1.5.24) of sine, cosine and tangent restricted to the branches of Theorem 4.5.13:

Function	Domain	Range (principal values)
$\arcsin x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	all reals	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Thus $y = \arcsin x$ means $\sin y = x$ *and* $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, and similarly for the others.

For $|x| \ge 1$ the angle $\theta = \arccos(1/x)$ in $[0, \pi]$ satisfies $\sec\theta = x$. It lies in $[0, \frac{\pi}{2})$ for $x \ge 1$ and in $(\frac{\pi}{2}, \pi]$ for $x \le -1$. Section 9.4 writes it in this form on both branches.

Figure 4.5.3 shows the restricted sine and its inverse as reflections in the line $y = x$.

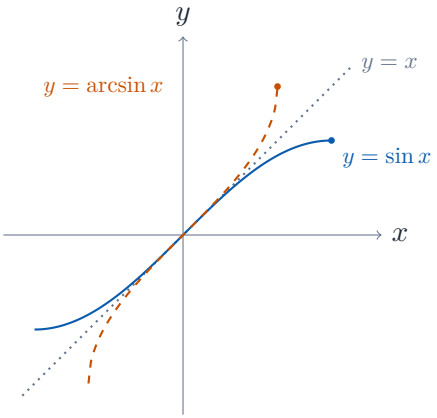


Figure 4.5.3: Sine restricted to $[-\pi/2, \pi/2]$ and its inverse arcsin on $[-1, 1]$. Reflection in the line $y = x$ exchanges the two graphs and maps the endpoint $(\pi/2, 1)$ to $(1, \pi/2)$.

Example 4.5.15. $\arcsin\frac{1}{2} = \frac{\pi}{6}$, $\arccos(-1) = \pi$ and $\arctan 1 = \frac{\pi}{4}$. As a non-example, $\arcsin\frac{1}{2}$ is not $\frac{5\pi}{6}$, even though $\sin\frac{5\pi}{6} = \frac{1}{2}$, because $\frac{5\pi}{6}$ lies outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Common misconception

The identity $\arcsin(\sin\theta) = \theta$ holds only for θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For instance $\arcsin(\sin\frac{3\pi}{4}) = \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. The other order, $\sin(\arcsin x) = x$, is valid for every $x \in [-1, 1]$. The notation $\sin^{-1} x$ means $\arcsin x$ and never $1/\sin x = \csc x$.

Later use

Read backwards, the derivatives of arcsin and arctan give the standard target integrals of Proposition 9.1.7 for use after completing the square. In trigonometric substitution the range restrictions above determine the sign when you simplify $\sqrt{\cos^2\theta} = |\cos\theta|$ back to $+\cos\theta$ (Table 9.4.1).

Study guide

Work in radians. Read every value off the unit circle: the reference table gives the first quadrant and the symmetries of Theorem 4.5.7 give the rest, with signs fixed by the quadrant. Derive the double-angle and power-reduction formulas from Theorem 4.5.9 whenever you are unsure of them. To evaluate an inverse trigonometric function, find an angle with the right value *in the principal range*. To simplify an expression such as $\cos(\arcsin x)$, name the angle $\theta = \arcsin x$, use the Pythagorean identity, and fix the sign of the root from the range of θ.

Worked Example 4.5.1

Solve $2\sin^2 x - \sin x - 1 = 0$ for $0 \le x < 2\pi$.
With $s = \sin x$ the equation $2s^2 - s - 1 = (2s+1)(s-1) = 0$ gives $\sin x = 1$ or $\sin x = -\frac{1}{2}$. By Theorem A.1.14(3) each point of the unit circle is $P(x)$ for exactly one $x \in [0, 2\pi)$. The only point with second coordinate 1 is $(0,1) = P(\frac{\pi}{2})$. The points with second coordinate $-\frac{1}{2}$ are $(\mp\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The half-turn gives $P(\frac{7\pi}{6}) = -P(\frac{\pi}{6}) = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and parity with periodicity gives $P(\frac{11\pi}{6}) = P(-\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The solutions are $\frac{\pi}{2}$, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Worked Example 4.5.2

Simplify $\cos(\arctan x)$ for real x.
Let $\theta = \arctan x$. Then $\tan\theta = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. By Theorem 4.5.7, $\sec^2\theta = 1 + \tan^2\theta = 1 + x^2$ and $\cos^2\theta = \dfrac{1}{1+x^2}$. On the range of arctan the cosine is positive, hence

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Problems

4.5.1 Find $\sin\frac{7\pi}{6}$, $\cos\frac{5\pi}{4}$ and $\tan(-\frac{\pi}{3})$. ⋆ Hint · Answer · Solution

4.5.2 Prove that $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ whenever all the tangents are defined and $\tan\alpha\tan\beta \neq 1$. ⋆⋆ Hint · Answer · Solution

4.5.3 Find the exact value of $\sin\frac{5\pi}{12}$. ⋆⋆ Hint · Answer · Solution

4.5.4 Solve $\cos 2x = \cos x$ for $0 \le x < 2\pi$. ⋆⋆ Hint · Answer · Solution

4.5.5 Evaluate $\arccos(-\frac{1}{2})$, $\arctan(-1)$, $\arcsin(\sin\frac{5\pi}{6})$ and $\sin(\arccos\frac{3}{5})$. ⋆⋆ Hint · Answer · Solution

4.5.6 Find the amplitude, period, horizontal shift and range of $y = 3\sin(2x - \frac{\pi}{3}) + 1$. ⋆⋆ Hint · Answer · Solution

4.5.7 Prove that $\sin(\arccos x) = \sqrt{1-x^2}$ for $x \in [-1,1]$ and that $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$ for $x \in (-1,1)$. ⋆⋆⋆ Hint · Answer · Solution

4.5.8 Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for every $x \in [-1,1]$.
⋆⋆⋆ Hint · Answer · Solution

4.5.9 Write the rotation $R_{3/5,4/5}$ of Definition 3.3.21 as multiplication by a complex number, use it to rotate the point $(1,2)$, and check the result with the matrix of $R_{3/5,4/5}$.
⋆ Hint · Answer · Solution

4.5.10 With the multiplication rules of geometric algebra in the remark on the origin of the trigonometric functions, let $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ and $B = \mathbf{e}_1\mathbf{e}_2$. Show that $\mathbf{v}B = -y\mathbf{e}_1 + x\mathbf{e}_2$, the quarter turn $R_{0,1}(x,y)$. Then show that the rotor with $\theta = \pi$ sends $\mathbf{v}$ to $-\mathbf{v}$.
⋆⋆ Hint · Answer · Solution

4.6 Finite Sums and Binomial Coefficients

Overview

Objectives: Read and write sigma notation; reindex a sum and check the shift on its first term; distinguish the legal splits of a sum from the illegal ones; prove closed forms by induction and evaluate telescoping and geometric sums; simplify factorial ratios with the recursion; expand $(a+b)^n$ by the binomial theorem; keep track of the signs in an alternating sum.
Prerequisites: Mathematical induction (Theorem 1.4.2); the special products (Section 4.3).
Key results: Linearity and range splitting (Theorems 4.6.5 and 4.6.6); the closed forms Eqs. (4.6.6) to (4.6.9); the telescoping sum Eq. (4.6.10); the factorial recursion Eq. (4.6.13); the binomial theorem (Theorem 4.6.14).

Notation

$\sum_{k=m}^{n} a_k$	the sum $a_m + a_{m+1} + \cdots + a_n$
$n!$	the factorial $n(n-1)\cdots 2\cdot 1$, with $0! = 1$
$\binom{n}{k}$	the binomial coefficient

A proof by induction usually concerns a sum or a product with n terms. The area under a curve is first approximated by a finite sum (Chapter 8) and Part IV studies sums with infinitely many terms. All three need a notation for a sum whose number of terms is a variable and a list of the manipulations that such a sum permits. This section sets up that notation, proves the closed forms that recur throughout the book, and develops the factorials and binomial coefficients on which the series of Part IV are built.

4.6.1 Sigma notation

For an indexed list of numbers $a_m, a_{m+1}, \dots, a_n$,

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n. \tag{4.6.1}$$

Here k is the *index*, m the *lower limit*, n the *upper limit* and a_k the *general term*. The general term is the formula that produces each summand once k is set. Formally, Eq. (4.6.1) is defined by recursion on n: the sum up to $n = m$ is a_m and the sum up to $n+1$ is the sum up to n plus a_{n+1}. The index is a bookkeeping device internal to the sum whose name never appears in the answer, as in

$$\sum_{k=1}^{4} k^2 = \sum_{j=1}^{4} j^2 = \sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = 30,$$

with any letter not already in use equally valid as an index. An index used this way is called a *dummy variable*.

When the upper limit falls below the lower one, the sum has no terms at all and its value is

$$\sum_{k=m}^{n} a_k = 0 \qquad \text{whenever } n < m. \tag{4.6.2}$$

The value in Eq. (4.6.2) is a convention chosen so that formulas splitting a sum into pieces stay true when one piece happens to be empty. The partial sum of the first n terms of a series is the empty sum at $n = 0$. The convention gives $s_0 = 0$ and excludes the tempting value $s_0 = a_1$.

4.6.2 Reindexing

Two sums written differently can contain the same terms. Adding a constant j to the index and subtracting it inside the general term leaves every summand where it was,

$$\sum_{k=m}^{n} a_k = \sum_{k=m+j}^{n+j} a_{k-j}, \tag{4.6.3}$$

for any integer j. Read right to left the same identity lowers the limits. The commonest instance starts a sum at 1 that was written from 0:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} a_{n-1}.$$

A power series differentiated term by term loses its constant term and needs its index shifted back to a standard starting value. Hence Sections 11.1 and 11.2 reindex on almost every page. Checking a reindex takes one line: write out the first summand of each side and confirm they agree.

Example 4.6.3 (Index shift checked on the first term)**.** In $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n-1}$ let $k = n - 1$. Then $n = k + 1$ and the lower limit $n = 2$ becomes $k = 1$:

$$\sum_{n=2}^{\infty} \frac{x^n}{n-1} = \sum_{k=1}^{\infty} \frac{x^{k+1}}{k}.$$

For the first term the left side gives $x^2/1$ at $n = 2$ and the right side gives $x^2/1$ at $k = 1$. Agreement at the second term $x^3/2$ as well confirms the shift. As a non-example, writing $\sum_{k=2}^{\infty} x^{k+1}/k$ on the right shifts the general term without shifting the limit: its first term is $x^3/2$ and the term x^2 is lost.

4.6.4 Legal and illegal splits of a sum

Two operations pass through a finite sum untouched.

Theorem 4.6.5 (Linearity of a finite sum)**.** *For numbers a_k, b_k and a constant c,*

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k, \qquad \sum_{k=m}^{n} c\, a_k = c \sum_{k=m}^{n} a_k. \tag{4.6.4}$$

Proof. By induction on $n \geq m$ (Theorem 1.4.2). For $n < m$ both sides are 0 by Eq. (4.6.2). At $n = m$ both statements read $a_m + b_m = a_m + b_m$ and $ca_m = ca_m$. If the first holds up to n, then adding $a_{n+1} + b_{n+1}$ to both sides and regrouping by the associative and commutative laws gives it up to $n + 1$. If the second holds up to n, then adding ca_{n+1} and using the distributive law gives it up to $n + 1$. ∎

Theorem 4.6.6 (Splitting the range of a sum)**.** *For any index p with $m \le p < n$,*

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{p} a_k + \sum_{k=p+1}^{n} a_k. \tag{4.6.5}$$

Proof. By induction on $n > p$. At $n = p+1$ the second sum on the right is a_{p+1} and the identity is the recursive definition of the sum. Passing from n to $n+1$ adds a_{n+1} to the left side and to the second sum on the right. ∎

Theorem 4.6.6 licenses the remark made throughout Part IV that changing finitely many terms of a series cannot change whether it converges: the altered terms all sit in one finite piece with a finite value.

No other operation passes through a sum.

Common misconception

A sum of products is not the product of the sums:

$$\sum_{k=m}^{n} a_k b_k \neq \Big(\sum_{k=m}^{n} a_k\Big)\Big(\sum_{k=m}^{n} b_k\Big).$$

At $n = m+1$ the left side is $a_m b_m + a_{m+1} b_{m+1}$ and the right side contains the two cross terms $a_m b_{m+1} + a_{m+1} b_m$ as well.

Common misconception

A sum of quotients is not the quotient of the sums:

$$\sum_{k=m}^{n} \frac{a_k}{b_k} \neq \frac{\sum_{k=m}^{n} a_k}{\sum_{k=m}^{n} b_k}.$$

Taking $a_k = b_k = 1$ over $m = 1$ to $n = 3$ makes the left side 3 and the right side 1. A *common denominator* does distribute: $\sum_k a_k/c = (\sum_k a_k)/c$ is Eq. (4.6.4) with the constant $1/c$.

Common misconception

Theorem 4.6.5 is stated for a *finite* sum. For an infinite series it applies only once both pieces are known to converge on their own. If $\sum a_k$ and $\sum b_k$ both converge then $\sum(a_k + b_k) = \sum a_k + \sum b_k$. If either diverges, the equation is a statement about a quantity that does not exist. The standard trap is splitting a convergent series into its positive terms and its negative terms: for the alternating harmonic series both halves diverge and the split proves nothing about the convergent series (Example 4.6.17). Theorem 10.6.8 describes the consequences of taking such a series apart.

Later use

Every law and every failure listed in this subsection has an integral counterpart. The index k plays the role of t in $\int_a^x f(t)\,dt$ (Eq. (8.3.1)), a function of x in which no t appears. Splitting the range is the discrete form of additivity of the integral over adjacent intervals (Theorem 8.2.15). Integration by parts (Theorem 9.2.1) is needed because the integral of a product is not the product of the integrals. A common denominator distributes over an

integral as it does over a sum (Eq. (9.1.1)). Section 9.1 lists the legal moves on an integrand as this subsection lists them for a sum.

4.6.7 Closed forms

Four sums should be known by sight.

Theorem 4.6.8 (Power sums)**.** *For every positive integer n and every constant c,*

$$\sum_{k=1}^{n} c = nc, \tag{4.6.6}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \tag{4.6.7}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{4.6.8}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{4.6.9}$$

Proof. The sum Eq. (4.6.6) has n terms equal to c. The other three formulas are Theorem 1.4.6, proved by induction in Section 1.4: the first in the text and the second and third in Problem 1.4.1 and Problem 1.4.2. ∎

Example 4.6.9. By Theorem 4.6.5 and Eq. (4.6.7), $\sum_{k=1}^{n}(2k-1) = 2\cdot\dfrac{n(n+1)}{2} - n = n^2$: the sum of the first n odd numbers is n^2. As a non-example, $\sum_{k=1}^{n} k^2$ is not $\left(\sum_{k=1}^{n} k\right)^2$: at $n = 2$ the first is 5 and the second is 9.

Later use

With a regular partition Eqs. (4.6.7) and (4.6.8) turn a Riemann sum Eq. (8.2.2) into a number. Areas were computed this way before the Fundamental Theorem made it unnecessary (Chapter 8).

A fifth pattern has no closed form of its own and reduces any sum of the right shape to its two end terms. A sum whose general term is a difference of consecutive values of one sequence *telescopes*:

$$\sum_{k=m}^{n} (b_k - b_{k+1}) = b_m - b_{n+1}. \tag{4.6.10}$$

By Theorem 4.6.5 the left side is $\sum_{k=m}^{n} b_k - \sum_{k=m}^{n} b_{k+1}$, and by Eq. (4.6.3) the second sum is $\sum_{k=m+1}^{n+1} b_k$. Splitting off b_m from the first and b_{n+1} from the second by Theorem 4.6.6 leaves two copies of $\sum_{k=m+1}^{n} b_k$ that cancel. Every interior value $b_{m+1}, \dots, b_n$ appears once with each sign.

Taking $b_k = r^k$ gives $(1-r)\sum_{k=0}^{n} r^k = \sum_{k=0}^{n}(r^k - r^{k+1}) = 1 - r^{n+1}$. Dividing by $1-r$ for $r \neq 1$ gives the finite geometric sum

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}. \tag{4.6.11}$$

Section 10.2 treats geometric and telescoping series, with infinitely many terms, as families of their own.

4.6.10 Factorials

For a positive integer n, the *factorial* $n!$ is the product of the integers from 1 to n, together with the convention that the empty product is 1:

$$n! = n(n-1)(n-2)\cdots 2\cdot 1, \qquad 0! = 1. \tag{4.6.12}$$

The choice $0! = 1$ is a convention of the same kind as Eq. (4.6.2). It lets one formula cover a case that would otherwise need special handling. Every Maclaurin series in Section 11.2 has general term $f^{(n)}(0)x^n/n!$ whose value at $n = 0$ is $f(0)x^0/0!$, equal to $f(0)$ exactly when $0! = 1$. With any other value the series would need its first term written out separately from the rest.

Two consequences of Eq. (4.6.12) do most of the work in Part IV: the recursion

$$(n+1)! = (n+1)\,n!, \tag{4.6.13}$$

and the ratio that follows from it at once,

$$\frac{(n+1)!}{n!} = n+1. \tag{4.6.14}$$

Each problem on the ratio test of Section 10.5 uses Eq. (4.6.14) once at the step where most marks are lost. Students cancel the factorials as though $(n+1)!/n!$ were $(n+1)/n$, or expand both products in full and lose the thread. Neither is needed. Expand the larger factorial with Eq. (4.6.13) until the smaller one appears as a factor and then cancel it.

Example 4.6.11 (Factorial ratios by the recursion)**.** Each ratio below is simplified by writing the larger factorial in terms of the smaller with Eq. (4.6.13) and without expanding either product.

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)\,n!}{n!} = (n+2)(n+1),$$

$$\frac{n!}{(n+3)!} = \frac{n!}{(n+3)(n+2)(n+1)\,n!} = \frac{1}{(n+3)(n+2)(n+1)},$$

$$\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)\,(2n)!}{(2n)!} = (2n+2)(2n+1).$$

The last ratio is the pattern of the cosine and sine series where the index of the factorial moves in steps of two. Treat $2n$ as a single block. The factorial above it is two steps up and two factors come out. As a non-example, $(2n)!$ is not $2\cdot n!$: at $n = 2$ the first is 24 and the second is 4.

Growth rates

For any fixed $p > 0$ and any fixed $r > 1$,

$$n^p \ll r^n \ll n! \qquad \text{as } n \to \infty,$$

meaning $\lim_{n\to\infty} \dfrac{n^p}{r^n} = 0$ and $\lim_{n\to\infty} \dfrac{r^n}{n!} = 0$. A power grows more slowly than any exponential and every exponential more slowly than the factorial. The logarithm sits below all three: $\ln n \ll n^p$ for every $p > 0$. Lemma 5.2.25 and Theorem 5.2.26 prove the comparisons of a power with an exponential and with a logarithm. Theorem 10.1.4 proves $r^n/n! \to 0$.

Later use

The growth ordering settles the limit comparisons of Section 10.4 at a glance: a general term with a factorial in its denominator is eventually smaller than one with only a power in its denominator, whatever the first few terms look like. The same ordering explains why the ratio test of Section 10.5 settles every series built from factorials immediately and why the exponential series of Section 11.2 converges for every real x: the factorial in the denominator grows faster than the power in the numerator however large x is.

4.6.12 Binomial coefficients

For integers $0 \le k \le n$, the *binomial coefficient* is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{4.6.15}$$

For example $\binom{4}{2} = \dfrac{24}{2 \cdot 2} = 6$ and $\binom{n}{0} = \binom{n}{n} = 1$.

Lemma 4.6.13 (Pascal's rule)**.** *For integers* $1 \le k \le n$, $\displaystyle\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Proof. Over the common denominator $k!\,(n-k+1)!$, using $k! = k\,(k-1)!$ and $(n-k+1)! = (n-k+1)\,(n-k)!$,

$$\frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} = \frac{n!\,(k + (n-k+1))}{k!\,(n-k+1)!} = \frac{(n+1)!}{k!\,(n+1-k)!}. \qquad \blacksquare$$

Theorem 4.6.14 (Binomial theorem)**.** *For all real* a, b *and every integer* $n \ge 0$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{4.6.16}$$

Proof. By induction on n. At $n = 0$ both sides are 1. Suppose (4.6.16) holds for n. Then

$$(a+b)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}.$$

By Eq. (4.6.3) the second sum is $\sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^k$. Splitting off the term $k = 0$ of the first sum and $k = n+1$ of the second (Theorem 4.6.6) and combining the rest by Theorem 4.6.5 and Lemma 4.6.13 gives

$$a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k. \qquad \blacksquare$$

In particular Eq. (4.3.3) is the case $n = 2$ of (4.6.16) read with $\binom{2}{0}, \binom{2}{1}, \binom{2}{2} = 1, 2, 1$. Cancelling the $(n-k)!$ in Eq. (4.6.15) with the tail of $n!$ leaves a form that never mentions a factorial of a difference,

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}, \tag{4.6.17}$$

a quotient of exactly k descending factors by $k!$.

Equation (4.6.17) is the form used in Section 11.4 because its right-hand side requires only that n be a number: the descending product $n(n-1)\cdots(n-k+1)$ makes sense for a fractional

or negative n even though $n!$ on its own does not. For any real m and any integer $k \geq 1$ the definition extends to

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}, \qquad \binom{m}{0} = 1, \tag{4.6.18}$$

the coefficient on which the binomial series of Section 11.4 is built. When m is a nonnegative integer, the product in Eq. (4.6.18) contains the factor $m - m = 0$ once k exceeds m. Every coefficient beyond the mth then vanishes and the series stops at the finite expansion of $(1+x)^m$. When m is not a nonnegative integer, no factor is ever zero and the expansion never terminates.

Example 4.6.15 (Binomial coefficient at a fractional upper index)**.** With $m = \frac{1}{2}$, Eq. (4.6.18) gives

$$\binom{1/2}{1} = \frac{1/2}{1} = \frac{1}{2}, \qquad \binom{1/2}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} = \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2} = -\frac{1}{8},$$

$$\binom{1/2}{3} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} = \frac{3/8}{6} = \frac{1}{16}.$$

These first coefficients of $\sqrt{1+x}$ alternate in sign from the second onwards because every factor after the first is negative.

4.6.16 Sign bookkeeping

The sign of an alternating series comes from a power of -1 in one of two standard forms that differ in which terms they make negative:

$$(-1)^n = \begin{cases} +1, & n \text{ even}, \\ -1, & n \text{ odd}, \end{cases} \qquad (-1)^{n+1} = \begin{cases} -1, & n \text{ even}, \\ +1, & n \text{ odd}. \end{cases} \tag{4.6.19}$$

Thus $\sum_{n=1}^{\infty}(-1)^n a_n$ opens with $-a_1$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ opens with $+a_1$. The two differ only by an overall factor of -1. The choice affects a numerical answer and leaves convergence unchanged.

Shifting the index flips the sign according to

$$(-1)^{n+1} = (-1)\,(-1)^n. \tag{4.6.20}$$

This is the one rule needed to keep the signs of a reindexed alternating series as they were before the shift. The check is the same as for any reindex: read off the first term on both sides and confirm the signs agree.

Example 4.6.17 (Illegal split of a series)**.** A tempting first move on $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is to gather the positive terms and the negative terms separately,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \stackrel{?}{=} \left(1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots\right) - \left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\cdots\right),$$

and then to cancel or compare the two bracketed pieces. This move is illegal because Theorem 4.6.5 splits an infinite series only when both pieces converge and here neither does. The second bracket is $\frac{1}{2}$ times the harmonic series that Section 10.2 shows to diverge. The first bracket is termwise larger than the second and diverges too. The written equation sets a finite number equal to a difference of two infinities and asserts nothing.

The series on the left converges by the alternating series test of Section 10.6 and has the value $\ln 2$. No information about that value can be recovered from the split above. By Theorem 10.6.8 a reordering of this particular series can produce any real number.

Study guide

Before manipulating a sum, write out its first two terms. A reindex is correct when the first terms agree on both sides. A sign convention is correct when the first sign agrees. Only linearity and range splitting pass through a finite sum. They pass through an infinite series only when every piece converges. Prove a closed form by induction: check $n = 1$ and then add the next term to the formula for n and factor. Simplify a factorial ratio by expanding the larger factorial with Eq. (4.6.13) until the smaller one appears. Expand $(a+b)^n$ by Theorem 4.6.14. Use the descending form Eq. (4.6.18) when the upper index is not a nonnegative integer.

Worked Example 4.6.1

Find a closed form for $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

By Proposition 4.2.20, $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. The general term is $b_k - b_{k+1}$ with $b_k = 1/k$. By Eq. (4.6.10),

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = b_1 - b_{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Worked Example 4.6.2

Find the coefficient of x^3 in $(2x-1)^5$.

By Theorem 4.6.14 with $a = 2x$ and $b = -1$, the term containing x^3 has $n - k = 3$, that is $k = 2$:

$$\binom{5}{2}(2x)^3(-1)^2 = 10 \cdot 8x^3 = 80x^3.$$

The coefficient is 80.

Problems

4.6.1 Rewrite $\displaystyle\sum_{n=3}^{\infty}(n-2)x^n$ as a sum whose index starts at 0, and check the first term.

⋆ Hint · Answer · Solution

4.6.2 Find a closed form for $\displaystyle\sum_{k=1}^{n}(2k-1)^3$, the sum of the cubes of the first n odd numbers.

⋆⋆ Hint · Answer · Solution

4.6.3 Find a closed form for $\displaystyle\sum_{k=1}^{n} k(k+1)$. ⋆⋆ Hint · Answer · Solution

4.6.4 Simplify $\dfrac{(2n+1)!}{(2n-1)!}$, $\dfrac{n!}{(n-2)!}$ for $n \geq 2$, and $\dfrac{((n+1)!)^2}{n!\,(n+2)!}$.

⋆⋆ Hint · Answer · Solution

4.6.5 Use Eq. (4.6.18) to show that $\binom{-1}{k} = (-1)^k$ for every integer $k \geq 0$, and compute $\binom{-2}{3}$. ⋆⋆ Hint · Answer · Solution

4.6.6 Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for $n \geq 0$ and $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n \geq 1$.

⋆⋆⋆ Hint · Answer · Solution

4.6.7 Prove that $\left(1 + \frac{1}{n}\right)^n < 3$ for every integer $n \geq 1$. ⋆⋆⋆ Hint · Answer · Solution

4.6.8 Evaluate $\displaystyle\sum_{k=0}^{9} 2^k$ and find a closed form for $\displaystyle\sum_{k=1}^{n} 3^{-k}$. ⋆⋆ Hint · Answer · Solution

Chapter 5

Limits and Continuity

Every theorem of the differential and integral calculus is a statement about a limit. The derivative is the limit of a difference quotient, the integral is the limit of a sum, and the sum of an infinite series is the limit of its partial sums. Before any of these can be proved, the word *limit* needs a meaning precise enough to argue with: a statement with quantifiers that can be checked, negated and combined, in the manner of Section 1.2.

The chapter builds that meaning in two steps. Limits of sequences come first because a sequence approaches its limit in only one way, through the growth of its index. Limits of functions follow and are reduced to sequences by Theorem 5.2.7. Continuity is then a limit that agrees with a value. On a closed interval the completeness of the real numbers (Assumption 2.4.5) turns continuity into the three theorems used throughout the book: the Intermediate Value Theorem, the Extreme Value Theorem and the Heine–Cantor theorem on uniform continuity.

The chapter also completes constructions that Chapter 4 began. Continuity and the Intermediate Value Theorem show that every exponential a^x maps $\mathbb{R}$ onto $(0,\infty)$, prove the power law $(a^x)^y = a^{xy}$ for irrational y (Theorem 5.4.12) and give $e^x = \lim(1+x/n)^n$ (Theorem 5.4.15). They also prove that the logarithm, roots and inverse trigonometric functions are continuous.

The definitions use the absolute value only as a distance. Section 3.6 replaced it by a norm. The limit of a sequence and continuity at a point are therefore stated in a normed space as well, at the end of Sections 5.1 and 5.3, with $\mathbb{R}$ and $|\cdot|$ as the first case. Every proof there is the real proof with $\|\cdot\|$ in place of $|\cdot|$.

5.1 Limits of Sequences

Overview

Objectives: Read a sequence as a function on the positive integers; state and use the ε–N definition of convergence (Definition 5.1.3); prove that a limit is unique and that a convergent sequence is bounded; prove and apply the limit laws (Theorem 5.1.9) and the Sandwich Theorem (Theorem 5.1.11); prove convergence from monotonicity and completeness (Theorem 5.1.18); show that $\left(1+\frac{1}{n}\right)^n$ converges (Theorem 5.1.20); extract a convergent subsequence from a bounded sequence (Theorem 5.1.25); state convergence and the Cauchy property in a normed space (Definitions 5.1.30 and 5.1.34) and give a Cauchy sequence with no limit (Example 5.1.36).

Prerequisites: Negating quantified statements (Section 1.2); proof by induction (Theorem 1.4.2); the triangle inequality (Lemma 2.3.8); the supremum (Definition 2.4.3) and the completeness axiom (Assumption 2.4.5); the Archimedean property (Theorem 2.4.10); Bernoulli's inequality (Theorem 2.3.11); norms and the equivalence of norms on $\mathbb{R}^m$ (Definitions 3.6.2 and 3.6.6 and Theorem 3.6.15).

Key results: The definition Eq. (5.1.1); uniqueness and boundedness (Theorems 5.1.6 and 5.1.7); the limit laws (Theorem 5.1.9); the Sandwich Theorem (Theorem 5.1.11); $r^n \to 0$

for $|r| < 1$ (Proposition 5.1.14); the Monotone Convergence Theorem (Theorem 5.1.18); the existence of e (Theorem 5.1.20); the Bolzano–Weierstrass theorem (Theorem 5.1.25); the Cauchy criterion (Theorem 5.1.28); limits in a normed space (Theorem 5.1.31); Cauchy sequences in $\mathbb{R}^m$ converge (Proposition 5.1.35).

Notation

$\{a_n\}$	the sequence whose nth term is a_n
$a_n \to L$	the sequence $\{a_n\}$ converges to L (Definition 5.1.3)
$\lim_{n\to\infty} a_n$	the limit of a convergent sequence
$\{a_{n_k}\}$	a subsequence of $\{a_n\}$, with $n_1 < n_2 < \cdots$
e	the limit of $(1 + \frac{1}{n})^n$ (Theorem 5.1.20)
$\mathbf{v}_n \to \mathbf{v}$	convergence in a normed space (Definition 5.1.30)

The decimal expansion $1,\ 1.4,\ 1.41,\ 1.414, \ldots$ of $\sqrt{2}$ is a list of rational numbers that approach $\sqrt{2}$ without reaching it. The same question arises for every list of this kind produced by a calculator, a numerical method or an infinite series: does the list settle down to a single number and, if so, to which one? A sequence is the mathematical name for such a list.

Definition 5.1.1 (Sequence)**.** A *sequence* of real numbers is a function a whose domain is the set of positive integers, written $\{a_n\}$ or $a_1, a_2, a_3, \ldots$ with a_n in place of $a(n)$. The number a_n is the *nth term*. A sequence may be given by a formula in n, as in $a_n = 1/n$, or by a *recursion* that names the first term and builds each later term from its predecessors, as in $a_1 = 1$ and $a_{n+1} = 1 + a_n/2$.

Like any function on the positive integers (Definition 1.5.8), a sequence records the order of its terms and allows repetition. The sequence $(-1)^n$ and its set of terms $\{1, -1\}$ are therefore different objects. Starting the index at 1 is a convention that changes nothing when a sequence is indexed from $n = 0$ instead.

5.1.2 Convergence

The informal phrase "a_n gets close to L" has two defects. Every term of $a_n = 1/n$ is close to -0.001 in some sense. The terms of $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$ come as close to $L = 0$ as we like and then move away again. The definition repairs both by letting the tolerance be arbitrary and by asking only that the terms *eventually* stay within it.

Definition 5.1.3 (Limit of a sequence)**.** The sequence $\{a_n\}$ *converges* to the number L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, if for every $\varepsilon > 0$ there is an integer N such that

$$n > N \implies |a_n - L| < \varepsilon. \tag{5.1.1}$$

A sequence that converges to no number *diverges*. It *diverges to infinity*, written $a_n \to \infty$, if for every M there is an N with $a_n > M$ for all $n > N$. It diverges to $-\infty$ if $-a_n \to \infty$.

Read (5.1.1) as a challenge and a response. A challenger names a tolerance ε, however small. The response is an index N beyond which every term lies in the band $(L-\varepsilon, L+\varepsilon)$ of Fig. 5.1.1. The order of the quantifiers matters: N is chosen after ε and may depend on it. Because altering

$a_1, \dots, a_{1000}$ changes nothing beyond the index $N = 1000$, no finite number of terms can decide a limit.

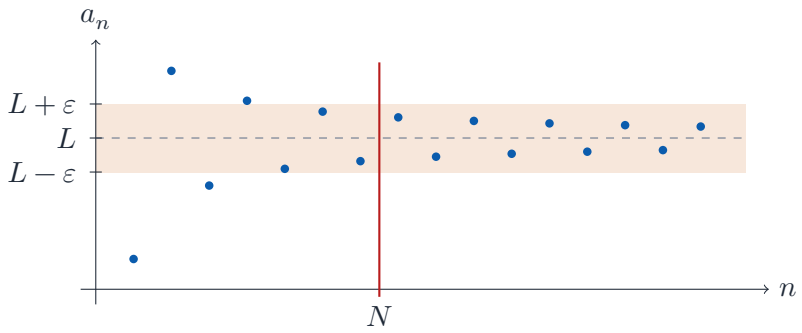


Figure 5.1.1: Convergence of a sequence in one picture: beyond the index N every point (n, a_n) lies inside the band of half-width ε about L. A smaller ε narrows the band and in general moves N to the right.

Example 5.1.4. The sequence $a_n = 1/n$ converges to 0. Given $\varepsilon > 0$ the Archimedean property (Theorem 2.4.10) supplies an integer $N > 1/\varepsilon$. For $n > N$ we have $0 < 1/n < 1/N < \varepsilon$ and hence $|a_n - 0| < \varepsilon$.

Example 5.1.5. The sequence $a_n = (-1)^n$ diverges. Suppose that $a_n \to L$ and take $\varepsilon = 1$ in (5.1.1). For some N, every $n > N$ has $|a_n - L| < 1$. Choosing an even $n > N$ and the odd index $n + 1$, the triangle inequality (Lemma 2.3.8) gives

$$2 = |a_n - a_{n+1}| \le |a_n - L| + |L - a_{n+1}| < 1 + 1 = 2,$$

a contradiction. Every term lies in $[-1, 1]$. The sequence is bounded and still has no limit.

Theorem 5.1.6 (Uniqueness of limits)**.** *A sequence converges to at most one number.*

Proof. Suppose $a_n \to L$ and $a_n \to M$ with $L \neq M$ and put $\varepsilon = \frac{1}{2}|L - M| > 0$. There are indices N_1 and N_2 with $|a_n - L| < \varepsilon$ for $n > N_1$ and $|a_n - M| < \varepsilon$ for $n > N_2$. For any n larger than both,

$$|L - M| \le |L - a_n| + |a_n - M| < 2\varepsilon = |L - M|.$$

No number is less than itself. ∎

Uniqueness justifies writing $\lim_{n\to\infty} a_n$ for *the* limit.

Theorem 5.1.7 (Convergent sequences are bounded)**.** *If $a_n \to L$, there is a number B with $|a_n| \le B$ for every n.*

Proof. With $\varepsilon = 1$ there is an N such that $|a_n - L| < 1$ for $n > N$. For those n the triangle inequality gives $|a_n| \le |a_n - L| + |L| < 1 + |L|$. The number $B = \max\{|a_1|, \dots, |a_N|, 1 + |L|\}$ bounds these terms together with the finitely many remaining terms $a_1, \dots, a_N$. ∎

The contrapositive of Theorem 5.1.7 is a quick test for divergence: an unbounded sequence such as $a_n = n$ or $a_n = (-2)^n$ diverges. The sequence $(-1)^n$ shows that the converse fails.

5.1.8 Limit laws

The limit laws assemble complicated limits from simple ones. Their proofs show how the tolerance ε is shared between the pieces.

Theorem 5.1.9 (Algebra of sequence limits)**.** *If $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to L + M$, $a_n b_n \to LM$ and $c\,a_n \to cL$ for any constant c. If $M \neq 0$, then $b_n \neq 0$ for every n beyond some index N_1, and the sequence $\{a_n/b_n\}_{n>N_1}$ converges to L/M.*

Proof. Let $\varepsilon > 0$.
(i) Sum. Choose N_1 and N_2 with $|a_n - L| < \varepsilon/2$ for $n > N_1$ and $|b_n - M| < \varepsilon/2$ for $n > N_2$. For $n > N = \max\{N_1, N_2\}$,

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

(ii) Product. By Theorem 5.1.7 some $B > 0$ has $|a_n| \leq B$ for every n. Adding and subtracting $a_n M$,

$$|a_n b_n - LM| = |a_n(b_n - M) + M(a_n - L)| \leq B|b_n - M| + |M|\,|a_n - L|. \tag{5.1.2}$$

Choose N_1 with $|b_n - M| < \varepsilon/(2B)$ for $n > N_1$ and N_2 with $|a_n - L| < \varepsilon/(2|M| + 1)$ for $n > N_2$. For $n > \max\{N_1, N_2\}$ the right-hand side of (5.1.2) is less than $\varepsilon/2 + \varepsilon/2 = \varepsilon$. The constant multiple is the product with the constant sequence $b_n = c$ of limit c.
(iii) Quotient. By the product law it suffices to show $1/b_n \to 1/M$. Taking $\varepsilon = |M|/2$ in the definition gives an N_1 such that every $n > N_1$ has $|b_n - M| < |M|/2$ and hence, by the triangle inequality, $|b_n| \geq |M| - |M - b_n| > |M|/2$. For those n the term b_n is nonzero and

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n|\,|M|} < \frac{2}{|M|^2}\,|b_n - M|. \tag{5.1.3}$$

Choosing N_2 with $|b_n - M| < \frac{1}{2}|M|^2\varepsilon$ for $n > N_2$ makes the right-hand side of (5.1.3) less than ε for $n > \max\{N_1, N_2\}$. The first step showed $b_n \neq 0$ for $n > N_1$. The quotient a_n/b_n is therefore defined for those n and the finitely many earlier terms play no part in its limit. ∎

Inequalities between terms pass to the limit in the weak form.

Theorem 5.1.10 (Limits preserve weak inequalities)**.** *If $a_n \to L$, $b_n \to M$ and $a_n \leq b_n$ for every n beyond some index, then $L \leq M$. In particular, if every term of a convergent sequence lies in a closed interval $[\alpha, \beta]$, so does its limit.*

Proof. The sequence $b_n - a_n$ converges to $M - L$ by Theorem 5.1.9 and is eventually nonnegative. If $M - L < 0$, then $\varepsilon = L - M$ gives an n beyond the index with $|(b_n - a_n) - (M - L)| < L - M$. That inequality forces $b_n - a_n < 0$, contrary to hypothesis. The second statement applies the first to the constant sequences α and β. ∎

Strict inequalities in the limit

Strict inequalities become weak in the limit. Every term of $1/n$ is positive while the limit 0 is not. Theorem 5.1.10 concludes $L \leq M$ even when $a_n < b_n$ for every n.

Theorem 5.1.11 (Sandwich Theorem for sequences)**.** *If $a_n \leq b_n \leq c_n$ for all n beyond some index, and $a_n \to L$ and $c_n \to L$, then $b_n \to L$.*

Proof. Let $\varepsilon > 0$. Choose N_1 with $|a_n - L| < \varepsilon$ for $n > N_1$ and N_2 with $|c_n - L| < \varepsilon$ for $n > N_2$. Let N be the largest of N_1, N_2 and the index beyond which the inequalities hold. For $n > N$,

$$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon,$$

and therefore $|b_n - L| < \varepsilon$. ∎

Theorem 5.1.11 proves the convergence of $\{b_n\}$ that Theorem 5.1.10 assumes.

Corollary 5.1.12 (Domination by a null sequence)**.** *If $|b_n| \le c_n$ for every n and $c_n \to 0$, then $b_n \to 0$.*

Proof. Apply Theorem 5.1.11 to $-c_n \le b_n \le c_n$. ∎

By Corollary 5.1.12 the sequence $(-1)^n/n$ tends to 0 because $|(-1)^n/n| = 1/n \to 0$.

5.1.13 Two basic limits

Two limits recur so often that we prove them once. Both rest on Bernoulli's inequality $(1+d)^n \ge 1 + nd$ for $d \ge -1$ (Theorem 2.3.11).

Proposition 5.1.14 (Geometric sequences)**.** *If $|r| < 1$, then $r^n \to 0$. If $|r| > 1$, then $\{r^n\}$ is unbounded and diverges.*

Proof. The case $r = 0$ is immediate. For $0 < |r| < 1$ write $1/|r| = 1 + d$ with $d > 0$. By Bernoulli's inequality (2.3.5), $(1 + d)^n \ge 1 + nd > nd$. Hence

$$0 < |r^n| = \frac{1}{(1+d)^n} < \frac{1}{nd}. \tag{5.1.4}$$

Since $1/(nd) \to 0$ by the example after Definition 5.1.3 and the constant multiple law, Corollary 5.1.12 gives $r^n \to 0$. For $|r| > 1$ write $|r| = 1 + d$ with $d > 0$. By the Archimedean property the lower bound in $|r^n| \ge 1 + nd$ exceeds any given number for large n. The sequence is therefore unbounded and diverges by Theorem 5.1.7. ∎

Proposition 5.1.15 (Roots of a fixed number)**.** *If $a > 0$ and $a^{1/n}$ denotes the positive nth root of a, then $a^{1/n} \to 1$.*

Proof. The positive nth root exists by Theorem 4.4.6. Suppose first that $a \ge 1$ and write $a^{1/n} = 1 + d_n$. By Lemma 4.4.11 (trivially when $a = 1$),

$$0 \le d_n \le \frac{a - 1}{n}. \tag{5.1.5}$$

Since the right-hand side of (5.1.5) tends to 0, Theorem 5.1.11 gives $d_n \to 0$ and hence $a^{1/n} = 1 + d_n \to 1$. If $0 < a < 1$, then $1/a > 1$ and $a^{1/n} = 1/(1/a)^{1/n} \to 1/1 = 1$ by the quotient law. ∎

Worked Example 5.1.1

Prove from Definition 5.1.3 that $a_n = \dfrac{2n + 1}{n + 3}$ converges to 2 and find an N that works for $\varepsilon = 0.01$.

The quantity to control is the distance from a_n to the proposed limit:

$$|a_n - 2| = \left|\frac{2n+1-2(n+3)}{n+3}\right| = \frac{5}{n+3} < \frac{5}{n}.$$

Given $\varepsilon > 0$, choose by the Archimedean property an integer $N \geq 5/\varepsilon$. Then (5.1.1) holds with $L = 2$: every $n > N$ satisfies $|a_n - 2| < 5/n < 5/N \leq \varepsilon$.
For $\varepsilon = 0.01$ the choice $N = 500$ works. Since the exact condition $5/(n+3) < 0.01$ reads $n > 497$, the value $N = 497$ also works. Because the definition asks for some N and never for the smallest one, a proof is free to use the simpler bound $5/n$ in place of $5/(n+3)$.

Terms that reach the limit

A convergent sequence may equal its limit infinitely often and may move away from it infinitely often. The sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots$ equals its limit 0 at every odd index and moves away from it at every even index. The definition (5.1.1) says nothing about monotone approach and is satisfied with $L = 0$.

5.1.16 Monotone sequences and completeness

Every limit so far was guessed first and then verified. For a sequence such as $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 2/a_n)$ there is no candidate in sight. An argument is needed that produces the limit from the sequence itself. Completeness supplies it.

Definition 5.1.17 (Monotone and bounded)**.** A sequence is *monotone* if it is either *nondecreasing*, with $a_n \leq a_{n+1}$ for every n, or *nonincreasing*, with $a_n \geq a_{n+1}$ for every n. It is *bounded above* if some M has $a_n \leq M$ for every n, and *bounded below* if some m has $a_n \geq m$ for every n.

The sequence $1 - 1/n$ is nondecreasing and bounded above by 1. The sequence $(-1)^n$ is bounded without being monotone. The sequence $a_n = n$ is monotone without being bounded above.

Theorem 5.1.18 (Monotone Convergence Theorem)**.** *A nondecreasing sequence that is bounded above converges to the supremum of its set of terms. A nonincreasing sequence that is bounded below converges to the infimum of its terms.*

Proof. Let $\{a_n\}$ be nondecreasing and bounded above with S the set of its terms. Then S is nonempty and bounded above and therefore has, by Assumption 2.4.5, a supremum $L = \sup S$ with $a_n \leq L$ for every n. Let $\varepsilon > 0$. Since $L - \varepsilon$ is smaller than L and hence not an upper bound for S (Lemma 2.4.4), some term a_N satisfies $a_N > L - \varepsilon$. Because the sequence is nondecreasing, every $n > N$ satisfies

$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$$

and hence $|a_n - L| < \varepsilon$. The nonincreasing case follows by applying this to the sequence $\{-a_n\}$ with supremum $-\inf S$. ∎

The proof of Theorem 5.1.18 uses completeness. Over the rational numbers the theorem fails: Worked Example 5.1.2 builds a nonincreasing sequence of positive rationals whose only possible limit is the irrational number $\sqrt{2}$ (Theorem 2.1.3).

Worked Example 5.1.2

Let $a_1 = 2$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$. Compute a_2, a_3 and a_4, show that the sequence converges, and find its limit.
Directly, $a_2 = \frac{1}{2}(2+1) = \frac{3}{2}$, $a_3 = \frac{1}{2}(\frac{3}{2} + \frac{4}{3}) = \frac{17}{12}$ and $a_4 = \frac{1}{2}(\frac{17}{12} + \frac{24}{17}) = \frac{577}{408} \approx 1.414216$.
1. Positive and bounded below. By induction every a_n is a positive rational. For every n,

$$a_{n+1}^2 - 2 = \frac{1}{4}\left(a_n^2 + 4 + \frac{4}{a_n^2}\right) - 2 = \frac{1}{4}\left(a_n - \frac{2}{a_n}\right)^2 \geq 0.$$

Together with $a_1^2 = 4$ this gives $a_n^2 \geq 2$ for every n.
2. Nonincreasing. Using $a_n^2 \geq 2$,

$$a_n - a_{n+1} = \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n} \geq 0.$$

By Theorem 5.1.18 the sequence converges to some L. Since $a_n^2 \geq 2$ and $a_n > 0$ give $a_n \geq 1$, Theorem 5.1.10 gives $L \geq 1$. Since the shifted sequence $\{a_{n+1}\}$ also converges to L, the limit laws applied to the recursion give $L = \frac{1}{2}(L + 2/L)$, that is $L^2 = 2$. Hence $a_n \to \sqrt{2}$. The order of the argument matters: the recursion identifies the limit only after convergence is proved.

5.1.19 Number e

Section 4.4 defines the number e as the limit of $(1 + \frac{1}{n})^n$. The definition is legitimate only once the limit is known to exist. Theorem 5.1.18 proves existence although the limit has no closed form.

Theorem 5.1.20 (Existence of e)**.** *The sequence $a_n = (1 + \frac{1}{n})^n$ is increasing and satisfies $2 \leq a_n < 3$ for every n. It therefore converges to a limit*

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{5.1.6}$$

with $2 < e \leq 3$.

Proof. The binomial expansion of $(1 + 1/n)^n$, with the coefficients Eq. (4.6.17), gives

$$a_n = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = \sum_{k=0}^{n}\frac{1}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k} = \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right), \tag{5.1.7}$$

where the empty product for $k = 0$ and $k = 1$ is 1.
1. Monotonicity. Passing from n to $n+1$ in (5.1.7) replaces each factor $1 - j/n$ by the larger factor $1 - j/(n+1)$. None of the first $n+1$ terms of the sum decreases. The expansion of a_{n+1} also has a further positive term with $k = n+1$. Hence $a_n < a_{n+1}$.
2. Boundedness. Every factor $1 - j/n$ in (5.1.7) lies in $[0, 1]$. Induction gives $k! \geq 2^{k-1}$ for $k \geq 1$ because each factor $2, 3, \ldots, k$ is at least 2. Therefore

$$a_n \leq 1 + \sum_{k=1}^{n}\frac{1}{k!} \leq 1 + \sum_{k=1}^{n}\frac{1}{2^{k-1}} = 1 + \left(2 - \frac{1}{2^{n-1}}\right) < 3, \tag{5.1.8}$$

where the geometric sum is the telescoping sum Eq. (4.6.10) with $b_k = 2^{2-k}$. Since $a_1 = 2$ and the sequence increases, $a_n \geq 2$.

By Theorem 5.1.18 the sequence converges to its supremum e. Theorem 5.1.10 and $a_n < 3$ give $e \leq 3$. As the supremum of the terms, $e \geq a_2 = \frac{9}{4} > 2$. ∎

The value is $e = 2.71828\ldots$. The expansion in the proof also suggests the series $e = \sum_{k\geq 0} 1/k!$ established in Section 11.3.

5.1.21 Subsequences and the Bolzano–Weierstrass theorem

Even a divergent bounded sequence has a convergent part. The sequence $(-1)^n$ has terms that settle at 1 if we keep only the even indices and at -1 if we keep only the odd ones.

Definition 5.1.22 (Subsequence)**.** Given a sequence $\{a_n\}$ and a strictly increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$, written $\{a_{n_k}\}$ and indexed by k, is a *subsequence* of $\{a_n\}$.

The even-indexed terms a_{2k} form a subsequence, with $n_k = 2k$. The list $a_2, a_1, a_3, \ldots$ is not a subsequence because its indices do not increase. Since $n_1 \geq 1$ and each index exceeds its predecessor, induction gives $n_k \geq k$ for every k.

Theorem 5.1.23 (Subsequences of a convergent sequence)**.** *If $a_n \to L$, then every subsequence $\{a_{n_k}\}$ also converges to L.*

Proof. Given $\varepsilon > 0$, choose N with $|a_n - L| < \varepsilon$ for $n > N$. For $k > N$ we have $n_k \geq k > N$ and hence $|a_{n_k} - L| < \varepsilon$. ∎

By Theorem 5.1.23 a sequence with two subsequences converging to different limits diverges. This is the quickest proof that $(-1)^n$ diverges.

Lemma 5.1.24 (Monotone subsequences)**.** *Every sequence has a monotone subsequence.*

Proof. Call an index m a *peak* if $a_m \geq a_n$ for every $n > m$. If there are infinitely many peaks $m_1 < m_2 < \cdots$, then $a_{m_1} \geq a_{m_2} \geq \cdots$ and the peaks give a nonincreasing subsequence. Otherwise there is an index N beyond which no index is a peak. Put $n_1 = N + 1$. Because n_1 is not a peak, some $n_2 > n_1$ has $a_{n_2} > a_{n_1}$. Because n_2 is not a peak, some $n_3 > n_2$ has $a_{n_3} > a_{n_2}$. Continuing inductively produces an increasing subsequence. ∎

Theorem 5.1.25 (Bolzano–Weierstrass)**.** *Every bounded sequence has a convergent subsequence. If every term lies in $[\alpha, \beta]$, the limit of the subsequence lies in $[\alpha, \beta]$.*

Proof. By Lemma 5.1.24 the sequence has a monotone subsequence. Being bounded together with the whole sequence, it converges by Theorem 5.1.18. The statement about $[\alpha, \beta]$ is Theorem 5.1.10. ∎

The Bolzano–Weierstrass theorem is the form of completeness used in Sections 5.4 and 5.5 to prove the Extreme Value Theorem and the Heine–Cantor theorem. The theorem asserts only that at least one convergent subsequence exists. A bounded sequence may have many, with different limits.

5.1.26 Cauchy sequences

Definition 5.1.3 mentions the limit L. A criterion that refers only to the terms themselves is useful when no candidate for L is available and the sequence is not monotone.

Definition 5.1.27 (Cauchy sequence)**.** A sequence $\{a_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an N such that $|a_m - a_n| < \varepsilon$ for all $m, n > N$.

Theorem 5.1.28 (Cauchy criterion)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. If $a_n \to L$, choose N with $|a_n - L| < \varepsilon/2$ for $n > N$. For $m, n > N$ the triangle inequality gives $|a_m - a_n| \le |a_m - L| + |L - a_n| < \varepsilon$.

Conversely, let $\{a_n\}$ be Cauchy. With $\varepsilon = 1$ there is an N such that $|a_n - a_{N+1}| < 1$ for $n > N$. The argument of Theorem 5.1.7 then shows that the sequence is bounded. By Theorem 5.1.25 a subsequence converges to some L. Given $\varepsilon > 0$, choose N' with $|a_m - a_n| < \varepsilon/2$ for $m, n > N'$ and then k with $n_k > N'$ and $|a_{n_k} - L| < \varepsilon/2$. For $n > N'$,

$$|a_n - L| \le |a_n - a_{n_k}| + |a_{n_k} - L| < \varepsilon. \qquad \blacksquare$$

5.1.29 Sequences in a normed space

The definition (5.1.1) uses the absolute value only to measure the distance $|a_n - L|$ between a term and the limit. A norm (Definition 3.6.2) measures the distance $\|\mathbf{u} - \mathbf{v}\|$ between two vectors (Definition 3.6.12). Putting that distance in place of $|a_n - L|$ defines convergence for vectors, for complex numbers and for functions with one definition. The proofs of this section that use only the triangle inequality and the rule $|ca| = |c|\,|a|$ carry over unchanged.

Definition 5.1.30 (Convergence in a normed space)**.** Let V be a normed space (Definition 3.6.3). A sequence $\{\mathbf{v}_n\}$ in V *converges* to $\mathbf{v} \in V$, written $\mathbf{v}_n \to \mathbf{v}$, if the real sequence $\|\mathbf{v}_n - \mathbf{v}\|$ tends to 0: for every $\varepsilon > 0$ there is an N with $\|\mathbf{v}_n - \mathbf{v}\| < \varepsilon$ for every $n > N$.

With $V = \mathbb{R}$ and the absolute value as the norm, Definition 5.1.30 is Definition 5.1.3. Replacing a norm by an equivalent one (Definition 3.6.14) changes no limit: if $\|\mathbf{v}\|' \le M\|\mathbf{v}\|$, then $\|\mathbf{v}_n - \mathbf{v}\| \to 0$ forces $\|\mathbf{v}_n - \mathbf{v}\|' \to 0$ by Corollary 5.1.12.

Theorem 5.1.31 (Limits in a normed space)**.** *Let $\mathbf{v}_n \to \mathbf{v}$ and $\mathbf{w}_n \to \mathbf{w}$ in a normed space V and let $c_n \to c$ in $\mathbb{R}$. Then the limit $\mathbf{v}$ is unique, the sequence $\|\mathbf{v}_n\|$ is bounded, $\mathbf{v}_n + \mathbf{w}_n \to \mathbf{v} + \mathbf{w}$, $c_n\mathbf{v}_n \to c\mathbf{v}$ and $\|\mathbf{v}_n\| \to \|\mathbf{v}\|$.*

Proof. Each claim is the real proof of this section with the norm in place of the absolute value. The triangle inequality and homogeneity of the norm (Definition 3.6.2) give

$$\|\mathbf{v} - \mathbf{v}'\| \le \|\mathbf{v} - \mathbf{v}_n\| + \|\mathbf{v}_n - \mathbf{v}'\|, \qquad \|(\mathbf{v}_n + \mathbf{w}_n) - (\mathbf{v} + \mathbf{w})\| \le \|\mathbf{v}_n - \mathbf{v}\| + \|\mathbf{w}_n - \mathbf{w}\| \tag{5.1.9}$$

$$\|c_n\mathbf{v}_n - c\mathbf{v}\| \le |c_n|\,\|\mathbf{v}_n - \mathbf{v}\| + |c_n - c|\,\|\mathbf{v}\|, \qquad \big|\|\mathbf{v}_n\| - \|\mathbf{v}\|\big| \le \|\mathbf{v}_n - \mathbf{v}\|. \tag{5.1.10}$$

If $\mathbf{v}_n \to \mathbf{v}'$ as well, the first inequality of (5.1.9) and Theorem 5.1.11 give $\|\mathbf{v} - \mathbf{v}'\| = 0$ and hence $\mathbf{v} = \mathbf{v}'$ by (N1). The right-hand sides of the second inequality of (5.1.9) and the first of (5.1.10) tend to 0 by Theorem 5.1.9, using that $\{c_n\}$ is bounded (Theorem 5.1.7). The last inequality of (5.1.10) is Proposition 3.6.4(3). It gives $\|\mathbf{v}_n\| \to \|\mathbf{v}\|$ and hence the boundedness of $\|\mathbf{v}_n\|$ by Theorem 5.1.7. $\blacksquare$

Example 5.1.32 (Coordinates in $\mathbb{R}^m$ and in $\mathbb{C}$)**.** In $\mathbb{R}^m$ with the norm $\|\cdot\|_\infty$ of Definition 3.6.6, every coordinate satisfies $|x_i - y_i| \le \|\mathbf{x}-\mathbf{y}\|_\infty \le \sum_j |x_j - y_j|$. A sequence of vectors therefore converges if and only if each coordinate sequence converges, by Corollary 5.1.12 and the sum law. By Theorem 3.6.15 the same holds for every norm on $\mathbb{R}^m$. The complex numbers with $|z|$ are the plane $\mathbb{R}^2$ with the Euclidean norm. Hence $z_n \to z$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$.

Example 5.1.33 (Two norms on one space (non-example))**.** On the space $\mathcal{P}$ of all polynomials compare the sup norm on $[0,1]$ with the coefficient norm $\|\cdot\|_c$ of the caution after Theorem 3.6.15. The polynomials $p_n = \left(\frac{1}{2} - x\right)^n$ satisfy $\|p_n\|_\infty = 2^{-n} \to 0$ because $|\frac{1}{2} - x| \le \frac{1}{2}$ on $[0,1]$ with equality at 0 and 1. By the binomial theorem their coefficient norms are $\sum_k \binom{n}{k} 2^{k-n} = (3/2)^n$, unbounded by Proposition 5.1.14. The sequence converges to 0 in the sup norm and by Theorem 5.1.31 converges to nothing in the coefficient norm. On an infinite-dimensional space the choice of norm decides which sequences converge.

The Cauchy criterion (Theorem 5.1.28) proves the convergence of a real sequence from its own terms. The same property in a normed space has to be proved separately for each space: it holds in some spaces and fails in others.

Definition 5.1.34 (Cauchy sequence in a normed space)**.** A sequence $\{\mathbf{v}_n\}$ in a normed space V is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an N with $\|\mathbf{v}_m - \mathbf{v}_n\| < \varepsilon$ for all $m, n > N$.

Every convergent sequence is Cauchy: if $\mathbf{v}_n \to \mathbf{v}$ and $\|\mathbf{v}_n - \mathbf{v}\| < \varepsilon/2$ for $n > N$, then $\|\mathbf{v}_m - \mathbf{v}_n\| \le \|\mathbf{v}_m - \mathbf{v}\| + \|\mathbf{v} - \mathbf{v}_n\| < \varepsilon$ for $m, n > N$.

Proposition 5.1.35 (Cauchy sequences in $\mathbb{R}^m$)**.** *For every norm on $\mathbb{R}^m$, every Cauchy sequence in $\mathbb{R}^m$ converges.*

Proof. By Theorem 3.6.15 the given norm is equivalent to $\|\cdot\|_\infty$. Equivalent norms have the same Cauchy sequences and the same convergent sequences. Take the norm $\|\cdot\|_\infty$. Since $|x_i - y_i| \le \|\mathbf{x}-\mathbf{y}\|_\infty$, each coordinate sequence is a real Cauchy sequence and converges by Theorem 5.1.28. The vector of the coordinate limits is the limit of the sequence by the coordinate example above. ∎

Example 5.1.36 (Finitely supported sequences (non-example))**.** Let c_{00} be the vector space of real sequences $\mathbf{x} = (x_1, x_2, \ldots)$ with only finitely many nonzero terms, normed by $\|\mathbf{x}\|_\infty = \max_k |x_k|$. Let $\mathbf{s}_n$ be the sequence $(\frac{1}{2}, \frac{1}{4}, \ldots, 2^{-n}, 0, 0, \ldots)$. For $m > n$, $\|\mathbf{s}_m - \mathbf{s}_n\|_\infty = 2^{-n-1}$ and hence $\{\mathbf{s}_n\}$ is Cauchy. Suppose $\mathbf{s}_n \to \mathbf{x}$ in c_{00}. Since $|(\mathbf{s}_n)_k - x_k| \le \|\mathbf{s}_n - \mathbf{x}\|_\infty$, every coordinate converges and $x_k = 2^{-k}$ for every k. This $\mathbf{x}$ has infinitely many nonzero terms and is not in c_{00}. The Cauchy sequence $\{\mathbf{s}_n\}$ therefore has no limit in c_{00}.

The defect in Example 5.1.36 is the one that the completeness axiom (Assumption 2.4.5) removes from the rational numbers: a sequence crowds together and the point it crowds towards is missing from the space. Section 10.1 calls a normed space in which every Cauchy sequence converges *complete* (Definition 10.1.13). By Theorem 5.1.28 and Proposition 5.1.35 the spaces $\mathbb{R}$ and $\mathbb{R}^m$ are complete and c_{00} is not.

Study guide

The definition (5.1.1) is a statement about every ε. To prove $a_n \to L$ from it, start from an arbitrary $\varepsilon > 0$, simplify $|a_n - L|$ to an expression bounded by something simple such

as C/n, and choose N from that bound. To prove divergence, negate the definition as in Section 1.2: some ε defeats every N.
In practice you will rarely return to the definition. Reduce a limit to known ones with Theorem 5.1.9, bound it with Theorem 5.1.11, or, when no limit can be guessed, prove that the sequence is monotone and bounded and appeal to Theorem 5.1.18. Two errors are common. The first is concluding that a bounded sequence converges. The sequence $(-1)^n$ refutes it. The second is solving a recursion for its fixed point before proving that a limit exists: the recursion $a_{n+1} = 2a_n$ with $a_1 = 1$ has fixed point 0 and diverges to infinity.
In a normed space read every $|\cdot|$ above as $\|\cdot\|$. The definitions and the proofs that use only the triangle inequality carry over. The Cauchy criterion is the exception. It is Theorem 5.1.28 for $\mathbb{R}$ and Proposition 5.1.35 for $\mathbb{R}^m$. It fails in the space c_{00} of Example 5.1.36.

Problems

5.1.1 Let $a_n = \dfrac{3n+1}{n+2}$. Show that $|a_n - 3| = \dfrac{5}{n+2}$ and use it to find the smallest N such that $|a_n - 3| < 0.01$ for every $n > N$. ⋆ Hint · Answer · Solution

5.1.2 Use the limit laws to evaluate $\lim_{n\to\infty} \dfrac{n^3+2n}{4n^3-1}$ and $\lim_{n\to\infty} \left(\dfrac{n+1}{n}\right)^3$. ⋆ Hint · Answer · Solution

5.1.3 Find $\lim_{n\to\infty} (\sqrt{n^2+3n} - n)$. ⋆⋆ Hint · Answer · Solution

5.1.4 Prove from Definition 5.1.3 that $1/\sqrt{n} \to 0$. ⋆⋆ Hint · Answer · Solution

5.1.5 Use Theorem 5.1.11 and Proposition 5.1.15 to find $\lim_{n\to\infty} \left(2^n + 3^n\right)^{1/n}$. ⋆⋆ Hint · Answer · Solution

5.1.6 Let $a_1 = 1$ and $a_{n+1} = 3 - \dfrac{1}{a_n}$. Show that $1 \le a_n < 3$ and that the sequence is nondecreasing. Then find its limit. ⋆⋆ Hint · Answer · Solution

5.1.7 Use Theorem 5.1.20 to find $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^{2n}$ and $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^{n+1}$. ⋆⋆ Hint · Answer · Solution

5.1.8 Let $a_n = (-1)^n + \dfrac{1}{n}$. Find the limits of the subsequences $\{a_{2k}\}$ and $\{a_{2k-1}\}$ and deduce that $\{a_n\}$ diverges. ⋆⋆ Hint · Answer · Solution

5.1.9 Prove that if $a_n \to L$ then $|a_n| \to |L|$. Give an example showing that the converse fails. ⋆⋆ Hint · Answer · Solution

5.1.10 Prove that a convergent sequence of integers is eventually constant: there is an N with $a_n = a_{N+1}$ for every $n > N$. ⋆⋆⋆ Hint · Answer · Solution

5.1.11 In $\mathbb{R}^2$ let $\mathbf{v}_n = \left(\dfrac{1}{n}, \dfrac{2n+1}{n+3}\right)$. Find the limit $\mathbf{v}$ of the sequence and the least N such that $\|\mathbf{v}_n - \mathbf{v}\| < 0.01$ for every $n > N$, first for the ∞-norm and then for the 1-norm (Definition 3.6.6). ⋆⋆ Hint · Answer · Solution

5.1.12 Let $\{\mathbf{v}_n\}$ be a Cauchy sequence in a normed space V (Definition 5.1.34). Prove that $\{\|\mathbf{v}_n\|\}$ is bounded. Prove that if a subsequence $\{\mathbf{v}_{n_k}\}$ converges to $\mathbf{v}$, then $\mathbf{v}_n \to \mathbf{v}$. ⋆⋆⋆ Hint · Answer · Solution

5.2 Limits of Functions

Overview

Objectives: State and apply the ε–δ definition of $\lim_{x\to a} f(x) = L$ (Definition 5.2.1); prove a limit from the definition for linear and quadratic functions; reduce function limits to sequence limits (Theorem 5.2.7) and use the reduction to prove that a limit fails to exist; work with one-sided limits, limits at infinity and infinite limits; apply the limit laws and the Squeeze Theorem; evaluate $\lim_{x\to 0}(\sin x)/x$; compare logarithms, powers and exponentials.
Prerequisites: Limits of sequences (Section 5.1); the triangle inequality (Lemma 2.3.8); the exponential and logarithm (Section 4.4); radian measure and the trigonometric functions (Definition 4.5.2 and Section 4.5).
Key results: The definition Eq. (5.2.2); the sequential characterisation (Theorem 5.2.7); one-sided limits (Theorem 5.2.11); the limit laws (Theorem 5.2.13); the Squeeze Theorem (Theorem 5.2.15); $\lim_{x\to 0}(\sin x)/x = 1$ (Theorem 5.2.18); the growth ranking (Theorem 5.2.26).

Notation

$\lim_{x\to a} f(x) = L$	$f(x)$ tends to L as x tends to a (Definition 5.2.1)
$\lim_{x\to a^+} f(x)$	the limit from the right, through $x > a$ (Definition 5.2.10)
$\lim_{x\to a^-} f(x)$	the limit from the left, through $x < a$
$\lim_{x\to\infty} f(x)$	the limit at infinity (Definition 5.2.20)
$f(x) \ll g(x)$	$f(x)/g(x) \to 0$ as $x \to \infty$ (Theorem 5.2.26)

The velocity of a moving object at an instant is the limit of average velocities over shorter and shorter intervals. At the instant itself the average velocity is the meaningless quotient $0/0$. A limit of a function describes the values of $f(x)$ for x *near* a while ignoring the value at a (which may not exist). The definition follows the pattern of Definition 5.1.3, with the condition "n large" replaced by "x close to a".

We say that f is *defined near* a if there is an $r > 0$ such that $f(x)$ is defined whenever $0 < |x - a| < r$, that is, on $(a - r, a) \cup (a, a + r)$. The point a itself may or may not lie in the domain.

Definition 5.2.1 (Limit)**.** Let f be defined near a (except possibly at a). The *limit* of f at a is L, written

$$\lim_{x\to a} f(x) = L \qquad \text{or} \qquad f(x) \to L \text{ as } x \to a, \tag{5.2.1}$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \tag{5.2.2}$$

See Fig. 5.2.1 for the band picture.

Because the condition $0 < |x - a|$ in (5.2.2) excludes $x = a$, the value $f(a)$ plays no part whether or not it exists. The functions $f(x) = x + 1$ and $g(x) = (x^2 - 1)/(x - 1)$ agree for $x \neq 1$ and therefore have the same limit 2 at 1, although $g(1)$ is undefined.

Example 5.2.2. For $f(x) = 3x + 1$ we have $\lim_{x\to 2} f(x) = 7$. Given $\varepsilon > 0$ the choice $\delta = \varepsilon/3$ satisfies (5.2.2) because $|f(x) - 7| = 3|x - 2| < \varepsilon$ whenever $|x - 2| < \varepsilon/3$.

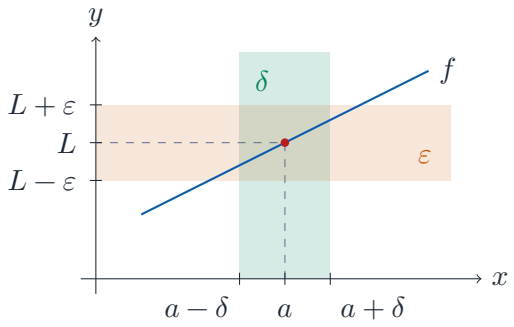


Figure 5.2.1: Definition of $\lim_{x\to a} f(x) = L$ in bands: for every horizontal ε-band about L there is a vertical δ-band about a whose points f maps into the ε-band.

Example 5.2.3. The sign function $\operatorname{sgn}(x) = x/|x|$, equal to 1 for $x > 0$ and -1 for $x < 0$, has no limit at 0. Suppose $\operatorname{sgn}(x) \to L$ and take $\varepsilon = 1$. For the corresponding δ, both $x = \delta/2$ and $x = -\delta/2$ satisfy $0 < |x| < \delta$ and therefore give $|1 - L| < 1$ and $|-1 - L| < 1$. The triangle inequality then gives $2 = |1 - (-1)| \le |1 - L| + |L + 1| < 2$. This is absurd.

Worked Example 5.2.1

Prove from Definition 5.2.1 that $\lim_{x\to 3} x^2 = 9$.
The distance to be controlled factors as

$$|x^2 - 9| = |x - 3|\,|x + 3|.$$

The factor $|x-3|$ is the one δ controls. The factor $|x+3|$ must be bounded. Agreeing first to take $\delta \le 1$ arranges this: $|x - 3| < 1$ gives $2 < x < 4$ and $|x + 3| < 7$. Given $\varepsilon > 0$ put

$$\delta = \min\Big\{1, \frac{\varepsilon}{7}\Big\}.$$

If $0 < |x-3| < \delta$, then $|x+3| < 7$ because $\delta \le 1$ and $|x-3| < \varepsilon/7$ because $\delta \le \varepsilon/7$. Hence $|x^2 - 9| < 7 \cdot \varepsilon/7 = \varepsilon$.
The minimum is the standard device for a nonlinear function. One entry confines x to a region where the troublesome factor is bounded and the other makes the product small.

Theorem 5.2.4 (Uniqueness of function limits)**.** *A function has at most one limit at a point.*

Proof. If $f(x) \to L$ and $f(x) \to M$ with $L \neq M$, take $\varepsilon = \frac{1}{2}|L - M|$ and the smaller of the two values of δ provided by the definition. Any x with $0 < |x - a| < \delta$ (one exists because f is defined near a) gives $|L - M| \le |L - f(x)| + |f(x) - M| < |L - M|$. This is absurd. ∎

Writing $x = a + h$ moves the point of approach to 0 without changing the definition.

Proposition 5.2.5 (Shift form of the limit)**.** *Writing $h = x - a$, we have $\lim_{x\to a} g(x) = L$ if and only if $\lim_{h\to 0} g(a + h) = L$.*

Proof. Since $g(x) = g(a + h)$ and the condition $0 < |x - a| < \delta$ on x is the condition $0 < |h| < \delta$ on $h = x - a$, the two ε–δ statements of Definition 5.2.1 are the same statement with the same δ for each ε. ∎

The shift form is the one in which the derivative is defined in Chapter 6.

5.2.6 Sequential characterisation

The definition of a function limit quantifies over all x near a. Sequences reduce it to the setting of Section 5.1 in which the limit laws are already proved.

Theorem 5.2.7 (Sequential characterisation of limits)**.** *Let f be defined near a. The limit of f at a is L if and only if $f(x_n) \to L$ for every sequence $\{x_n\}$ in the domain of f with $x_n \neq a$ for every n and $x_n \to a$.*

Proof. Suppose $f(x) \to L$ and let $x_n \to a$ with $x_n \neq a$. Given $\varepsilon > 0$ take δ from (5.2.2) and then N with $|x_n - a| < \delta$ for $n > N$. For $n > N$ we have $0 < |x_n - a| < \delta$ and hence $|f(x_n) - L| < \varepsilon$.

Conversely, suppose $f(x) \not\to L$. Negating Definition 5.2.1 as in Section 1.2, some $\varepsilon_0 > 0$ has the property that for every $\delta > 0$ there is an x with $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. Taking $\delta = 1/n$ for each n (and n large enough that $1/n$ is less than the radius r on which f is defined) gives points x_n with $0 < |x_n - a| < 1/n$ and $|f(x_n) - L| \geq \varepsilon_0$. Then $x_n \to a$ with $x_n \neq a$ and $f(x_n) \not\to L$. This proves the contrapositive of the converse. ∎

Theorem 5.2.7 is most often used in the negative direction: to show that $\lim_{x \to a} f(x)$ does not exist, exhibit two sequences approaching a along which f has different limits.

Example 5.2.8. The function $f(x) = \sin(1/x)$ has no limit at 0 (Fig. 5.2.2). The values are $f(x_n) = \sin(2\pi n) = 0$ along $x_n = 1/(2\pi n)$ and $f(y_n) = 1$ along $y_n = 1/(\frac{\pi}{2} + 2\pi n)$. Both sequences tend to 0 without reaching it. Since the limits 0 and 1 differ, no single L satisfies Theorem 5.2.7.

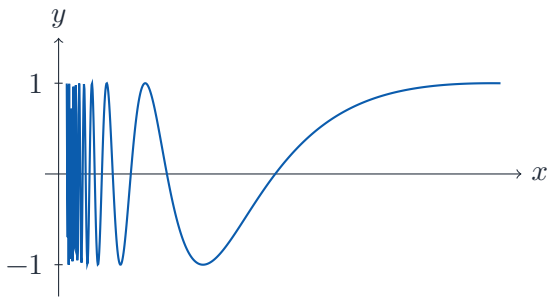


Figure 5.2.2: Graph of $y = \sin(1/x)$ for $x > 0$. Every interval $(0, \delta)$ contains points where the function equals 1 and points where it equals 0.

5.2.9 One-sided limits

The function $\operatorname{sgn}(x) = x/|x|$ has no limit at 0 and yet settles on each side. Restricting x to one side of a records this.

Definition 5.2.10 (One-sided limits)**.** Let f be defined on an interval $(a, a + r)$. We write $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ with $|f(x) - L| < \varepsilon$ whenever $a < x < a + \delta$. The limit from the left, $\lim_{x \to a^-} f(x)$, is defined in the same way with $a - \delta < x < a$.

For the sign function, $\lim_{x \to 0^+} \operatorname{sgn}(x) = 1$ and $\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$. Every result of this section, including Theorem 5.2.7 with sequences $x_n > a$, holds for one-sided limits with the same proof.

Theorem 5.2.11 (Two-sided limits from one-sided limits)**.** *Let f be defined near a. Then $\lim_{x \to a} f(x) = L$ if and only if both one-sided limits at a exist and equal L.*

Proof. A δ that works in (5.2.2) works for each side. Conversely, if δ_+ and δ_- work for the right and left limits with the same ε, then $\delta = \min\{\delta_+, \delta_-\}$ works in (5.2.2) because every x with $0 < |x - a| < \delta$ lies on one side of a or the other. ∎

5.2.12 Limit laws and the Squeeze Theorem

Theorem 5.2.13 (Limit laws)**.** *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$*, then*

$$\lim_{x\to a}(f(x) + g(x)) = L + M, \qquad \lim_{x\to a} cf(x) = cL, \qquad \lim_{x\to a} f(x)g(x) = LM, \tag{5.2.3}$$

and, if $M \neq 0$, $\lim_{x\to a} f(x)/g(x) = L/M$*. The same laws hold for one-sided limits.*

Proof. Let $x_n \to a$ with $x_n \neq a$ and every x_n in the common domain of f and g. By Theorem 5.2.7, $f(x_n) \to L$ and $g(x_n) \to M$. Theorem 5.1.9 then gives $f(x_n) + g(x_n) \to L + M$ and the analogous statements for cf and fg. Since the sequence was arbitrary, Theorem 5.2.7 gives the sum, multiple and product laws.

For the quotient, f/g must first be defined near a. Taking $\varepsilon = |M|/2$ in the definition of $g(x) \to M$ gives a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|g(x) - M| < |M|/2$ and hence $|g(x)| > |M|/2 > 0$. On that punctured interval f/g is defined and the sequence argument applies with the quotient law of Theorem 5.1.9. ∎

Since $\lim_{x\to a} x = a$ (take $\delta = \varepsilon$) and a constant tends to itself, repeated use of the sum and product laws gives the following.

Corollary 5.2.14 (Limits of polynomials and rational functions)**.** *If* p *and* q *are polynomials, then* $\lim_{x\to a} p(x) = p(a)$ *for every* a *and* $\lim_{x\to a} p(x)/q(x) = p(a)/q(a)$ *whenever* $q(a) \neq 0$*.*

When $q(a) = 0$, Corollary 5.2.14 is silent. If $p(a) = 0$ as well, the factor $x - a$ divides both polynomials. Cancelling it for $x \neq a$ is legitimate because the limit ignores the value at a:

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

Theorem 5.2.15 (Squeeze Theorem)**.** *If* $g(x) \leq f(x) \leq h(x)$ *for* $0 < |x - a| < r$ *and* $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$*, then* $\lim_{x\to a} f(x) = L$*. The same holds for one-sided limits.*

Proof. For any sequence $x_n \to a$ with $0 < |x_n - a| < r$, the sequences $g(x_n)$ and $h(x_n)$ tend to L by Theorem 5.2.7. Theorem 5.1.11 then gives $f(x_n) \to L$. Every sequence $x_n \to a$ with $x_n \neq a$ satisfies $|x_n - a| < r$ beyond some index. Its finitely many earlier terms do not affect the limit. Theorem 5.2.7 completes the proof. ∎

5.2.16 Trigonometric limits

The trigonometric functions enter calculus through one limit. We take from Section 4.5 the identities $\sin^2 x + \cos^2 x = 1$ and $1 - \cos x = 2\sin^2(x/2)$, the parities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. We also use the inequalities

$$\sin x < x < \tan x \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{5.2.4}$$

They are part (3) of Theorem 8.6.6, proved in Chapter 8 from the arc-length integral Eq. (4.5.1) without Proposition 5.2.17, Theorem 5.2.18 or any trigonometric derivative (see the logical order set out in Section 4.5). The arc length $\Theta(s)$ from $(1, 0)$ to $(\sqrt{1 - s^2}, s)$ satisfies $\Theta(s) > s$ for

$0 < s < 1$ by part (3) of Theorem 8.6.6. Since $\pi/2$ is the supremum of Θ on $[0, 1)$, $\pi/2 \geq 1$ and $|\sin x| \leq 1 \leq \pi/2 \leq |x|$ for $|x| \geq \pi/2$. The left inequality and the parity of sin give $|\sin x| \leq |x|$ for every real x.

Proposition 5.2.17 (Sine and cosine at zero)**.** $\lim_{x\to 0} \sin x = 0$ *and* $\lim_{x\to 0} \cos x = 1$.

Proof. From $-|x| \leq \sin x \leq |x|$ and Theorem 5.2.15, $\sin x \to 0$. From $0 \leq 1 - \cos x = 2\sin^2(x/2) \leq 2(x/2)^2 = x^2/2$ and the Squeeze Theorem, $1 - \cos x \to 0$. ∎

Theorem 5.2.18 (Fundamental trigonometric limit)**.**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{5.2.5}$$

Proof. For $0 < x < \pi/2$, dividing (5.2.4) by the positive number $\sin x$ gives $1 < x/\sin x < 1/\cos x$. Taking reciprocals,

$$\cos x < \frac{\sin x}{x} < 1. \tag{5.2.6}$$

Because $\cos x$, the constant 1 and $(\sin x)/x$ are all even functions of x, the same inequalities hold for $-\pi/2 < x < 0$. Since $\cos x \to 1$ by Proposition 5.2.17, Theorem 5.2.15 gives the result. ∎

Worked Example 5.2.2

Evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ and $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 2x}$.
For the first, the half-angle identity rewrites the quotient in terms of the fundamental limit:

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2(x/2)}{x^2} = \frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^2.$$

Since $t = x/2 \to 0$ with $t \neq 0$ as $x \to 0$, Theorem 5.2.18 gives $\sin(x/2)/(x/2) \to 1$ (formally, by Theorem 5.2.7: any sequence $x_n \to 0$ gives $t_n = x_n/2 \to 0$). The product law gives the limit $\frac{1}{2}$.
For the second, insert the factors that the fundamental limit requires:

$$\frac{\sin 5x}{\sin 2x} = \frac{\sin 5x}{5x} \cdot \frac{2x}{\sin 2x} \cdot \frac{5}{2}.$$

Since the first two factors tend to 1 by the same argument with $t = 5x$ and $t = 2x$, the limit is $\frac{5}{2}$.

5.2.19 Limits at infinity and infinite limits

Two further kinds of limit describe the behaviour of $f(x)$ as x grows without bound and the behaviour of a function that grows without bound near a point.

Definition 5.2.20 (Limits at infinity)**.** Let f be defined on an interval (c, ∞). We write $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is an M such that $|f(x) - L| < \varepsilon$ for every $x > M$. The limit as $x \to -\infty$ is defined in the same way with $x < M$.

Definition 5.2.20 is Definition 5.1.3 with a real variable in place of the integer n. In particular, if $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $a_n \to L$: the sequence is the restriction of f to the positive integers. Section 10.1 uses this to evaluate sequence limits by calculus.

Definition 5.2.21 (Infinite limits)**.** Let f be defined near a. We write $\lim_{x\to a} f(x) = \infty$ if for every M there is a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$. Similarly $\lim_{x\to a} f(x) = -\infty$ means $-f(x) \to \infty$. One-sided infinite limits and $\lim_{x\to\infty} f(x) = \infty$ are defined in the same way.

A function with an infinite limit has no limit in the sense of Definition 5.2.1: the symbol ∞ is not a number. The statement $\lim f(x) = \infty$ records how the limit fails to exist. The line $x = a$ is then a *vertical asymptote* of the graph. Similarly $y = L$ is a *horizontal asymptote* when $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$.

Example 5.2.22. $\lim_{x\to\infty} 1/x = 0$: given ε, take $M = 1/\varepsilon$. $\lim_{x\to 0^+} 1/x = \infty$: given $M > 0$, take $\delta = 1/M$ because $0 < x < 1/M$ gives $1/x > M$. Likewise $\lim_{x\to 0^-} 1/x = -\infty$. From both sides $\lim_{x\to 0} 1/x^2 = \infty$.

The sequential characterisation, the limit laws and the Squeeze Theorem hold for limits at infinity with the same proofs, using sequences $x_n \to \infty$. For a rational function at infinity, dividing above and below by the highest power of x in the denominator reduces every term to a constant or a multiple of $1/x^k$.

The Monotone Convergence Theorem has a version for functions, used for improper integrals in Section 9.8.

Proposition 5.2.23 (Monotone limits of functions)**.** *Let $a < \beta \le \infty$ and let F be nondecreasing on $[a, \beta)$ and bounded above. Then $\lim_{x\to\beta^-} F(x)$ (read as $\lim_{x\to\infty} F(x)$ when $\beta = \infty$) exists and equals $\sup\{F(x) : a \le x < \beta\}$.*

Proof. Let L be the supremum (Assumption 2.4.5) and let $\varepsilon > 0$. By Lemma 2.4.4 some $x_0 \in [a, \beta)$ has $F(x_0) > L - \varepsilon$. For every x with $x_0 < x < \beta$, monotonicity gives $L - \varepsilon < F(x_0) \le F(x) \le L$. Hence $|F(x) - L| < \varepsilon$ whenever $0 < \beta - x < \delta$ with $\delta = \beta - x_0$ for finite β, and whenever $x > M$ with $M = x_0$ for $\beta = \infty$. ∎

Worked Example 5.2.3

Evaluate $\displaystyle\lim_{x\to\infty} \frac{2x^3 - x + 4}{x^3 + 7x}$ and $\displaystyle\lim_{x\to\infty} (\sqrt{x^2 + x} - x)$.
Dividing above and below by x^3,

$$\frac{2x^3 - x + 4}{x^3 + 7x} = \frac{2 - 1/x^2 + 4/x^3}{1 + 7/x^2} \to \frac{2}{1} = 2,$$

by the limit laws and $1/x^k \to 0$.
In the second limit both terms tend to infinity. A difference of the form $\infty - \infty$ cannot be evaluated term by term. Rationalising for $x > 0$,

$$\sqrt{x^2 + x} - x = \frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x} = \frac{x}{\sqrt{x^2 + x} + x} = \frac{1}{\sqrt{1 + 1/x} + 1}.$$

Since $1 \le \sqrt{1 + 1/x} \le 1 + 1/x$ for $x > 0$, the Squeeze Theorem gives $\sqrt{1 + 1/x} \to 1$. The limit is $\frac{1}{2}$.

Arithmetic with infinity

The symbols $\infty - \infty$, $0 \cdot \infty$, ∞/∞ and $0/0$ are *indeterminate forms*: they describe limits whose value depends on the functions involved and cannot be computed from the form alone. The two limits $\sqrt{x^2+x} - x \to \frac{1}{2}$ and $\sqrt{x^2+x^{3/2}} - x \to \infty$ both have the form $\infty - \infty$. Rewrite the expression before taking the limit.

5.2.24 Growth ranking

Logarithms, powers and exponentials all tend to infinity with x at different rates. The comparison is used throughout Part IV and in every improper integral of Section 9.8. The proof uses four facts from Section 4.4: exp is increasing (Theorem 4.4.16), $e^{x+y} = e^x e^y$ (Theorem 4.4.16), ln is the inverse of exp on $(0,\infty)$ (Definition 4.4.22), and $x^p = e^{p \ln x}$ for $x > 0$ (Theorem 4.4.24). The last two rest on the surjectivity of exp and on the power law for irrational exponents. Theorem 5.4.12 proves both in Section 5.4 from continuity alone without using Theorem 5.2.26. For $p > 0$ the power x^p is then increasing in x.

Lemma 5.2.25 (Powers against geometric growth)**.** *If $a > 1$ and $k \geq 0$ is an integer, then $n^k/a^n \to 0$.*

Proof. Write $a = 1 + d$ with $d > 0$. Since every term of the binomial expansion of $(1+d)^n$ is nonnegative, for $n > k+1$ the single term with d^{k+1} gives

$$a^n \geq \binom{n}{k+1} d^{k+1} = \frac{n(n-1)\cdots(n-k)}{(k+1)!}\, d^{k+1}.$$

For $n \geq 2k$ each of the $k+1$ factors $n-j$ with $0 \leq j \leq k$ is at least $n - k \geq n/2$. Hence

$$0 < \frac{n^k}{a^n} \leq \frac{n^k\,(k+1)!\,2^{k+1}}{n^{k+1} d^{k+1}} = \frac{(k+1)!\,2^{k+1}}{d^{k+1}} \cdot \frac{1}{n}, \tag{5.2.7}$$

which tends to 0. Corollary 5.1.12 completes the proof. ∎

Theorem 5.2.26 (Growth ranking of elementary functions at infinity)**.** *For every real p and every $q > 0$,*

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\ln x}{x^q} = 0. \tag{5.2.8}$$

In words, as $x \to \infty$,

$$\ln x \;\ll\; x^q \;\ll\; e^x \qquad (q > 0). \tag{5.2.9}$$

Proof. **1.** Powers against the exponential. Choose an integer $k \geq \max\{p, 0\}$. For $x \geq 1$ we have $x^p \leq x^k$. Let m be the integer with $m \leq x < m+1$. Since exp and $t \mapsto t^k$ are increasing,

$$0 < \frac{x^p}{e^x} \leq \frac{(m+1)^k}{e^m} = e \cdot \frac{(m+1)^k}{e^{m+1}}. \tag{5.2.10}$$

By Lemma 5.2.25 with $a = e > 1$ (Theorem 5.1.20), given $\varepsilon > 0$ there is an N with $j^k/e^j < \varepsilon/e$ for every integer $j > N$. If $x > \max\{N, 1\}$, then $j = m+1 > x > N$ and (5.2.10) gives $x^p/e^x < \varepsilon$.

2. Logarithm against powers. For $x > 1$ put $t = \ln x > 0$. Then $x^q = e^{qt}$ and

$$\frac{\ln x}{x^q} = \frac{t}{e^{qt}} = \frac{1}{q} \cdot \frac{qt}{e^{qt}}.$$

By step 1 with $p = 1$, given $\varepsilon > 0$ there is an $S > 0$ with $s/e^s < q\varepsilon$ for every $s > S$. If $x > e^{S/q}$, then $qt = q \ln x > S$ because ln is increasing. Therefore $0 < \ln x / x^q < \varepsilon$. ∎

Theorem 5.2.26 is often quoted in the reciprocal forms $e^x/x^p \to \infty$ and $x^q/\ln x \to \infty$. L'Hôpital's rule (Theorem 7.2.16) gives a second proof in Chapter 7 and resolves the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ in general. The ranking (5.2.9) settles every convergence question in Section 9.8 that compares a logarithm, a power and an exponential.

Study guide

A solution to a limit problem should name which of four methods it uses. Substitute, when the function is a polynomial or a rational function with nonzero denominator (Corollary 5.2.14). Simplify, by factoring, cancelling or rationalising, when substitution gives $\frac{0}{0}$ or $\infty - \infty$. Squeeze, when an oscillating factor such as $\sin(1/x)$ is bounded. Reduce to a known limit such as Theorem 5.2.18 or Theorem 5.2.26.
To show that a limit does not exist, use two sequences (Theorem 5.2.7) or two different one-sided limits (Theorem 5.2.11). An ε–δ proof has the shape of Worked Example 5.2.1: factor $|f(x) - L|$, bound the factor that is not $|x - a|$ by restricting $\delta \le 1$, and take δ to be a minimum.

Problems

5.2.1 Prove from Definition 5.2.1 that $\lim_{x\to 2}(5x - 3) = 7$. Give δ as a function of ε.
⋆ Hint · Answer · Solution

5.2.2 Evaluate $\displaystyle\lim_{x\to -1} \frac{x^3 + 1}{x + 1}$ and $\displaystyle\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 9}$.
⋆ Hint · Answer · Solution

5.2.3 Prove from Definition 5.2.1 that $\lim_{x\to 4} \sqrt{x} = 2$.
⋆⋆ Hint · Answer · Solution

5.2.4 Let $f(x) = \dfrac{|x - 3|}{x - 3}$. Find both one-sided limits at 3 and decide whether $\lim_{x\to 3} f(x)$ exists.
⋆⋆ Hint · Answer · Solution

5.2.5 Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$ and $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$.
⋆⋆ Hint · Answer · Solution

5.2.6 Show that $\displaystyle\lim_{x\to 0} x \sin\frac{1}{x} = 0$, although $\lim_{x\to 0} \sin(1/x)$ does not exist.
⋆⋆ Hint · Answer · Solution

5.2.7 Find $\displaystyle\lim_{x\to 2^+} \frac{x + 1}{x - 2}$, $\displaystyle\lim_{x\to 2^-} \frac{x + 1}{x - 2}$ and $\displaystyle\lim_{x\to \infty} \frac{3x + 1}{x^2 + 1}$.
⋆⋆ Hint · Answer · Solution

5.2.8 Use Theorem 5.2.26 to find $\displaystyle\lim_{x\to \infty} \frac{x^{10}}{e^{x/2}}$ and $\displaystyle\lim_{x\to \infty} \frac{(\ln x)^2}{x}$.
⋆⋆⋆ Hint · Answer · Solution

5.2.9 Use Theorem 5.2.7 to prove that $\lim_{x\to 0} \cos(1/x)$ does not exist.
⋆⋆⋆ Hint · Answer · Solution

5.3 Continuity

Overview

Objectives: State the definition of continuity at a point and on a set (Definition 5.3.1); prove continuity from the definition and from the sequential criterion (Theorem 5.3.5); show that sums, products, quotients and compositions of continuous functions are continuous (Theorem 5.3.6); establish the continuity of polynomials, rational functions, the exponential and the trigonometric functions (Theorem 5.3.10); classify a discontinuity as removable, jump, infinite or oscillating (Definition 5.3.14); test continuity between normed spaces with sequences (Theorem 5.3.12).
Prerequisites: Limits of sequences and functions (Sections 5.1 and 5.2); the exponential and trigonometric functions (Sections 4.4 and 4.5); composition of functions (Definition 1.5.20); continuity between normed spaces and bounded linear maps (Definition 3.6.17 and Theorem 3.6.21).
Key results: The definition Definition 5.3.1; the sequential criterion (Theorem 5.3.5); the algebra of continuous functions (Theorem 5.3.6); limits through a continuous function (Theorem 5.3.7); persistence of sign (Lemma 5.3.8); continuity of the elementary functions (Theorem 5.3.10); the sequential criterion between normed spaces (Theorem 5.3.12).

A limit at a ignores the value $f(a)$. For most functions met in practice the limit and the value agree. This agreement makes it safe to compute a limit by substitution, to approximate $f(a)$ by $f(x)$ for x near a, and to read a measured input of finite precision into a formula. Every theorem of the differential and integral calculus has continuity among its hypotheses.

Definition 5.3.1 (Continuity)**.** Let f be defined on a set D (usually an interval) and let $a \in D$. The function f is *continuous at* a if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in D \text{ and } |x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon. \tag{5.3.1}$$

When f is defined near a, this says that $f(a)$ exists and $\lim_{x \to a} f(x) = f(a)$. At an endpoint of an interval it says that the one-sided limit from inside the interval equals $f(a)$. The function is *continuous on* D if it is continuous at each point of D, and *continuous* if it is continuous on its domain.

Continuity at a makes three demands: $f(a)$ is defined, the limit at a exists, and the two are equal.

Example 5.3.2. The absolute value is continuous at every a. The reverse triangle inequality $\big||x| - |a|\big| \le |x - a|$ follows from Lemma 2.3.8 applied to $x = (x - a) + a$ and to $a = (a - x) + x$. It shows that $\delta = \varepsilon$ works.

Example 5.3.3. The function equal to $(x^2 - 1)/(x - 1)$ for $x \neq 1$ and to 0 at $x = 1$ is not continuous at 1: its limit there is 2 and its value is 0. The sign function $\operatorname{sgn}(x) = x/|x|$, with $\operatorname{sgn}(0) = 0$, is not continuous at 0 because it has no limit there.

5.3.4 Sequential criterion and algebra of continuous functions

Theorem 5.3.5 (Continuous function theorem)**.** *If $a_n \to L$ and f is continuous at L and defined at every a_n, then $f(a_n) \to f(L)$. Conversely, if f is defined on D, $L \in D$ and $f(a_n) \to f(L)$ for every sequence $\{a_n\}$ in D with $a_n \to L$, then f is continuous at L.*

Proof. Given $\varepsilon > 0$ take δ from Definition 5.3.1 at L, and then N with $|a_n - L| < \delta$ for $n > N$. For $n > N$ the point a_n lies in the domain within δ of L and therefore satisfies $|f(a_n) - f(L)| < \varepsilon$.

For the converse, suppose that f is not continuous at L. Negating the definition, some $\varepsilon_0 > 0$ has, for each n, a point $a_n \in D$ with $|a_n - L| < 1/n$ and $|f(a_n) - f(L)| \geq \varepsilon_0$. Then $a_n \to L$ while $f(a_n) \not\to f(L)$. ∎

By Theorem 5.3.5 a limit may be taken inside a continuous function. Since the exponential and the square root are continuous (Theorem 5.3.10 and Corollary 5.4.13), every sequence $a_n \to L$ has $e^{a_n} \to e^L$ and, when its terms are nonnegative, $\sqrt{a_n} \to \sqrt{L}$.

Theorem 5.3.6 (Algebra of continuous functions)**.** *If f and g are continuous at a, then so are $f + g$, $f - g$, fg, and, when $g(a) \neq 0$, f/g. If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a.*

Proof. Let $x_n \to a$ with every x_n in the common domain. By Theorem 5.3.5, $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Theorem 5.1.9 then gives $f(x_n) + g(x_n) \to f(a) + g(a)$ and the analogous statements for $f - g$, fg and f/g. Since the sequence was arbitrary, the converse half of Theorem 5.3.5 gives continuity of each combination at a. For the quotient, a is one of the points where $g \neq 0$ that make up the domain of f/g.

For the composition, continuity of g gives $g(x_n) \to g(a)$ and then continuity of f at $g(a)$ with Theorem 5.3.5 gives $f(g(x_n)) \to f(g(a))$. The converse half of the theorem gives continuity of $f \circ g$ at a. ∎

Problem 5.3.4 asks for a direct ε–δ proof of the sum rule. The argument is the tolerance splitting of Theorem 5.1.9: apply Definition 5.3.1 to f and to g with tolerance $\varepsilon/2$ and take the smaller δ.

Theorem 5.3.7 (Limits through a continuous function)**.** *If $\lim_{x \to a} g(x) = b$, f is continuous at b and $f \circ g$ is defined near a, then*

$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big) = f(b). \tag{5.3.2}$$

The same holds for one-sided limits and for limits at infinity.

Proof. If $x_n \to a$ with $x_n \neq a$, then $g(x_n) \to b$ by Theorem 5.2.7 and $f(g(x_n)) \to f(b)$ by Theorem 5.3.5. Theorem 5.2.7 gives the conclusion. ∎

Because the value of g at a plays no part in Theorem 5.3.7, the theorem applies to $g(x) = (\sin x)/x$ at $a = 0$ although $g(0)$ is undefined. For instance $\lim_{x \to 0} e^{(\sin x)/x} = e^1$.

Lemma 5.3.8 (Persistence of sign)**.** *If f is continuous at c and $f(c) > N$, there is a $\delta > 0$ such that $f(x) > N$ for every x in the domain with $|x - c| < \delta$. The same holds with $<$ in place of $>$.*

Proof. Take $\varepsilon = f(c) - N > 0$ in Definition 5.3.1. For $|x - c| < \delta$, $f(x) > f(c) - \varepsilon = N$. The other case applies this to $-f$ and $-N$. ∎

5.3.9 Continuity of the elementary functions

The functions of Chapter 4 are continuous wherever they are defined. For polynomials this follows from the algebra of limits alone. For the exponentials and the trigonometric functions the proof uses a few facts about the functions themselves, taken from Sections 4.4 and 4.5:

- for a base $a > 0$: $a^x > 0$, $a^0 = 1$ and $a^{x+y} = a^x a^y$ for all real x, y, with $x \mapsto a^x$ increasing when $a > 1$. These are parts (1), (2) and (4) of Theorem 4.4.16, proved in full in Section 4.4 without any appeal to continuity;

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$, together with the facts listed before Proposition 5.2.17.

Theorem 5.3.10 (Continuity of the elementary functions)**.** *The following functions are continuous at every point of their domains: polynomials; rational functions; the absolute value; the exponential a^x for each base $a > 0$, in particular e^x; $\sin x$, $\cos x$ and $\tan x$.*

Proof. **(i)** Rational functions. Corollary 5.2.14 states exactly that $\lim_{x\to a} p(x) = p(a)$ for every a and that $\lim_{x\to a} p(x)/q(x) = p(a)/q(a)$ at every point of the domain $\{x : q(x) \neq 0\}$ of p/q.
(ii) Absolute value. This is the first example after Definition 5.3.1.
(iii) Exponentials. Let $a > 1$. We first show $\lim_{h\to 0} a^h = 1$. The functional equation gives $(a^{1/n})^n = a^1 = a$. Hence $a^{1/n}$ is the positive nth root of a and tends to 1 by Proposition 5.1.15. Let $0 < \varepsilon < 1$ and choose n with $a^{1/n} < 1 + \varepsilon$. For $0 < h < 1/n$, monotonicity gives $1 = a^0 < a^h < a^{1/n} < 1 + \varepsilon$. For $-1/n < h < 0$, $a^h = 1/a^{-h}$ lies between $1/(1+\varepsilon)$ and 1 with $1/(1+\varepsilon) > 1 - \varepsilon$. Hence $|a^h - 1| < \varepsilon$ whenever $0 < |h| < 1/n$. Since ε was arbitrary, $a^h \to 1$. At any point c, by Proposition 5.2.5 and the product law,

$$\lim_{x\to c} a^x = \lim_{h\to 0} a^c a^h = a^c \cdot 1 = a^c.$$

For $0 < a < 1$ the function $a^x = 1/(1/a)^x$ is the reciprocal of a positive continuous function. The function $1^x = 1$ is constant. The base e is greater than 1 by Theorem 5.1.20.
(iv) Sine and cosine. By the addition formula, Proposition 5.2.5 and Proposition 5.2.17,

$$\lim_{h\to 0} \sin(a+h) = \lim_{h\to 0} (\sin a \cos h + \cos a \sin h) = \sin a \cdot 1 + \cos a \cdot 0 = \sin a,$$

and the same computation with the cosine formula gives $\cos(a+h) \to \cos a$. The tangent $\tan = \sin / \cos$ is a quotient of continuous functions and is continuous at every point of its domain $\{x : \cos x \neq 0\}$. ∎

The logarithm, roots, real powers $x^r = e^{r \ln x}$ and the inverse trigonometric functions are continuous as well. Their proofs use the Intermediate Value Theorem and are given in Corollary 5.4.13. Combining Theorems 5.3.6 and 5.3.10, every function built from these by finitely many sums, products, quotients and compositions is continuous on its domain. A limit of such a function at a point of its domain is computed by substitution.

Continuity on the domain

The function $1/x$ is continuous because it is continuous at every point of its domain $\{x : x \neq 0\}$. It cannot be extended to a function continuous at 0 because it has no finite limit there. Statements such as "$1/x$ is discontinuous at 0" refer to this failure to extend. Say on which set a function is continuous whenever the domain has a gap.

5.3.11 Continuity between normed spaces

Definition 3.6.17 states continuity for a function $F : V \to W$ between normed spaces with $\|\mathbf{v} - \mathbf{v}_0\|$ and $\|F(\mathbf{v}) - F(\mathbf{v}_0)\|$ in place of $|x - a|$ and $|f(x) - f(a)|$. For $V = W = \mathbb{R}$ with the absolute value it is Definition 5.3.1 with $D = \mathbb{R}$. The sequential criterion and its proof are the same.

Theorem 5.3.12 (Sequential criterion in normed spaces)**.** *Let V and W be normed spaces, $F : V \to W$ and $\mathbf{v}_0 \in V$. Then F is continuous at $\mathbf{v}_0$ if and only if $F(\mathbf{v}_n) \to F(\mathbf{v}_0)$ for every sequence $\mathbf{v}_n \to \mathbf{v}_0$ in V (Definition 5.1.30).*

Proof. Suppose F is continuous at $\mathbf{v}_0$ and $\mathbf{v}_n \to \mathbf{v}_0$. Given $\varepsilon > 0$ take δ from Definition 3.6.17 and then N with $\|\mathbf{v}_n - \mathbf{v}_0\| < \delta$ for $n > N$. For $n > N$, $\|F(\mathbf{v}_n) - F(\mathbf{v}_0)\| < \varepsilon$.

Conversely, suppose F is not continuous at $\mathbf{v}_0$. Negating Definition 3.6.17, some $\varepsilon_0 > 0$ has, for each n, a point $\mathbf{v}_n$ with $\|\mathbf{v}_n - \mathbf{v}_0\| < 1/n$ and $\|F(\mathbf{v}_n) - F(\mathbf{v}_0)\| \geq \varepsilon_0$. Then $\mathbf{v}_n \to \mathbf{v}_0$ while $F(\mathbf{v}_n) \not\to F(\mathbf{v}_0)$. ∎

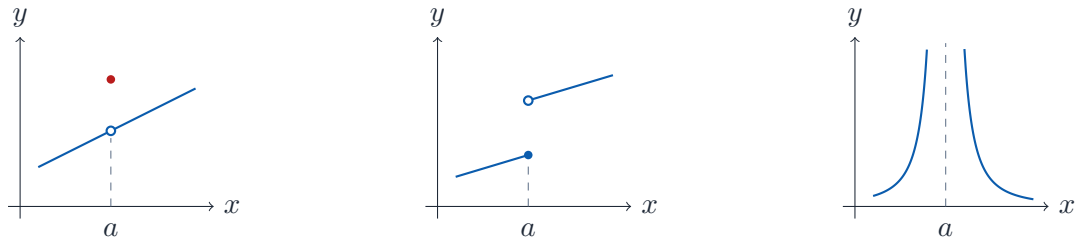
A bounded linear map T (Definition 3.6.18) is continuous by Theorem 3.6.21. By Theorem 5.3.12 it therefore takes limits to limits: $\mathbf{v}_n \to \mathbf{v}$ gives $T\mathbf{v}_n \to T\mathbf{v}$. For the evaluation $E_p f = f(p)$ on $C([a,b])$ with the sup norm, bounded with $\|E_p\| = 1$ by Section 3.6, this says that $\|f_n - f\|_\infty \to 0$ gives $f_n(p) \to f(p)$ at every point p. As a non-example, give the space $\mathcal{P}$ of polynomials the sup norm on $[0,1]$ and let $F(p) = p(2)$. The polynomials $p_n = x^n/n$ satisfy $\|p_n\|_\infty = 1/n \to 0$ while $F(p_n) = 2^n/n \geq 1$ by Bernoulli's inequality $2^n \geq 1+n$ (Theorem 2.3.11). Hence $F(p_n) \not\to F(0) = 0$ and F is not continuous at 0, in agreement with the unboundedness of evaluation at 2 shown in Section 3.6.

5.3.13 Types of discontinuity

When a function defined near a fails to be continuous at a, the one-sided limits describe how it fails (Fig. 5.3.1).

Definition 5.3.14 (Types of discontinuity)**.** Let f be defined near a and not continuous at a (or undefined at a). A point outside the domain is admitted. There the type describes the behaviour of f near a and says whether a value at a would make f continuous, as in the caution after Theorem 5.3.10. The point a is

- a *removable discontinuity* if $\lim_{x\to a} f(x)$ exists. The function becomes continuous at a when $f(a)$ is defined or redefined to be this limit.
- a *jump discontinuity* if both one-sided limits exist and are different.
- an *infinite discontinuity* if at least one one-sided limit is ∞ or $-\infty$.
- an *oscillating discontinuity* if some one-sided limit exists neither as a number nor as ∞ or $-\infty$.



(a) Removable (b) Jump (c) Infinite

Figure 5.3.1: Three kinds of discontinuity at a. A hollow circle marks a point missing from the graph and a filled dot the value $f(a)$.

The four types are exhaustive. If both one-sided limits exist and are finite, equal limits give a removable discontinuity and different limits give a jump. Otherwise some one-sided limit is infinite or fails to exist. Given any value at 0 the function $\sin(1/x)$ of Fig. 5.2.2 has an oscillating discontinuity there. The types can overlap. The function equal to $1/x$ for $x > 0$ and to $\sin(1/x)$ for $x < 0$ has a discontinuity at 0 that is both infinite and oscillating.

Worked Example 5.3.1

Find the constant c for which

$$f(x) = \begin{cases} cx^2 + 1, & x < 2, \\ 3x - c, & x \geq 2 \end{cases}$$

is continuous on $\mathbb{R}$.

On $(-\infty, 2)$ and on $(2, \infty)$ the function agrees with a polynomial near each point and is continuous there by Theorem 5.3.10. At 2 the value is $f(2) = 6 - c$, the right-hand limit is $6 - c$, and the left-hand limit is $\lim_{x \to 2^-}(cx^2 + 1) = 4c + 1$. By Theorem 5.2.11, f is continuous at 2 exactly when $4c + 1 = 6 - c$, that is $c = 1$. For any other c the point 2 is a jump discontinuity.

Worked Example 5.3.2

Find and classify the discontinuities of $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

The function is rational and continuous on its domain by Theorem 5.3.10. Since the denominator factors as $(x - 1)(x - 2)$, the points to examine are 1 and 2 at which f is undefined. For $x \neq 1, 2$,

$$f(x) = \frac{(x - 1)(x + 1)}{(x - 1)(x - 2)} = \frac{x + 1}{x - 2}.$$

At 1 the limit is $\dfrac{1 + 1}{1 - 2} = -2$ by Corollary 5.2.14. The point 1 is therefore a removable discontinuity that the definition $f(1) = -2$ removes. At 2 the numerator tends to 3 while the denominator tends to 0 through positive values from the right and negative values from the left. Hence $\lim_{x \to 2^{\pm}} f(x) = \pm\infty$ and 2 is an infinite discontinuity with a vertical asymptote.

Drawing without lifting the pen

The slogan that a continuous function can be drawn without lifting the pen fails even on an interval. The function $f(x) = x \sin(1/x)$, with $f(0) = 0$, is continuous on $\mathbb{R}$ (at 0 by Problem 5.3.5 and elsewhere by Theorems 5.3.6 and 5.3.10). Its graph oscillates infinitely often in every interval $(0, \delta)$ and no pen traces it. The function of Problem 5.3.9 is continuous at exactly one point. Prove continuity from Definition 5.3.1 or from the theorems of this section.

Study guide

To prove continuity, assemble the function from pieces already known to be continuous (Theorem 5.3.10) with Theorem 5.3.6. Then check the remaining points (usually the junctions of a piecewise definition) by computing both one-sided limits and the value. To prove discontinuity, exhibit a sequence $x_n \to a$ with $f(x_n) \not\to f(a)$ (Theorem 5.3.5). Between normed spaces the same test is Theorem 5.3.12. A linear map is continuous exactly when it is bounded (Theorem 3.6.21).

When classifying, simplify first and then compute the one-sided limits at each point where the function is undefined or defined by a separate rule. Name the type and give its evidence.

A removable point has a common limit, a jump has two different limits and an infinite discontinuity has an infinite one-sided limit.

Problems

5.3.1 Find k such that $f(x) = \begin{cases} x^2 - kx, & x < 1, \\ 2x + k, & x \geq 1 \end{cases}$ is continuous on $\mathbb{R}$.

⋆ Hint · Answer · Solution

5.3.2 Find and classify the discontinuities of $f(x) = \dfrac{x^2 - 4}{x^2 - x - 2}$.

⋆ Hint · Answer · Solution

5.3.3 Evaluate $\lim_{x \to \pi} e^{\sin x + \cos x}$ and $\lim_{x \to 0} \cos\left(\dfrac{\pi \sin x}{2x}\right)$. ⋆ Hint · Answer · Solution

5.3.4 Prove directly from Definition 5.3.1 that if f and g are continuous at a, then $f + g$ is continuous at a. ⋆⋆ Hint · Answer · Solution

5.3.5 Let $f(x) = x \sin(1/x)$ and $g(x) = \sin(1/x)$ for $x \neq 0$, with $f(0) = g(0) = 0$. Show that f is continuous at 0 and that g has an oscillating discontinuity at 0.

⋆⋆ Hint · Answer · Solution

5.3.6 Show that the floor function $\lfloor x \rfloor$, the largest integer not exceeding x, has a jump discontinuity at every integer m and find its one-sided limits there.

⋆⋆ Hint · Answer · Solution

5.3.7 Show that if f and g are continuous on D, then so are $|f|$ and $\max\{f, g\}$.

⋆⋆ Hint · Answer · Solution

5.3.8 Let $\chi_{\mathbb{Q}}(x) = 1$ for rational x and $\chi_{\mathbb{Q}}(x) = 0$ for irrational x. Prove that $\chi_{\mathbb{Q}}$ is discontinuous at every point. ⋆⋆⋆ Hint · Answer · Solution

5.3.9 Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x. Prove that f is continuous at 0 and at no other point. ⋆⋆⋆ Hint · Answer · Solution

5.3.10 Give $C([0,1])$ the sup norm and let $Q(f) = f^2$. Prove that

$$\|f^2 - g^2\|_\infty \leq (\|f\|_\infty + \|g\|_\infty)\|f - g\|_\infty$$

and deduce that $Q \colon C([0,1]) \to C([0,1])$ is continuous at every f. Show that Q is not linear. ⋆⋆ Hint · Answer · Solution

5.4 Theorems on Continuous Functions

Overview

Objectives: Prove the Intermediate Value Theorem from completeness (Theorem 5.4.2) and use it to locate roots, to prove that nth roots exist and to approximate a root by bisection; prove the Extreme Value Theorem from the Bolzano–Weierstrass theorem (Theorem 5.4.6) and recognise when its hypotheses fail; prove that a continuous strictly monotone function on an interval has a continuous inverse (Theorem 5.4.9) and deduce the continuity of the logarithm, roots, real powers and inverse trigonometric functions; complete the theory of exponentials begun in Section 4.4.

Prerequisites: Continuity (Section 5.3); the supremum and completeness (Definition 2.4.3 and Assumption 2.4.5); the Bolzano–Weierstrass theorem (Theorem 5.1.25); inverse functions (Definition 1.5.24).

Key results: The Intermediate Value Theorem (Theorem 5.4.2); existence of nth roots (Corollary 5.4.3); continuous images of intervals (Corollary 5.4.4); the Extreme Value Theorem (Theorem 5.4.6); continuity of inverses (Theorem 5.4.9 and Corollary 5.4.13); exponentials as bijections onto $(0,\infty)$ (Theorem 5.4.12); $e^x = \lim(1+x/n)^n$ (Theorem 5.4.15).

A continuous function on an interval has properties that no finite table of its values can reveal. It cannot pass from a negative value to a positive one without vanishing in between. On a closed bounded interval it has a largest and a smallest value. Both statements fail over the rational numbers: the function x^2-2 is negative at 1 and positive at 2 and has no rational zero. On $[1,2]\cap\mathbb{Q}$ the function $-(x^2-2)^2$ has supremum 0 and attains it at no rational point. Their proofs must therefore use completeness, either directly through the supremum or through the Bolzano–Weierstrass theorem.

5.4.1 Intermediate Value Theorem

Theorem 5.4.2 (Intermediate Value Theorem)**.** *If f is continuous on $[a,b]$ and N lies between $f(a)$ and $f(b)$, then $f(c)=N$ for some $c\in[a,b]$ (Fig. 5.4.1).*

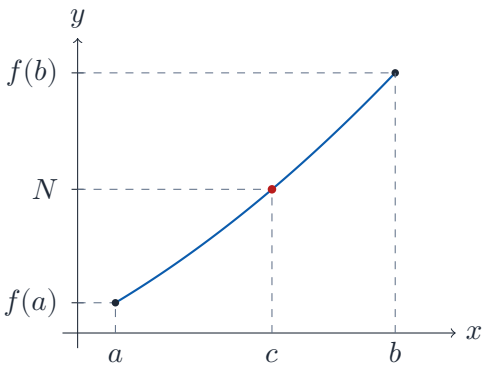


Figure 5.4.1: A continuous curve from $f(a)<N$ to $f(b)>N$ must meet $y=N$.

Proof. If N equals $f(a)$ or $f(b)$, take $c=a$ or $c=b$. If $f(a)>N>f(b)$, apply the remaining case to $-f$ and $-N$. Assume therefore that $f(a)<N<f(b)$ and let

$$S=\{x\in[a,b]:f(x)<N\}.$$

Because S contains a and is bounded above by b, completeness (Assumption 2.4.5) gives it a supremum $c = \sup S$ (Definition 2.4.3) with $a \le c \le b$. We show that $f(c) < N$ and $f(c) > N$ are both impossible.

Suppose $f(c) < N$. Then $c \neq b$ because $f(b) > N$. By persistence of sign (Lemma 5.3.8) there is a $\delta > 0$ with $f(x) < N$ for every $x \in [a, b]$ with $|x - c| < \delta$. The point $x = \min\{c + \delta/2,\, b\}$ lies in $[a, b]$, exceeds c and satisfies $f(x) < N$. Then $x \in S$ and $x > \sup S$. This is absurd.

Suppose $f(c) > N$. Then $c \neq a$ and Lemma 5.3.8 gives a $\delta > 0$ with $f(x) > N$ for every $x \in [a, b]$ with $|x - c| < \delta$. Since $c - \delta$ is not an upper bound of S, some $s \in S$ has $s > c - \delta$. Also $s \le c$ because c is an upper bound. Then $|s - c| < \delta$ and $f(s) > N$. This contradicts $s \in S$.

Hence $f(c) = N$. ∎

The proof uses completeness to obtain the supremum c. Over the rationals the argument collapses because the set $\{x \in [1, 2] \cap \mathbb{Q} : x^2 < 2\}$ has no rational supremum.

Corollary 5.4.3 (Existence of nth roots)**.** *For every $y \ge 0$ and every integer $n \ge 1$ there is exactly one $x \ge 0$ with $x^n = y$.*

Proof. This is Theorem 4.4.6 with the case $y = 0$ added. The Intermediate Value Theorem gives a second proof. The polynomial $f(x) = x^n$ is continuous on $[0, 1 + y]$, with $f(0) = 0 \le y$ and $f(1 + y) = (1 + y)^n \ge 1 + y > y$ because $1 + y \ge 1$. The Intermediate Value Theorem gives $x \in [0, 1 + y]$ with $x^n = y$. If $0 \le u < v$, then $u^n < v^n$. Two different nonnegative numbers therefore have different nth powers and the root is unique. ∎

A set $J \subseteq \mathbb{R}$ is an interval exactly when it contains every point between any two of its points. If J has this property, put $\alpha = \inf J$ and $\beta = \sup J$ (either may be infinite). For every y with $\alpha < y < \beta$, the definitions of infimum and supremum give points $y_1, y_2 \in J$ with $y_1 < y < y_2$. By the property of J, the point y lies in J. The set J is then one of the intervals of Definition 2.1.8 with endpoints α and β.

Corollary 5.4.4 (Continuous images of intervals)**.** *If f is continuous on an interval I, then its image $f(I)$ is an interval.*

Proof. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$ be points of $f(I)$ with $y_1 < y < y_2$. Applied on the closed interval with endpoints x_1 and x_2 (contained in I), Theorem 5.4.2 gives a point c with $f(c) = y$. Hence $y \in f(I)$. ∎

Worked Example 5.4.1

Show that $x^3 - x - 1 = 0$ has a root in $[1, 2]$ and locate it in an interval of length $\frac{1}{8}$ by bisection.

Since the polynomial $f(x) = x^3 - x - 1$ is continuous with $f(1) = -1 < 0$ and $f(2) = 5 > 0$, Theorem 5.4.2 with $N = 0$ gives a root in $[1, 2]$. *Bisection* evaluates f at the midpoint and keeps the half on which f changes sign:

interval	midpoint m	$f(m)$	new interval
$[1, 2]$	1.5	0.875	$[1, 1.5]$
$[1, 1.5]$	1.25	-0.296875	$[1.25, 1.5]$
$[1.25, 1.5]$	1.375	0.224609375	$[1.25, 1.375]$

The root lies in the interval $[1.25, 1.375]$ of length $\frac{1}{8}$. After k steps the interval has length 2^{-k}. The left endpoints form a nondecreasing bounded sequence with $f < 0$ and the right endpoints a nonincreasing one with $f > 0$. By Theorem 5.1.18 both converge. Their

difference tends to 0 and the two limits are therefore one number c. The continuity of f with Theorem 5.1.10 gives $f(c) \leq 0 \leq f(c)$. The root is 1.3247 ....

Worked Example 5.4.2

Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point c with $f(c) = c$. Deduce that $\cos x = x$ has a solution in $[0, \pi/2]$.
The function $g(x) = f(x) - x$ is continuous on $[0,1]$ by Theorem 5.3.6. Because the values of f lie in $[0,1]$, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Since 0 lies between $g(1)$ and $g(0)$, Theorem 5.4.2 gives c with $g(c) = 0$, that is $f(c) = c$.
For the equation, apply the same idea to $h(x) = \cos x - x$ on $[0, \pi/2]$. This function is continuous by Theorem 5.3.10 with $h(0) = 1 > 0$ and $h(\pi/2) = -\pi/2 < 0$. A solution therefore lies in $(0, \pi/2)$. Numerically it is 0.7391 ....

5.4.5 Extreme Value Theorem

Theorem 5.4.6 (Extreme Value Theorem)**.** *If f is continuous on a closed bounded interval $[a,b]$, then f is bounded and attains a maximum and a minimum value on $[a,b]$ (Fig. 5.4.2).*

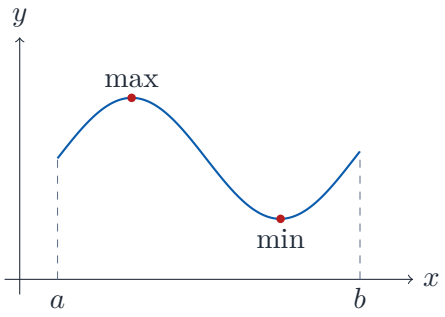


Figure 5.4.2: On a closed interval a continuous function attains a highest and a lowest value.

Proof. **1.** Boundedness. Suppose f is not bounded above on $[a,b]$. Then for each n there is $x_n \in [a,b]$ with $f(x_n) > n$. By the Bolzano–Weierstrass theorem (Theorem 5.1.25) a subsequence x_{n_k} converges to some $c \in [a,b]$. By continuity and Theorem 5.3.5, $f(x_{n_k}) \to f(c)$. The convergent sequence $f(x_{n_k})$ is therefore bounded (Theorem 5.1.7) in contradiction with $f(x_{n_k}) > n_k \geq k$. The same argument applied to $-f$ shows that f is bounded below.
2. Maximum attained. The set of values $f([a,b])$ is nonempty and bounded above and has a supremum M by Assumption 2.4.5. For each n the number $M - 1/n$ is not an upper bound. Hence some $x_n \in [a,b]$ has

$$M - \frac{1}{n} < f(x_n) \leq M.$$

A subsequence x_{n_k} converges to some $c \in [a,b]$ by Theorem 5.1.25. Along it, $f(x_{n_k}) \to f(c)$ by continuity and $f(x_{n_k}) \to M$ by the Sandwich Theorem because $n_k \to \infty$. Uniqueness of limits gives $f(c) = M$.
3. Minimum attained. Apply the previous step to $-f$. ∎

Example 5.4.7 (Necessity of the hypotheses of the Extreme Value Theorem)**.** On the half-open interval $(0,1]$ the continuous function $1/x$ is unbounded. On $[0,1)$ the function $f(x) = x$ is bounded and no point attains its supremum 1. On $[-1,1]$ the function equal to $1 - |x|$ for

$x \neq 0$ and to 0 at 0 is bounded and discontinuous at 0 and never attains its supremum 1. On the unbounded closed interval $[0, \infty)$ the function x has no maximum.

The Extreme Value Theorem asserts existence and gives no method for locating the extreme points. Differential calculus locates them (Part II). The error bounds for numerical integration in Section 9.7 rely on the maximum that the theorem guarantees.

5.4.8 Continuous inverses

A function that is strictly increasing on an interval is injective with an inverse (Definition 1.5.24) defined on its image. The logarithm, the nth root and the inverse trigonometric functions all arise in this way.

Theorem 5.4.9 (Continuity of inverse functions)**.** *Let f be continuous and strictly increasing on an interval I. Then $J = f(I)$ is an interval and the inverse function $f^{-1} : J \to I$ is continuous and strictly increasing. The same holds with "decreasing" in place of "increasing".*

Proof. J is an interval by Corollary 5.4.4. Write $g = f^{-1}$. If $y < y'$ in J and $g(y) \geq g(y')$, then applying the increasing function f gives $y \geq y'$. This is absurd. Hence g is strictly increasing.

Let $y_0 = f(x_0) \in J$ and $\varepsilon > 0$. If x_0 is not the right endpoint of I, pick a point $x_+ \in I$ with $x_0 < x_+ < x_0 + \varepsilon$ and set $\delta_+ = f(x_+) - y_0 > 0$. If x_0 is the right endpoint of I, then y_0 is the largest point of J and we set $\delta_+ = 1$. Define x_- and δ_- symmetrically on the left and let $\delta = \min\{\delta_+, \delta_-\}$. Let $y \in J$ with $|y - y_0| < \delta$. If $y > y_0$, then x_0 is not the right endpoint and $y < y_0 + \delta_+ = f(x_+)$. Applying the increasing function g gives $x_0 < g(y) < x_+ < x_0 + \varepsilon$. If $y < y_0$, the symmetric argument gives $x_0 - \varepsilon < g(y) < x_0$. In every case $|g(y) - x_0| < \varepsilon$.

For a decreasing f, apply the result to the increasing function $-f$ and use the identity $f^{-1}(y) = (-f)^{-1}(-y)$. ∎

5.4.10 Exponentials completed

Section 4.4 proved the laws of exponents for real exponents with one exception: the power law $(a^x)^y = a^{xy}$ for irrational y. It also deferred the fact that a^x takes every positive value. Both need continuity. Theorem 5.3.10 proved the continuity of $x \mapsto a^x$ from parts (1), (2) and (4) of Theorem 4.4.16 alone. The argument below therefore uses nothing that depends on the facts it proves.

Lemma 5.4.11 (Continuous functions agreeing on the rationals)**.** *If f and g are continuous on $\mathbb{R}$ and $f(r) = g(r)$ for every rational r, then $f = g$.*

Proof. Let x be real. By Theorem 2.4.12 each interval $(x, x + 1/n)$ contains a rational r_n. Then $r_n \to x$ and Theorem 5.3.5 gives $f(x) = \lim f(r_n) = \lim g(r_n) = g(x)$. ∎

Theorem 5.4.12 (Power law and bijectivity of real exponentials)**.** *Let $a > 0$.*

1. *$(a^x)^y = a^{xy}$ for all real x and y. This completes part (5) of Theorem 4.4.16.*
2. *If $a \neq 1$, then $x \mapsto a^x$ is a bijection from $\mathbb{R}$ onto $(0, \infty)$. This completes Theorem 4.4.17.*

Proof. (1) Fix x and put $b = a^x > 0$. The functions $y \mapsto b^y$ and $y \mapsto a^{xy}$ are continuous on $\mathbb{R}$, the first by Theorem 5.3.10 and the second as the composition of $y \mapsto xy$ with $t \mapsto a^t$ (Theorem 5.3.6). The proof of Theorem 4.4.16 shows that they agree at every rational y. By Lemma 5.4.11 they agree everywhere.

(2) Injectivity is part (4) of Theorem 4.4.16 with Theorem 4.1.19, and every value is positive by part (1). For surjectivity let $a > 1$ and $y > 0$. By Theorem 2.4.10 there is an integer

$N \geq 1$ with $1 + N(a-1) > \max\{y, 1/y\}$. Bernoulli's inequality (Theorem 2.3.11) gives $a^N \geq 1 + N(a-1) > y$ and $a^{-N} = 1/a^N < y$. The Intermediate Value Theorem applied to the continuous function a^x on $[-N, N]$ (Theorem 5.3.10) gives $x \in [-N, N]$ with $a^x = y$. For $0 < a < 1$ write $a^x = (1/a)^{-x}$ and apply the case just proved to the base $1/a > 1$. ∎

With Theorem 5.4.12 the logarithm of Definition 4.4.22 is defined on all of $(0, \infty)$. The identities $a^x = e^{x \ln a}$ and $x^r = e^{r \ln x}$ of Theorem 4.4.24 rest on the power law and hold for all real exponents.

Corollary 5.4.13 (Continuity of the inverse elementary functions)**.** *The following are continuous on the stated domains: the nth root $x^{1/n}$ on $[0, \infty)$; the natural logarithm on $(0, \infty)$; the power $x^r = e^{r \ln x}$ on $(0, \infty)$ for every real r;* arcsin *and* arccos *on* $[-1, 1]$*;* arctan *on* $\mathbb{R}$*.*

Proof. The function x^n is continuous and strictly increasing on $[0, \infty)$ with image $[0, \infty)$ by Corollary 5.4.3. Its inverse $x^{1/n}$ is continuous by Theorem 5.4.9. The exponential is continuous (Theorem 5.3.10) and strictly increasing on $\mathbb{R}$ with image $(0, \infty)$ (Theorem 5.4.12) and inverse ln. The power $e^{r \ln x}$ is a composition of continuous functions (Theorem 5.3.6). Finally sin is strictly increasing on $[-\pi/2, \pi/2]$, cos is strictly decreasing on $[0, \pi]$, and tan is strictly increasing on $(-\pi/2, \pi/2)$ (Theorem 4.5.13). The images of the first two are $[-1, 1]$ by the Intermediate Value Theorem, since $\sin(\pm\pi/2) = \pm 1$, $\cos 0 = 1$ and $\cos \pi = -1$. For tan let $y > 0$. Since cos is continuous with $\cos(\pi/2) = 0$, there is $c \in (\pi/4, \pi/2)$ with $\cos c < 1/(2y)$. Then $\sin c > \sin(\pi/4) > \frac{1}{2}$ and $\tan c > y$. Because tan is odd, $\tan(-c) < -y$. The Intermediate Value Theorem on $[-c, c]$ shows that y, $-y$ and 0 are values of tan. The image is therefore $\mathbb{R}$. The inverses are arcsin, arccos and arctan. ∎

With Corollary 5.4.13 every function of Chapter 4 is known to be continuous on its domain.

5.4.14 Exponential as a limit

Definition 4.4.19 defines e as the limit of $(1 + \frac{1}{n})^n$. The same limit with x/n in place of $1/n$ gives e^x for every real x. Chapter 6 differentiates the exponential from (5.4.1).

Theorem 5.4.15 (Exponential as a limit)**.** *For every real x,*

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n. \tag{5.4.1}$$

Proof. Write $a_n(x) = (1 + x/n)^n$. The case $x = 0$ is trivial.
1. Convergence for $x > 0$. The binomial expansion (5.1.7) of the proof of Theorem 5.1.20, with x^k in each term, gives

$$a_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \tag{5.4.2}$$

As there, passing from n to $n+1$ enlarges every term and adds a positive one. The sequence $a_n(x)$ is therefore increasing. Let $N \geq x$ be an integer. Bernoulli's inequality gives $(1 + 1/n)^N \geq 1 + N/n \geq 1 + x/n$, hence $a_n(x) \leq \left((1 + 1/n)^n\right)^N < 3^N$ by Theorem 5.1.20. By Theorem 5.1.18 the limit $E(x) = \lim a_n(x)$ exists. Since $a_n(x)$ increases with x for each n, Theorem 5.1.10 gives $E(x) \leq E(y)$ whenever $0 < x \leq y$.
2. Rational $x > 0$. Let $x = p/q$ with positive integers p and q. Along the indices $n = pk$,

$$a_{pk}(x) = \left(1 + \frac{1}{qk}\right)^{pk} = c_k^{p/q}, \qquad c_k = \left(1 + \frac{1}{qk}\right)^{qk},$$

by the power law of Theorem 4.4.10. The sequence c_k is a subsequence of the sequence of Theorem 5.1.20 and tends to e (Theorem 5.1.23). The function $t \mapsto t^{p/q} = (t^{1/q})^p$ is continuous on $(0, \infty)$ (Corollary 5.4.13 and Theorem 5.3.6). Hence $a_{pk}(x) \to e^{p/q}$ by Theorem 5.3.5. Since $\{a_{pk}(x)\}$ is a subsequence of the convergent sequence $\{a_n(x)\}$, $E(p/q) = e^{p/q}$.

3. Real $x > 0$. For every rational r with $0 < r < x$, $e^r = E(r) \leq E(x)$. Since the sequence increases from $a_1(x) = 1 + x$, $E(x) \geq 1 + x > 1$. Rationals $r \leq 0$ therefore have $e^r \leq 1 < E(x)$. By Definition 4.4.13, e^x is the supremum of the e^r over rationals $r < x$. Hence $e^x \leq E(x)$. Conversely, for each $m \geq 1$ choose a rational s with $x < s < x + 1/m$ (Theorem 2.4.12). Then $E(x) \leq E(s) = e^s = e^x e^{s-x} \leq e^x e^{1/m}$. Letting $m \to \infty$ with $e^{1/m} \to 1$ (Proposition 5.1.15) gives $E(x) \leq e^x$.

4. Negative x. Let $x < 0$ and $n > |x|$. Then $a_n(x)\, a_n(-x) = (1 - x^2/n^2)^n$ with $0 < x^2/n^2 < 1$, and Bernoulli's inequality gives

$$1 \geq \left(1 - \frac{x^2}{n^2}\right)^n \geq 1 - \frac{x^2}{n}.$$

The product therefore tends to 1 (Theorem 5.1.11). Since $a_n(-x) \to e^{-x} > 0$ by the case already proved, $a_n(x) \to 1/e^{-x} = e^x$ by Theorem 5.1.9. ∎

Study guide

The two theorems of this section are existence theorems with hypotheses that must be checked every time. For the Intermediate Value Theorem, name the function, state that it is continuous on the closed interval, compute the values at the endpoints and say which number lies between them. For the Extreme Value Theorem, check that the interval is closed and bounded and that the function is continuous on all of it. The examples after Theorem 5.4.6 show what fails when a hypothesis is dropped.

To show that an equation $F(x) = G(x)$ has a solution, apply the Intermediate Value Theorem to $F - G$ with $N = 0$. To show that the solution is unique as well, show that $F - G$ is strictly monotone.

Problems

5.4.1 Show that $x^5 + x - 1 = 0$ has a solution in $(0, 1)$. ⋆ Hint · Answer · Solution

5.4.2 Apply three steps of bisection to $f(x) = x^2 - 3$ on $[1, 2]$. In which interval of length $\frac{1}{8}$ does $\sqrt{3}$ lie? ⋆ Hint · Answer · Solution

5.4.3 Find the maximum and minimum values of $f(x) = |x^2 - 1|$ on $[-2, 2]$ and the points where they are attained. ⋆ Hint · Answer · Solution

5.4.4 Decide whether each function attains a maximum and a minimum on the given interval and explain the result with reference to Theorem 5.4.6: (a) x^2 on $[-1, 2]$; (b) $1/x$ on $(0, 1]$; (c) x on $[0, 1)$. ⋆⋆ Hint · Answer · Solution

5.4.5 Show that $e^x = 3 - x$ has exactly one real solution and that it lies in $(0, 1)$. ⋆⋆ Hint · Answer · Solution

5.4.6 Evaluate $\lim_{x \to 1} \arctan x$, $\lim_{x \to 2} \sqrt{x^2 + 5}$ and $\lim_{x \to 0} \ln(1 + \sin x)$. ⋆⋆ Hint · Answer · Solution

5.4.7 Prove that every polynomial of odd degree has a real root.

⋆⋆ Hint · Answer · Solution

5.4.8 Let f be continuous on $[0,1]$ with $f(0) = f(1)$. Prove that there is a $c \in [0, \frac{1}{2}]$ with $f(c) = f(c + \frac{1}{2})$.

⋆⋆⋆ Hint · Answer · Solution

5.4.9 Prove that a continuous injective function on an interval I is strictly monotone.

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5.5 Uniform Continuity

Overview

Objectives: State the definition of uniform continuity (Definition 5.5.1) and compare its quantifiers with those of continuity; prove uniform continuity with a Lipschitz bound (Proposition 5.5.3); disprove it with two sequences (Proposition 5.5.6); prove the Heine–Cantor theorem (Theorem 5.5.10).
Prerequisites: Continuity (Definition 5.3.1); the Bolzano–Weierstrass theorem (Theorem 5.1.25); negating quantified statements (Section 1.2).
Key results: The definition Definition 5.5.1; Lipschitz functions are uniformly continuous (Proposition 5.5.3); the sequential test for failure (Proposition 5.5.6); a continuous function on a closed bounded interval is uniformly continuous (Theorem 5.5.10).

In Definition 5.3.1 the number δ is chosen after the point a and may depend on it. For $f(x) = 1/x$ on $(0, 1)$ and a fixed tolerance ε, the admissible δ shrinks to zero as a approaches 0 because the graph becomes steeper there (Fig. 5.5.1). To control a Riemann sum everywhere with one fine partition, the theory of the integral in Chapter 8 requires a single δ that works at every point of an interval at once. Uniform continuity is the name for that property.

Definition 5.5.1 (Uniform continuity)**.** A function f defined on a set D is *uniformly continuous on* D if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in D \text{ and } |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon. \tag{5.5.1}$$

The difference from continuity on D lies entirely in the order of the quantifiers:

$$\begin{aligned} &\text{continuous on } D\text{:} \\ &\qquad \forall a \in D \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in D : \;\; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon, \\ &\text{uniformly continuous on } D\text{:} \\ &\qquad \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall a \in D \;\; \forall x \in D : \;\; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \end{aligned} \tag{5.5.2}$$

In the second statement δ is chosen before a and cannot depend on it. A uniformly continuous function is therefore continuous at every point of D. The converse fails.

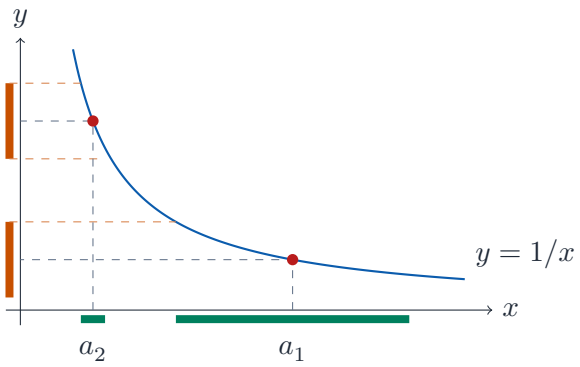


Figure 5.5.1: Same tolerance at two points of $y = 1/x$. Orange bands on the y-axis have the common half-width $\varepsilon = \frac{1}{2}$. Green windows on the x-axis are the largest symmetric intervals mapped into them. Near 0 the window is much narrower.

5.5.2 Proving uniform continuity

Proposition 5.5.3 (Lipschitz functions are uniformly continuous)**.** *If there is a constant $K > 0$ with $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in D$, then f is uniformly continuous on D.*

Proof. Given $\varepsilon > 0$ take $\delta = \varepsilon/K$. Then $|x - y| < \delta$ gives $|f(x) - f(y)| \le K|x - y| < \varepsilon$. ∎

A function satisfying the hypothesis of Proposition 5.5.3 is called *Lipschitz* with constant K.

Example 5.5.4. Every linear function $f(x) = mx + c$ is Lipschitz on $\mathbb{R}$ with constant $|m|$ (any $K > 0$ if $m = 0$). The sine is Lipschitz with constant 1. Together with the bound $|\sin t| \le |t|$, the identity $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$ from the addition formulas of Section 4.5 gives $|\sin x - \sin y| \le 2 \cdot \left|\frac{x-y}{2}\right| = |x - y|$.

Worked Example 5.5.1

Show that $f(x) = x^2$ is uniformly continuous on $[0, 3]$ and give δ in terms of ε.
For $x, y \in [0, 3]$,

$$|x^2 - y^2| = |x + y|\,|x - y| \le 6|x - y|,$$

because $x + y \le 6$. Thus f is Lipschitz on $[0, 3]$ with constant 6 and Proposition 5.5.3 gives $\delta = \varepsilon/6$. The same δ works at every point of $[0, 3]$. The argument uses the bound $|x + y| \le 6$ that the boundedness of the interval provides. Proposition 5.5.6 shows that on all of $\mathbb{R}$ no δ works.

5.5.5 Disproving uniform continuity

Negating Definition 5.5.1 produces a test with sequences.

Proposition 5.5.6 (Sequential test for uniform continuity)**.** *A function f is* not *uniformly continuous on D if and only if there are an $\varepsilon_0 > 0$ and sequences $\{x_n\}$, $\{y_n\}$ in D with $x_n - y_n \to 0$ and $|f(x_n) - f(y_n)| \ge \varepsilon_0$ for every n.*

Proof. The negation of Definition 5.5.1 reads: some $\varepsilon_0 > 0$ has, for every $\delta > 0$, points $x, y \in D$ with $|x - y| < \delta$ and $|f(x) - f(y)| \ge \varepsilon_0$. Taking $\delta = 1/n$ gives the sequences. Conversely, given such sequences and any $\delta > 0$, choose n with $|x_n - y_n| < \delta$. The pair x_n, y_n violates the definition with $\varepsilon = \varepsilon_0$. ∎

Example 5.5.7. The function x^2 is not uniformly continuous on $\mathbb{R}$. With $x_n = n + 1/n$ and $y_n = n$ the differences $x_n - y_n = 1/n$ tend to 0 while $x_n^2 - y_n^2 = 2 + 1/n^2 \ge 2$. Take $\varepsilon_0 = 2$.

Example 5.5.8. The function $1/x$ is not uniformly continuous on $(0, 1)$. With $x_n = 1/n$ and $y_n = 1/(n + 1)$ in $(0, 1)$ for $n \ge 2$, the differences $x_n - y_n = 1/(n(n + 1))$ tend to 0 while $|1/x_n - 1/y_n| = 1$.

In both examples the function is continuous at every point of its domain. The failure comes from the domain: unbounded in the first, not closed in the second.

Worked Example 5.5.2

Show that $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$. Then find for each $a \in (0, 1)$ a δ that proves continuity at a. Explain why the second answer does not give uniform continuity on $(0, 1)$.

For $x, y \geq 1$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leq |x-y|,$$

so f is Lipschitz with constant 1 on $[1, \infty)$ and $\delta = \varepsilon$ works there.
At $a \in (0,1)$, restrict first to $|x-a| < a/2$. On that interval $x > a/2$ and $xa > a^2/2$. Consequently

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{xa} < \frac{2}{a^2}|x-a|,$$

and $\delta = \min\{a/2,\ a^2\varepsilon/2\}$ proves continuity at a. This δ tends to 0 as $a \to 0^+$. The second example before this worked example shows that no positive δ serves all $a \in (0,1)$ at once. On $[1, \infty)$ the bound $1/(xa) \leq 1$ lets one δ suffice.

5.5.9 Heine–Cantor theorem

On a closed bounded interval the two notions coincide.

Theorem 5.5.10 (Heine–Cantor)**.** *A function continuous on a closed bounded interval $[a,b]$ is uniformly continuous on $[a,b]$.*

Proof. Suppose f is continuous on $[a,b]$ and not uniformly continuous. By Proposition 5.5.6 there are $\varepsilon_0 > 0$ and sequences $\{x_n\}$, $\{y_n\}$ in $[a,b]$ with $x_n - y_n \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. By the Bolzano–Weierstrass theorem (Theorem 5.1.25) a subsequence x_{n_k} converges to some $c \in [a,b]$. Then

$$y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k}) \to c - 0 = c$$

by Theorems 5.1.9 and 5.1.23. Continuity at c and Theorem 5.3.5 give $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$. Hence $f(x_{n_k}) - f(y_{n_k}) \to 0$. This contradicts $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0$ for every k. ∎

The proof uses both halves of the hypothesis on the interval. Boundedness lets the Bolzano–Weierstrass theorem extract a convergent subsequence. Closedness puts its limit c inside the interval on which f is continuous. The examples x^2 on $\mathbb{R}$ and $1/x$ on $(0,1)$ show that neither half can be dropped.

Uniform continuity and integrability

Chapter 8 uses Theorem 5.5.10 to prove that a continuous function on $[a,b]$ is integrable (Theorem 8.2.13). Given $\varepsilon > 0$, a single δ makes the oscillation of f on every subinterval of length less than δ smaller than ε. Take two partitions that fine and a Riemann sum Eq. (8.2.2) for each. Both sums lie within $\varepsilon(b-a)$ of any Riemann sum of the common refinement and therefore within $2\varepsilon(b-a)$ of each other. The sums therefore converge as the partition is refined.

Continuity on an interval

Continuity on an interval does not imply uniform continuity unless the interval is closed and bounded. A common error is to apply Theorem 5.5.10 on $(0,1]$ or on $[0,\infty)$. The function $\sqrt{x}$ is uniformly continuous on $[0,\infty)$ (Problem 5.5.3) while $1/x$ is not uniformly continuous on $(0,1]$. A proof is needed in each case.

Study guide

To prove uniform continuity on a closed bounded interval, check continuity and cite Theorem 5.5.10. On any other set, look for a Lipschitz bound: factor $f(x)-f(y)$ so that $|x-y|$ appears and bound the remaining factor uniformly over the set, as in Worked Example 5.5.1. To disprove uniform continuity, find two sequences that approach each other while their images stay a fixed distance apart (Proposition 5.5.6). Such sequences are usually easy to find near a point where the graph becomes vertical and towards infinity when the graph steepens without bound.

Problems

5.5.1 Show that $f(x)=4x-7$ is uniformly continuous on $\mathbb{R}$ and give δ in terms of ε.

⋆ Hint · Answer · Solution

5.5.2 Show that $f(x)=x^3$ is uniformly continuous on $[-2,2]$ by finding a Lipschitz constant.

⋆ Hint · Answer · Solution

5.5.3 Prove that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$. ⋆⋆ Hint · Answer · Solution

5.5.4 Prove that $f(x)=\dfrac{1}{1+x^2}$ is Lipschitz on $\mathbb{R}$ with constant 1.

⋆⋆ Hint · Answer · Solution

5.5.5 Use Proposition 5.5.6 to show that $\ln x$ is not uniformly continuous on $(0,1)$.

⋆⋆ Hint · Answer · Solution

5.5.6 Show that $\sin(x^2)$ is not uniformly continuous on $[0,\infty)$, although it is bounded and continuous. ⋆⋆⋆ Hint · Answer · Solution

5.5.7 Let f be uniformly continuous on D and let $\{x_n\}$ be a Cauchy sequence in D (Definition 5.1.27). Prove that $\{f(x_n)\}$ is a Cauchy sequence and deduce again that $1/x$ is not uniformly continuous on $(0,1)$. ⋆⋆⋆ Hint · Answer · Solution

5.5.8 Prove that a function uniformly continuous on a bounded interval I is bounded on I. ⋆⋆⋆ Hint · Answer · Solution

Part II

Differential Calculus

Chapter 6

Derivative

Two questions start differential calculus. The first is geometric: what is the line tangent to a curve at a point, when the curve is not a circle and the tangent cannot be described as the line meeting it once? The second is physical: what is the speed of a moving body at one instant, when speed has been defined only as distance divided by elapsed time over an interval? Both ask for the same number. A secant line through two points of a graph and the average speed over a time interval are both difference quotients. The answer to each question is the limit of a difference quotient as the second point approaches the first. Chapter 5 made that limit an exact statement on which this chapter builds.

The derivative is introduced in Section 6.1 as a limit and then recast as the slope of the best linear approximation to a function near a point. The second reading leads to the central construction of the book. The differential operator d applied to the coordinate function x produces $\mathrm{d}x$, the linear functional that returns the displacement it is given. Multiplying $\mathrm{d}x$ by a function f gives the 1-form $f(x)\,\mathrm{d}x$, one of the objects that integrals integrate. Every integral, substitution and change of variable in later chapters is read through the objects that Section 6.2 defines. The same section states the derivative of a map between normed spaces as multivariable calculus and analysis define it and shows that the derivative of one variable is its case $\mathbb{R} \to \mathbb{R}$. The rules of differentiation follow in Section 6.3, each proved from the definition and stated twice, once for f' and once for d. Section 6.4 differentiates the exponential, logarithmic, trigonometric and inverse trigonometric functions of Chapter 4. Section 6.5 applies the rules to curves given by equations and to quantities that change together in time. The chapter uses the limit laws and continuity theorems of Chapter 5 throughout. It cites one result of Chapter 7, that a function with zero derivative on an interval is constant (Corollary 7.2.2). Its proof uses the Extreme Value Theorem and, from this chapter, only Section 6.1 and the basic rules of Theorem 6.3.1. The derivative of the sine also rests on a result of Chapter 8. The limit $\lim_{h\to 0}(\sin h)/h = 1$ of Chapter 5 uses the inequality $\sin x < x < \tan x$, and Theorem 8.6.6 proves it from an arc-length integral of an algebraic function. That proof uses no trigonometric derivative.

6.1 Derivative as a Best Linear Approximation

Overview

Objectives: Compute a derivative from the limit of a difference quotient; state differentiability in the equivalent form $f(a+h) = f(a) + f'(a)h + o(h)$ and use it; prove that a differentiable function is continuous; compute one-sided derivatives and use them to decide differentiability at a corner; recognise the three standard ways a continuous function fails to be differentiable.

Prerequisites: The ε-δ limit (Definition 5.2.1) and its shift form (Proposition 5.2.5); one-sided limits and the limit laws (Section 5.2); continuity (Definition 5.3.1).

Key results: The derivative Eq. (6.1.2); differentiability as linear approximation with an $o(h)$ error (Theorem 6.1.9); the tangent line as the unique best linear approximation (Proposition 6.1.10); differentiability implies continuity (Theorem 6.1.11).

Notation

$Q(h)$	the difference quotient of f at a (Definition 6.1.2)
$f'(a)$	the derivative of f at a
f'	the derivative function $x \mapsto f'(x)$
$o(h)$	a quantity $r(h)$ with $r(h)/h \to 0$ as $h \to 0$ (Definition 6.1.7)
$f'_+(a)$, $f'_-(a)$	the right-hand and left-hand derivatives (Definition 6.1.13)

6.1.1 Tangent problem and rates of change

Let f be a function defined on an open interval containing the point a. For a number $h \neq 0$ small enough that $a+h$ also lies in that interval, the line through the two points $(a, f(a))$ and $(a+h, f(a+h))$ of the graph is a *secant line*. Its slope is the change in f divided by the change in the input. If $f(t)$ is the position of a body at time t, the same quotient is the average velocity of the body over the time interval from a to $a+h$. In both readings the quotient depends on two points. The tangent slope or the instantaneous velocity at a should be the number that the quotient approaches as the second point moves towards the first.

Definition 6.1.2 (Difference quotient and derivative)**.** Let f be defined on an open interval containing a. The *difference quotient* of f at a is

$$Q(h) = \frac{f(a+h) - f(a)}{h} \qquad (h \neq 0), \tag{6.1.1}$$

the slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$. The *derivative* of f at a is the limit of this quotient,

$$f'(a) = \lim_{h \to 0} Q(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{6.1.2}$$

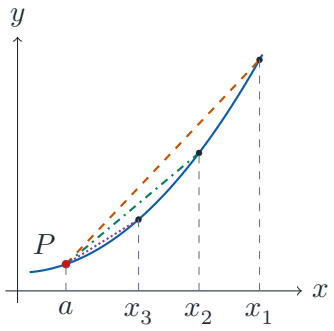
whenever the limit exists as a real number. By Proposition 5.2.5 this is exactly the ε-δ limit (Definition 5.2.1) read at $h \to 0$: for every $\varepsilon > 0$ there is a $\delta > 0$ with $0 < |h| < \delta \Rightarrow |Q(h) - f'(a)| < \varepsilon$.

Since Q is undefined at $h = 0$ the derivative is a limit of Q and never a value $Q(0)$. Writing $x = a + h$ turns Eq. (6.1.2) into the equivalent form

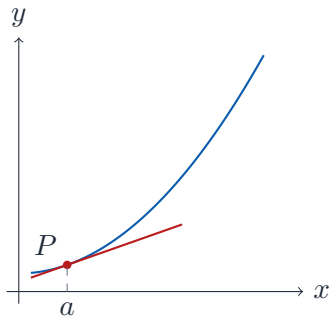
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \tag{6.1.3}$$

again by Proposition 5.2.5. Figure 6.1.1 shows the secants of slope $Q(h)$ turning into the tangent as $h \to 0$.

Definition 6.1.3 (Differentiable function)**.** A function f defined on an open interval containing a is *differentiable at a* when the limit Eq. (6.1.2) exists as a real number. It is *differentiable on an open interval I* when it is differentiable at every point of I. The *derivative function* of such



(a) Secants of slope $Q(h_i)$ from $P = (a, f(a))$ to $(x_i, f(x_i))$ with $x_i = a + h_i$ and $h_1 > h_2 > h_3$.



(b) Limiting tangent of slope $f'(a)$ at $P = (a, f(a))$.

Figure 6.1.1: As $h \to 0$ the feet $x_i = a + h_i$ move towards a and the secants Eq. (6.1.1) approach the tangent with slope equal to the derivative Eq. (6.1.2).

an f is the function f' on I whose value at each $x \in I$ is $f'(x)$. The *tangent line* to the graph of f at a is the line $y = f(a) + f'(a)(x - a)$.

Example 6.1.4 (Derivative of the square)**.** Let $f(x) = x^2$ and fix a. For $h \neq 0$,

$$Q(h) = \frac{(a+h)^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h.$$

The cancellation of h is legitimate because $h \neq 0$ throughout. Since $2a + h \to 2a$ as $h \to 0$ the function f is differentiable at every a with $f'(a) = 2a$. The derivative function is $f'(x) = 2x$. At $a = 3$ the tangent line is $y = 9 + 6(x - 3)$.

Example 6.1.5 (Non-example: a jump)**.** Let $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. At $a = 0$ and $h < 0$ the difference quotient $Q(h) = (0 - 1)/h = -1/h$ is unbounded as $h \to 0^-$. Since the limit Eq. (6.1.2) does not exist, f is not differentiable at 0. Theorem 6.1.11 below explains the failure: f is not even continuous at 0.

6.1.6 Linear approximation with an error of order little-o

The difference quotient describes the derivative through slopes. A second description compares the function with a linear one and asks how fast the error decreases. This second description generalises to functions of several variables and is the one from which Section 6.2 builds the operator d.

Definition 6.1.7 (Little-o)**.** Let r be a function defined for $0 < |h| < \eta$ for some $\eta > 0$. We write $r(h) = o(h)$ as $h \to 0$ and say that r is *little-o of* h when

$$\lim_{h \to 0} \frac{r(h)}{h} = 0. \tag{6.1.4}$$

An equation $A(h) = B(h) + o(h)$ means that the difference $A(h) - B(h)$ is little-o of h.

The symbol $o(h)$ names the property of a function that its ratio with h tends to 0. Because it never names one particular function the equation $o(h) - o(h) = 0$ is false: h^2 and h^3 are both $o(h)$ with the non-zero difference $h^2 - h^3$. The valid rules follow from the limit laws. If $r_1(h) = o(h)$, $r_2(h) = o(h)$ and c is a constant, then $r_1(h) + r_2(h) = o(h)$ and $c\, r_1(h) = o(h)$. If $b(h) \to 0$, then $b(h) \cdot h = o(h)$.

Example 6.1.8 (Little-o functions of h)**.** The functions h^2, h^3 and $|h|^{3/2}$ are all $o(h)$, because their ratios with h are h, h^2 and $\pm|h|^{1/2}$, each tending to 0. The function $5h$ is *not* $o(h)$ because $5h/h = 5$. Since $\sin(1/h)$ has no limit at 0, the function $h\sin(1/h)$ is not $o(h)$ either.

Theorem 6.1.9 (Differentiability as linear approximation)**.** *Let f be defined on an open interval containing a. Then f is differentiable at a if and only if there is a number L such that*

$$f(a+h) = f(a) + L\,h + o(h) \qquad \text{as } h \to 0. \tag{6.1.5}$$

When such an L exists it is unique and equal to $f'(a)$.

Proof. Define $r(h) = f(a+h) - f(a) - Lh$ for $h \neq 0$. Dividing by h,

$$\frac{r(h)}{h} = Q(h) - L. \tag{6.1.6}$$

If f is differentiable at a, take $L = f'(a)$. Then Eq. (6.1.2) and (6.1.6) give $r(h)/h = Q(h) - L \to 0$, that is, $r(h) = o(h)$. Conversely, if Eq. (6.1.5) is true for some L, then (6.1.6) and the limit laws give $Q(h) = L + r(h)/h \to L$ and hence $f'(a) = L$. Uniqueness follows because the limit of Q, when it exists, is a single number. ∎

Theorem 6.1.9 splits the increment $f(a+h) - f(a)$ into a part $f'(a)h$ that is linear in h and a remainder that is small compared with h. Among all linear functions of h only one leaves a remainder of that size.

Proposition 6.1.10 (Tangent line as the best linear approximation)**.** *Let f be differentiable at a and let m be a real number. The error $E_m(h) = f(a+h) - f(a) - mh$ is $o(h)$ if and only if $m = f'(a)$.*

Proof. By Theorem 6.1.9, $E_m(h) = (f'(a) - m)h + r(h)$ with $r(h) = o(h)$. Hence $E_m(h)/h \to f'(a) - m$. The limit is 0 exactly when $m = f'(a)$. ∎

Every line through $(a, f(a))$ approximates the graph with an error that tends to 0 when f is continuous at a. The tangent line is the one line whose error tends to 0 faster than h itself. The linear map (Definition 3.3.2) $h \mapsto f'(a)h$ that produces it is the *differential* of f at a, the object that Section 6.2 turns into the 1-form $\mathrm{d}f$.

Theorem 6.1.11 (Differentiable implies continuous)**.** *If f is differentiable at a, then f is continuous at a.*

Proof. For $h \neq 0$, $f(a+h) - f(a) = h \cdot Q(h)$. As $h \to 0$ the first factor tends to 0 and the second to $f'(a)$. By the limit laws the product tends to $0 \cdot f'(a) = 0$. By Proposition 5.2.5 and Definition 5.3.1 the resulting limit $\lim_{h\to 0} f(a+h) = f(a)$ is continuity at a. ∎

Common misconception

The converse of Theorem 6.1.11 is false. The function $|x|$ is continuous at 0 and is not differentiable there (Example 6.1.16 below). Continuity says that $f(a+h) - f(a) \to 0$. Differentiability says that this difference divided by h has a limit.

6.1.12 One-sided derivatives

At a corner of a graph the secants from the right and the secants from the left settle on different slopes. One-sided limits separate the two.

Definition 6.1.13 (One-sided derivatives)**.** The *right-hand derivative* and the *left-hand derivative* of f at a are

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \qquad f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}, \tag{6.1.7}$$

whenever these one-sided limits exist. A function is *differentiable on a closed interval* $[a,b]$ when it is differentiable on (a,b) and the one-sided derivatives $f'_+(a)$ and $f'_-(b)$ exist.

Proposition 6.1.14 (Two-sided derivative from one-sided derivatives)**.** *Let f be defined on an open interval containing a. Then f is differentiable at a if and only if $f'_+(a)$ and $f'_-(a)$ both exist and are equal, with common value $f'(a)$.*

Proof. A limit as $h \to 0$ exists and equals L exactly when both one-sided limits exist and equal L (Section 5.2). Applied to the difference quotient Q this is the statement. ∎

6.1.15 Three ways to fail

Example 6.1.16 (Corner)**.** Let $f(x) = |x|$ and $a = 0$. The difference quotient is $Q(h) = h/h = 1$ for $h > 0$ and $Q(h) = -h/h = -1$ for $h < 0$. Hence $f'_+(0) = 1$ and $f'_-(0) = -1$. Since these differ, $|x|$ is not differentiable at 0 by Proposition 6.1.14. Away from 0 it is differentiable with derivative 1 for $x > 0$ and -1 for $x < 0$.

Example 6.1.17 (Vertical tangent)**.** Let $f(x) = x^{1/3}$ be the real cube root and let $a = 0$. For $h \neq 0$,

$$Q(h) = \frac{h^{1/3}}{h} = \frac{1}{|h|^{2/3}}.$$

This quotient is positive and exceeds any bound M once $0 < |h| < M^{-3/2}$. As the quotient tends to $+\infty$ it has no real limit and f is not differentiable at 0. The secants become steeper without bound and the graph has a vertical tangent at the origin.

Example 6.1.18 (Oscillation)**.** Let $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Since $|f(x)| \le |x|$, the squeeze theorem (Section 5.2) gives $f(x) \to 0 = f(0)$ and hence continuity at 0. Its difference quotient at 0 is

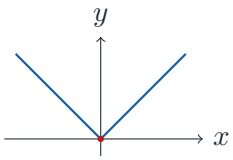
$$Q(h) = \frac{h\sin(1/h) - 0}{h} = \sin\frac{1}{h}.$$

The quotient equals 1 along $h_n = 1/(\frac{\pi}{2} + 2\pi n)$ and -1 along $k_n = 1/(\frac{3\pi}{2} + 2\pi n)$. Since both sequences tend to 0, Q has no limit at 0 and f is not differentiable there. The secants from the origin swing between slopes 1 and -1 and never settle.

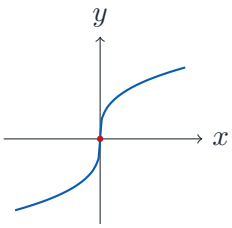
Figure 6.1.2 shows the three graphs near the origin.

Caution

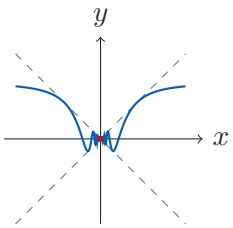
A formula for $f'(x)$ valid for $x \neq a$ says nothing on its own about $f'(a)$. For $f(x) = x^2\sin(1/x)$ with $f(0) = 0$ the formula $f'(x) = 2x\sin(1/x) - \cos(1/x)$ for $x \neq 0$ has no limit at 0, yet f is differentiable at 0 with $f'(0) = 0$ (Problem 6.1.6). At an exceptional point return to the definition Eq. (6.1.2).



(a) Corner of $|x|$.



(b) Vertical tangent of $x^{1/3}$.



(c) Oscillation of $x\sin(1/x)$ between $\pm x$.

Figure 6.1.2: Continuous functions that are not differentiable at 0. The dashed lines $y = \pm x$ bound the oscillating graph.

Worked Example 6.1.1

Find the derivative of $f(x) = \sqrt{x}$ at a point $a > 0$ from the definition and the tangent line at $a = 4$.

For $h \neq 0$ with $a + h > 0$, multiply the numerator and the denominator of the difference quotient by the positive number $\sqrt{a+h} + \sqrt{a}$:

$$Q(h) = \frac{\sqrt{a+h} - \sqrt{a}}{h} = \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} = \frac{1}{\sqrt{a+h} + \sqrt{a}}.$$

Because the square root is continuous at a (Corollary 5.4.13), the denominator tends to $2\sqrt{a} \neq 0$ and

$$f'(a) = \frac{1}{2\sqrt{a}}.$$

At $a = 4$ the slope is $\frac{1}{4}$ and the tangent line is $y = 2 + \frac{1}{4}(x - 4)$. At $a = 0$ the quotient $1/\sqrt{h}$ for $h > 0$ is unbounded: the right-hand derivative of $\sqrt{x}$ at 0 does not exist.

Worked Example 6.1.2

Use Theorem 6.1.9 to differentiate $f(x) = x^3$ at an arbitrary point a.

Expand the increment:

$$f(a+h) = (a+h)^3 = a^3 + 3a^2h + (3ah^2 + h^3).$$

The bracket $r(h) = 3ah^2 + h^3$ satisfies $r(h)/h = 3ah + h^2 \to 0$. Hence $r(h) = o(h)$ and the expansion has the form Eq. (6.1.5) with $L = 3a^2$. By Theorem 6.1.9, f is differentiable at a with $f'(a) = 3a^2$. The linear part $3a^2h$ is the tangent-line increment. The remainder $3ah^2 + h^3$ is the vertical gap between the curve and its tangent above the point $a + h$.

Worked Example 6.1.3

Find constants m and c for which

$$f(x) = \begin{cases} x^2, & x \leq 1, \\ mx + c, & x > 1, \end{cases}$$

is differentiable at 1.

Differentiability forces continuity by Theorem 6.1.11. Since $f(1) = 1$ and the right-hand limit is $m + c$, continuity requires $m + c = 1$. With this constraint the one-sided derivatives at 1 are

$$f'_-(1) = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^-} (2 + h) = 2,$$
$$f'_+(1) = \lim_{h \to 0^+} \frac{m(1+h) + c - 1}{h} = \lim_{h \to 0^+} \frac{mh}{h} = m.$$

Equality of the one-sided derivatives (Proposition 6.1.14) gives $m = 2$ and then $c = -1$. The line $y = 2x - 1$ is the tangent to $y = x^2$ at $x = 1$ and joins the parabola without a corner.

Study guide

Every derivative in this section comes from one of two computations. The first simplifies the difference quotient until the factor h in the denominator cancels and then lets $h \to 0$. The second expands $f(a + h)$ as $f(a) + Lh + r(h)$ and checks that $r(h)/h \to 0$. Write each step with $h \neq 0$ in view.

To prove that f is *not* differentiable at a, show that Q has no limit: two one-sided limits that differ, a quotient that is unbounded, or two sequences $h_n \to 0$ along which Q tends to different values. A function that is not continuous at a is never differentiable there by Theorem 6.1.11.

Problems

6.1.1 Use the definition Eq. (6.1.2) to find $f'(a)$ for $f(x) = 3x^2 - 5x$.

⋆ Hint · Answer · Solution

6.1.2 Use the definition to find $f'(a)$ for $f(x) = 1/x$ at a point $a \neq 0$.

⋆ Hint · Answer · Solution

6.1.3 Use the definition to find $f'(a)$ for $f(x) = 1/\sqrt{x}$ at a point $a > 0$.

⋆⋆ Hint · Answer · Solution

6.1.4 For $f(x) = x^3$, write $f(2 + h)$ in the form $f(2) + Lh + r(h)$, identify L and $r(h)$, and verify that $r(h) = o(h)$. Use the result to estimate $(2.01)^3$ and compare with the exact value.

⋆⋆ Hint · Answer · Solution

6.1.5 Compute the one-sided derivatives of $f(x) = |x^2 - 1|$ at $x = 1$ and decide whether f is differentiable there.

⋆⋆ Hint · Answer · Solution

6.1.6 Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable at 0 with $f'(0) = 0$.

⋆⋆ Hint · Answer · Solution

6.1.7 Find a and b for which $f(x) = ax + b$ for $x < 2$ and $f(x) = x^2$ for $x \geq 2$ is differentiable at 2.

⋆⋆ Hint · Answer · Solution

6.1.8 Find the tangent line to $y = \sqrt{x}$ at $x = 4$ and use it to estimate $\sqrt{4.1}$.

⋆⋆ Hint · Answer · Solution

6.1.9 Suppose $|f(x)| \leq x^2$ for every x. Prove that f is differentiable at 0 and find $f'(0)$.

⋆⋆ Hint · Answer · Solution

6.1.10 Let f be differentiable at a. Prove that

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

Show by an example that this symmetric limit can exist at a point where f is not differentiable. ⋆⋆⋆ Hint · Answer · Solution

6.2 Differential Operator and 1-Forms

Overview

Objectives: Read $\mathrm{d}x$ as the differential of the coordinate function, a linear functional on displacements; evaluate a 1-form at a point on a displacement; write any 1-form as $f(x)\,\mathrm{d}x$ and any differential as $\mathrm{d}f = f'(x)\,\mathrm{d}x$; read $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as an exact quotient of 1-forms; compute the pullback of a 1-form by a change of variable; state the derivative of a map between normed spaces and recognise the derivative of Section 6.1 as its case $\mathbb{R} \to \mathbb{R}$.
Prerequisites: Differentiability as linear approximation (Theorem 6.1.9); functions and composition (Definitions 1.5.8 and 1.5.20); tangent spaces, linear functionals and the dual space (Definitions 3.1.8 and 3.4.2 and Section 3.4); normed spaces and bounded linear maps (Definitions 3.6.12, 3.6.18 and 3.6.19).
Key results: The differential Eq. (6.2.2); the 1-form $\mathrm{d}x$ Eq. (6.2.4); every 1-form is a multiple of $\mathrm{d}x$ (Theorem 6.2.8) and $\mathrm{d}f = f'(x)\,\mathrm{d}x$ Eq. (6.2.6); the Leibniz quotient Eq. (6.2.7); the pullback Eq. (6.2.8); the derivative between normed spaces (Definition 6.2.14), its uniqueness (Theorem 6.2.15) and its one-variable case (Proposition 6.2.16).

Notation

$T_p\mathbb{R}$, $\mathbf{v}$	displacements at the point p, and one of them
$\omega_p(\mathbf{v})$	the value of the 1-form ω at p on $\mathbf{v}$
d	the differential operator $f \mapsto \mathrm{d}f$ (Definition 6.2.5)
$\mathrm{d}x$	the 1-form $\mathbf{v} \mapsto \mathbf{v}$ (Eq. (6.2.4))
$f(x)\,\mathrm{d}x$	the 1-form $\mathbf{v} \mapsto f(x)\,\mathbf{v}$ (Eq. (6.2.5))
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}}{\mathrm{d}x}f$	the quotient of 1-forms $\mathrm{d}y$ by $\mathrm{d}x$, equal to the derivative (Eq. (6.2.7))
$g^*\omega$	the pullback of ω by g (Definition 6.2.11)
$\mathrm{d}F_{\mathbf{a}}$	the differential at $\mathbf{a}$ of a map between normed spaces (Definition 6.2.14)

Theorem 6.1.9 attaches to a function f, at each point a where it is differentiable, a linear map $h \mapsto f'(a)h$ that approximates the increment $f(a+h)-f(a)$. The map changes from point to point. Collecting all of these linear maps into one object gives the differential $\mathrm{d}f$ of f. It is the first example of a 1-form. Every integral in this book is the integral of a 1-form built from one differential: the differential $\mathrm{d}x$ of the coordinate function x. This section defines the operator d, applies it to x, and shows that the integrand $f(x)\,\mathrm{d}x$ is a function multiplied by a linear functional. The definitions are the one-variable case of the calculus of differential forms on manifolds [1]. Every later chapter uses them unchanged.

6.2.1 Displacements and linear functionals

A point p of the line can be displaced by a number $\mathbf{v}$, written in bold to mark it as a vector. The displacements based at p form the *tangent space* $T_p\mathbb{R}$ of Definition 3.1.8. It is a copy of $\mathbb{R}$ attached to the point p. Its element (p, v) is identified with the number v throughout this chapter. Thus $\mathbf{v} = 1$ names $(p, 1)$, $\omega_p(1)$ means $\omega_p((p, 1))$, and $\mathbf{v}$ enters products, powers and limits as a number. The number h of Theorem 6.1.9 is such a displacement at a. Keeping the point and the displacement apart lets a single symbol describe the linear approximation at every

point at once. A linear functional on $T_p\mathbb{R}$ is a linear map $T_p\mathbb{R} \to \mathbb{R}$ (Definition 3.4.2) and an element of the dual space $(T_p\mathbb{R})^*$ of Definition 3.4.7.

Definition 6.2.2 (1-form)**.** A 1-*form* ω on an interval I assigns to each point $p \in I$ a linear functional $\omega_p \in (T_p\mathbb{R})^*$ on the tangent space $T_p\mathbb{R}$.

Example 6.2.3 (Linear functionals and non-examples)**.** The maps $\mathbf{v} \mapsto 3\mathbf{v}$ and $\mathbf{v} \mapsto 0$ are linear functionals on $T_p\mathbb{R}$. The map $\mathbf{v} \mapsto \mathbf{v} + 1$ sends 0 to 1. It is not a linear functional because every linear map sends $0 = 0 \cdot \mathbf{v}$ to $0 \cdot \lambda(\mathbf{v}) = 0$. The map $\mathbf{v} \mapsto |\mathbf{v}|$ takes the value 1 on both 1 and $-1 = (-1)\cdot 1$ and is therefore not linear either. The rule assigning to each point p the functional $\mathbf{v} \mapsto p^2\mathbf{v}$ is a 1-form on $\mathbb{R}$.

Linearity fixes a functional on $T_p\mathbb{R}$ by its value on the single displacement $\mathbf{v} = 1$:

$$\lambda(\mathbf{v}) = \lambda(\mathbf{v} \cdot 1) = \lambda(1)\,\mathbf{v}. \tag{6.2.1}$$

Every linear functional on the line is therefore multiplication by one number ((6.2.1) and Theorem 3.4.5(3)). The dual space $(T_p\mathbb{R})^*$ has dimension 1 with the dual basis $\mathbf{v} \mapsto \mathbf{v}$ (Theorem 3.4.8). A 1-form on I is accordingly determined by the function $p \mapsto \omega_p(1)$.

6.2.4 Operator d

Hubbard and Hubbard state differentiability itself in terms of a linear function:

> *a function f is differentiable at a point a if the increment Δf to the function is well approximated by a linear function of the increment h to the variable. This linear function is $f'(a)h$.*
>
> Hubbard and Hubbard [1]

The operator d takes a function to that linear function at every point at once.

Definition 6.2.5 (Differential)**.** Let f be differentiable on an open interval I. The *differential* of f is the 1-form $\mathrm{d}f$ on I given by

$$(\mathrm{d}f)_p(\mathbf{v}) = f'(p)\,\mathbf{v} \qquad (p \in I,\ \mathbf{v} \in T_p\mathbb{R}). \tag{6.2.2}$$

The map $f \mapsto \mathrm{d}f$ is the *differential operator* d, also called the *exterior derivative*. It takes a function (a 0-*form*) to a 1-form.

By Theorem 6.1.9 the differential at p is the linear part of the increment of f:

$$f(p + \mathbf{v}) - f(p) = (\mathrm{d}f)_p(\mathbf{v}) + o(\mathbf{v}) \qquad (\mathbf{v} \to 0). \tag{6.2.3}$$

The two sides of (6.2.3) are different numbers. The increment on the left is the change in the value of f. The differential $(\mathrm{d}f)_p(\mathbf{v})$ is the rise along the tangent line over the same displacement. Figure 6.2.1 shows both, with the $o(\mathbf{v})$ remainder as the gap between them.

Example 6.2.6 (Differential of the square)**.** For $f(x) = x^2$ the derivative $f'(p) = 2p$ found in Section 6.1 gives $(\mathrm{d}f)_p(\mathbf{v}) = 2p\,\mathbf{v}$. At $p = 3$ and $\mathbf{v} = 0.1$ the differential gives 0.6 against the increment $3.1^2 - 9 = 0.61$. Since $|x|$ is not differentiable at 0 (Example 6.1.16), it has no differential on any interval containing 0.

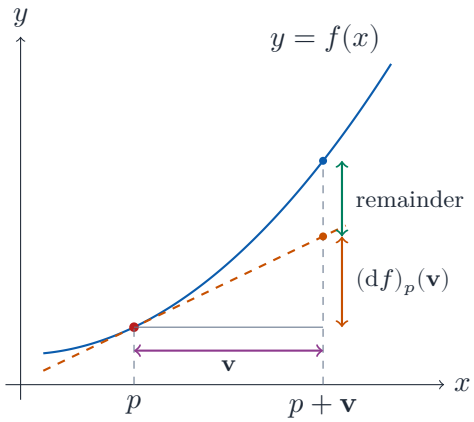


Figure 6.2.1: Value of the differential $\mathrm{d}f$ at p on the displacement $\mathbf{v}$: the rise $f'(p)\,\mathbf{v}$ along the tangent line. The increment $f(p+\mathbf{v})-f(p)$ exceeds it by the remainder of Theorem 6.1.9.

6.2.7 Differential $\mathrm{d}x$

The letter x also names a function: the *coordinate function* $x \mapsto x$ that returns the point it is given. Its difference quotient $((p+h)-p)/h$ equals 1 for every $h \neq 0$. Substituting the derivative 1 into Eq. (6.2.2) gives

$$(\mathrm{d}x)_p(\mathbf{v}) = \mathbf{v} \qquad \text{at every point } p. \tag{6.2.4}$$

The 1-form $\mathrm{d}x$ is the operator d applied to the coordinate function x. At each point it is the linear functional that returns the displacement it is given. The symbol $\mathrm{d}x$ names this functional and nothing else. At p it is the dual basis functional of the basis 1 of $T_p\mathbb{R}$ (Theorem 3.4.8). Reading off the displacement is the one-dimensional case of selecting a coordinate with the dot product (Definition 3.5.2):

> *The dot product of a vector $\vec{x}$ with a standard basis vector $\vec{e}_i$ selects the ith coordinate of $\vec{x}$: $\vec{x}\cdot\vec{e}_i = x_i$.*
>
> Hubbard and Hubbard [1]

A function f multiplies a 1-form pointwise. The product $f(x)\,\mathrm{d}x$ is the 1-form whose value at p is the functional $\mathrm{d}x$ scaled by the number $f(p)$:

$$\big(f(x)\,\mathrm{d}x\big)_p(\mathbf{v}) = f(p)\,\mathbf{v}. \tag{6.2.5}$$

Theorem 6.2.8 (Every 1-form is a multiple of $\mathrm{d}x$)**.** *For every 1-form ω on I there is exactly one function f on I with $\omega = f(x)\,\mathrm{d}x$, namely $f(p) = \omega_p(1)$. In particular for f differentiable on I,*

$$\mathrm{d}f = f'(x)\,\mathrm{d}x. \tag{6.2.6}$$

Proof. By linearity $\omega_p(\mathbf{v}) = \omega_p(1)\,\mathbf{v} = \omega_p(1)\,(\mathrm{d}x)_p(\mathbf{v})$ by Eq. (6.2.4). Hence $\omega = f(x)\,\mathrm{d}x$ with $f(p) = \omega_p(1)$. If also $\omega = g(x)\,\mathrm{d}x$, then evaluating both expressions on $\mathbf{v} = 1$ at p gives $f(p) = \omega_p(1) = g(p)$ for every p. For $\omega = \mathrm{d}f$, Eq. (6.2.2) gives $\omega_p(1) = f'(p)$ and hence Eq. (6.2.6). ∎

The uniqueness half of Theorem 6.2.8 says that an identity $f(x)\,\mathrm{d}x = g(x)\,\mathrm{d}x$ between 1-forms on I is the same statement as the identity $f = g$ between functions on I. In particular $f(x)\,\mathrm{d}x = 0$ exactly when f is zero at every point of I.

Integrand

In $\int f(x)\,\mathrm{d}x$ the integrand is the 1-form $f(x)\,\mathrm{d}x$: the function f multiplied by the linear functional $\mathrm{d}x$. Integration acts on 1-forms. The factor $\mathrm{d}x$ makes $f(x)\,\mathrm{d}x$ one of them. Chapter 8 defines the integral of a 1-form over an oriented interval (Definition 8.2.2) by evaluating it on the displacements of a partition.

6.2.9 Leibniz quotient

Because $(\mathrm{d}x)_p(1) = 1$ the 1-form $\mathrm{d}x$ is non-zero at every point. Theorem 6.2.8 therefore lets any 1-form be divided by it: the quotient $\omega/\mathrm{d}x$ is the unique function f with $\omega = f(x)\,\mathrm{d}x$. If $y = f(x)$, then $\mathrm{d}y = f'(x)\,\mathrm{d}x$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) \tag{6.2.7}$$

is a quotient of two 1-forms, equal to the function f'. The Leibniz notation for the derivative is therefore exact in one variable. The operator notation $\dfrac{\mathrm{d}}{\mathrm{d}x}f$ denotes the same function $\mathrm{d}f/\mathrm{d}x = f'$.

Caution

The quotient Eq. (6.2.7) is defined by Theorem 6.2.8 alone. It does not license operations on the symbols $\mathrm{d}y$ and $\mathrm{d}x$ as though they were numbers. Each later identity between Leibniz quotients, such as the chain rule $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$, is a theorem about 1-forms whose validity comes from its proof (Theorem 6.3.5).

6.2.10 Pullback

A change of variable $u = g(x)$ replaces a function of u by a function of x, namely $F \mapsto F \circ g$. A 1-form in u must be moved to the x-line as well, by a definition that accounts for the stretching of displacements by g.

Definition 6.2.11 (Pullback)**.** Let $g : I \to J$ be differentiable on the open interval I and let $\omega = f(u)\,\mathrm{d}u$ be a 1-form on J. The *pullback* of ω by g is the 1-form on I

$$g^*\omega = f\big(g(x)\big)\,g'(x)\,\mathrm{d}x. \tag{6.2.8}$$

The pullback evaluates ω at $g(p)$ on the image displacement $g'(p)\,\mathbf{v}$ of $\mathbf{v}$:

$$(g^*\omega)_p(\mathbf{v}) = \omega_{g(p)}\big(g'(p)\,\mathbf{v}\big). \tag{6.2.9}$$

The factor $g'(p)$ is the stretching of displacements by g at p that the linear approximation $g(p+\mathbf{v}) \approx g(p) + g'(p)\,\mathbf{v}$ describes. In particular $g^*(\mathrm{d}u) = g'(x)\,\mathrm{d}x = \mathrm{d}g$ by Eq. (6.2.6) applied to g.

Example 6.2.12 (Pullback by a square)**.** Let $g(x) = x^2$ on $\mathbb{R}$ and $\omega = u^3\,\mathrm{d}u$. Then $g^*\omega = (x^2)^3 \cdot 2x\,\mathrm{d}x = 2x^7\,\mathrm{d}x$. At $p = 1$ on $\mathbf{v} = 0.5$ the pullback gives 1, equal to $\omega_1(2 \cdot 0.5) = 1^3 \cdot 1$. Substituting $u = x^2$ into u^3 and writing $x^6\,\mathrm{d}x$ omits the factor $g'(x) = 2x$ and does not give the pullback.

The pullback is the operation behind every substitution in integration. When a substitution is written $u = g(x)$, the line $\mathrm{d}u = g'(x)\,\mathrm{d}x$ states $g^*(\mathrm{d}u) = \mathrm{d}g$: the differential of the new variable, pulled back to the x-line. The chain rule for differentials (Theorem 6.3.5 in the next

section) states that d commutes with pullback. Chapter 8 proves that pulling back leaves the integral of a 1-form unchanged.

6.2.13 Derivative between normed spaces

Theorem 6.1.9 describes $f'(a)$ through the linear map $h \mapsto f'(a)h$ and an error that is small compared with h. Neither ingredient needs the real line. Linear maps and the size $\|\mathbf{h}\|$ of a displacement make sense in every normed space (Section 3.6). The same description therefore defines a derivative there. Multivariable calculus and analysis take this as the definition of the derivative. This subsection states it, proves that the differential is unique and shows that the derivative of Section 6.1 is its case $\mathbb{R} \to \mathbb{R}$. Chapter 7 uses it once, to read differentiation D between the spaces $C^{k+1}([a,b])$ and $C^k([a,b])$ as its own derivative. No proof elsewhere in the book depends on it.

Definition 6.2.14 (Derivative of a map between normed spaces)**.** Let V and W be normed spaces, let $\mathbf{a} \in V$ and $r > 0$ and let F map the ball $B_r(\mathbf{a})$ (Definition 3.6.12) into W. The map F is *differentiable at* $\mathbf{a}$ if there is a bounded linear map $L\colon V \to W$ (Definition 3.6.18) such that the remainder

$$R(\mathbf{h}) = F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a}) - L\mathbf{h} \qquad (\|\mathbf{h}\| < r) \tag{6.2.10}$$

satisfies: for every $\varepsilon > 0$ there is a $\delta \in (0, r]$ with $\|R(\mathbf{h})\| \le \varepsilon\|\mathbf{h}\|$ whenever $\|\mathbf{h}\| < \delta$. The map L is the *differential* of F at $\mathbf{a}$, written $\mathrm{d}F_{\mathbf{a}}$. In short, $F(\mathbf{a}+\mathbf{h}) = F(\mathbf{a}) + \mathrm{d}F_{\mathbf{a}}\mathbf{h} + o(\|\mathbf{h}\|)$, where $o(\|\mathbf{h}\|)$ names a remainder with $\|R(\mathbf{h})\|/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to \mathbf{0}$.

Theorem 6.2.15 (Uniqueness of the derivative and continuity)**.** *If F is differentiable at $\mathbf{a}$, then the map L of Definition 6.2.14 is unique and F is continuous at $\mathbf{a}$ (Definition 3.6.17).*

Proof. **1.** Uniqueness. Suppose L_1 and L_2 both satisfy Definition 6.2.14 with remainders R_1 and R_2. Let $\mathbf{v} \in V$ be nonzero. For $0 < t < r/\|\mathbf{v}\|$ the two expansions of $F(\mathbf{a}+t\mathbf{v})$ give $t(L_1 - L_2)\mathbf{v} = R_2(t\mathbf{v}) - R_1(t\mathbf{v})$. Given $\varepsilon > 0$, take t so small that both remainders are at most $\varepsilon\|t\mathbf{v}\|$. Dividing by t gives $\|(L_1 - L_2)\mathbf{v}\| \le 2\varepsilon\|\mathbf{v}\|$. Since ε is arbitrary, $(L_1 - L_2)\mathbf{v} = \mathbf{0}$ (Problem 2.2.7 and (N1)). Hence $L_1 = L_2$.
2. Continuity. Take $\varepsilon = 1$ in Definition 6.2.14. For $\|\mathbf{h}\| < \delta$, Proposition 3.6.20(1) gives

$$\|F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a})\| \le \|L\mathbf{h}\| + \|R(\mathbf{h})\| \le (\|L\| + 1)\|\mathbf{h}\|.$$

Given $\varepsilon' > 0$, the right side is less than ε' once $\|\mathbf{h}\| < \min\{\delta, \varepsilon'/(\|L\|+1)\}$. ∎

Proposition 6.2.16 (Agreement with the derivative on the real line)**.** *Let $V = W = \mathbb{R}$ with the absolute value and let f be defined on an open interval containing a. Then f is differentiable at a in the sense of Definition 6.2.14 if and only if it is differentiable in the sense of Definition 6.1.3. In that case $\mathrm{d}f_a h = f'(a)\,h$ for every h. When f is differentiable on an open interval containing a, the map $\mathrm{d}f_a$ is the functional $(\mathrm{d}f)_a$ of Eq. (6.2.2) with $T_a\mathbb{R}$ read as $\mathbb{R}$.*

Proof. An open interval containing a contains a ball $(a-r, a+r)$. That ball is itself an open interval containing a. Every linear map $\mathbb{R} \to \mathbb{R}$ is $h \mapsto mh$ for one number m (Theorem 3.4.5) and is bounded. The condition on R in Definition 6.2.14 says $|R(h)/h| \le \varepsilon$ for $0 < |h| < \delta$, that is $R(h) = o(h)$ (Definition 6.1.7). Definition 6.2.14 for f is therefore Eq. (6.1.5) with $L = m$, and Theorem 6.1.9 gives the equivalence with $m = f'(a)$. ∎

Example 6.2.17 (Linear maps)**.** A bounded linear map $T\colon V \to W$ is differentiable at every $\mathbf{a}$ with $\mathrm{d}T_{\mathbf{a}} = T$, because $T(\mathbf{a}+\mathbf{h}) = T\mathbf{a} + T\mathbf{h}$ leaves the remainder 0. In one variable this is $(cx)' = c$. The evaluation E_p on $C([a,b])$ and the definite integral of Chapter 8 are examples.

Example 6.2.18 (Squaring a function)**.** Give $C([a,b])$ the sup norm (Definition 3.6.9) and let $S(f) = f^2$. For $f, h \in C([a,b])$,

$$S(f+h) = f^2 + 2fh + h^2.$$

The map $L_f h = 2fh$ is linear in h. It is bounded with $\|2fh\|_\infty \le 2\|f\|_\infty \|h\|_\infty$, since $|2f(x)h(x)| \le 2\|f\|_\infty\|h\|_\infty$ at every x. The remainder h^2 has $\|h^2\|_\infty = \|h\|_\infty^2$, at most $\varepsilon\|h\|_\infty$ once $\|h\|_\infty < \varepsilon$. Hence S is differentiable at every f with $\mathrm{d}S_f h = 2fh$. The formula is the one-variable rule $(x^2)' = 2x$ with the function f in place of the number x.

Example 6.2.19 (A norm at the origin (non-example))**.** Let $V \neq \{\mathbf{0}\}$ and $N(\mathbf{v}) = \|\mathbf{v}\|$ from V to $\mathbb{R}$. Suppose N were differentiable at $\mathbf{0}$ with differential L. Fix $\mathbf{v} \neq \mathbf{0}$. For $t \neq 0$ near 0, $|t|\|\mathbf{v}\| = tL\mathbf{v} + R(t\mathbf{v})$ with $R(t\mathbf{v})/t \to 0$. Dividing by $t > 0$ and letting $t \to 0^+$ gives $\|\mathbf{v}\| = L\mathbf{v}$. Dividing by $t < 0$ and letting $t \to 0^-$ gives $-\|\mathbf{v}\| = L\mathbf{v}$. Hence $\|\mathbf{v}\| = 0$. This contradicts $\mathbf{v} \neq \mathbf{0}$. For $V = \mathbb{R}$ this is the corner of $|x|$ (Example 6.1.16).

For a map $F\colon \mathbb{R}^n \to \mathbb{R}^m$ the differential $\mathrm{d}F_{\mathbf{a}}$ is a linear map with an $m \times n$ matrix (Definition 3.3.14), bounded by Corollary 3.6.22. Multivariable calculus shows that its entries are the partial derivatives of the coordinates of F. The chain rule, the inverse and implicit function theorems and the higher derivatives of multivariable calculus are statements about $\mathrm{d}F_{\mathbf{a}}$ [1]. Every other derivative in this book is that of a function from $\mathbb{R}$ to $\mathbb{R}$.

Forms on manifolds

On a manifold M of dimension n a 1-form assigns to each point p a linear functional on the tangent space T_pM. In coordinates $x_1, \dots, x_n$ every 1-form is $\sum_i f_i\,\mathrm{d}x_i$. This is the n-variable form of Theorem 6.2.8. The operator d sends a function to $\mathrm{d}f = \sum_i \dfrac{\partial f}{\partial x_i}\,\mathrm{d}x_i$, extends to a map from k-forms to $(k+1)$-forms with $\mathrm{d}(\mathrm{d}\omega) = 0$, and commutes with pullback as in Eq. (6.3.15). The single identity

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega, \tag{6.2.11}$$

the generalised Stokes theorem, contains the Fundamental Theorem of Calculus (Eq. (8.3.6)), Green's theorem, the classical Stokes theorem and the divergence theorem as special cases [1]. Only the quotient Eq. (6.2.7) is special to one variable: with two or more coordinates a 1-form has no unique quotient by $\mathrm{d}x_1$.

Study guide

Every computation in this section reduces to two facts. The first is that a 1-form on an interval is a function times $\mathrm{d}x$ (Theorem 6.2.8). The second is that $\mathrm{d}x$ returns the displacement it is given (Eq. (6.2.4)). To evaluate $f(x)\,\mathrm{d}x$ at p on $\mathbf{v}$, multiply $f(p)$ by $\mathbf{v}$. To find $\mathrm{d}f$, differentiate f and write $f'(x)\,\mathrm{d}x$. To pull back $f(u)\,\mathrm{d}u$ by $u = g(x)$, substitute $g(x)$ for u in f *and* replace $\mathrm{d}u$ by $\mathrm{d}g = g'(x)\,\mathrm{d}x$.
The common error is to treat $\mathrm{d}x$ as a number. It is a function of a point and a displacement. An equation between 1-forms is an equation between their coefficient functions. A Leibniz quotient is the coefficient of a 1-form with respect to $\mathrm{d}x$.

Between normed spaces the derivative is the bounded linear map $\mathrm{d}F_{\mathbf{a}}$ with $F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a}) - \mathrm{d}F_{\mathbf{a}}\mathbf{h}$ small compared with $\|\mathbf{h}\|$. To find it, expand $F(\mathbf{a}+\mathbf{h})$, collect the part linear in $\mathbf{h}$ and bound the rest by a constant times $\|\mathbf{h}\|^2$ for $\|\mathbf{h}\| \le 1$.

Worked Example 6.2.1

Let $f(x) = x^3$. Write $\mathrm{d}f$ as a multiple of $\mathrm{d}x$, evaluate it at $p = 2$ on $\mathbf{v} = 0.1$, and compare with the increment $f(2.1) - f(2)$.
Since $f'(x) = 3x^2$ by Worked Example 6.1.2, Eq. (6.2.6) gives $\mathrm{d}f = 3x^2\,\mathrm{d}x$. By Eq. (6.2.5),

$$(\mathrm{d}f)_2(0.1) = 3 \cdot 2^2 \cdot 0.1 = 1.2.$$

The increment is $2.1^3 - 8 = 9.261 - 8 = 1.261$. The difference 0.061 equals $6\mathbf{v}^2 + \mathbf{v}^3$ at $\mathbf{v} = 0.1$. This is the $o(\mathbf{v})$ remainder found in Worked Example 6.1.2 with $a = 2$.

Worked Example 6.2.2

Compute the pullback of $\omega = u^2\,\mathrm{d}u$ by $g(x) = 3x + 1$, evaluate it at $p = 0$ on $\mathbf{v} = 2$, and check the value against the pointwise description of the pullback.
Since $g'(x) = 3$, Eq. (6.2.8) gives

$$g^*\omega = (3x+1)^2 \cdot 3\,\mathrm{d}x = 3(3x+1)^2\,\mathrm{d}x.$$

At $p = 0$ on $\mathbf{v} = 2$ the value is $3 \cdot 1^2 \cdot 2 = 6$. Pointwise, g sends $p = 0$ to $g(0) = 1$ and the displacement 2 to $g'(0) \cdot 2 = 6$. Since $\omega_1(6) = 1^2 \cdot 6 = 6$ the two computations agree as (6.2.9) states.

Worked Example 6.2.3

Give $\mathbb{R}^2$ the ∞-norm (Definition 3.6.6) and let $F(x, y) = xy$. Show that F is differentiable at every point (a, b) with $\mathrm{d}F_{(a,b)}(h_1, h_2) = bh_1 + ah_2$. Evaluate the differential at $(2, 3)$ on $\mathbf{h} = (0.1, -0.2)$ and compare it with the increment of F.
Expanding the product,

$$F(a + h_1, b + h_2) = ab + (bh_1 + ah_2) + h_1h_2.$$

The map $L\mathbf{h} = bh_1 + ah_2$ is linear and bounded with $|L\mathbf{h}| \le (|a| + |b|)\|\mathbf{h}\|_\infty$. The remainder satisfies $|h_1h_2| \le \|\mathbf{h}\|_\infty^2 \le \varepsilon\|\mathbf{h}\|_\infty$ once $\|\mathbf{h}\|_\infty < \varepsilon$. By Definition 6.2.14 the differential is L. At $(2, 3)$ on $(0.1, -0.2)$ it gives $3(0.1) + 2(-0.2) = -0.1$. The increment is $F(2.1, 2.8) - F(2, 3) = 5.88 - 6 = -0.12$. The difference -0.02 is the remainder h_1h_2. Each partial derivative of F appears as a coefficient: b is the derivative of $x \mapsto xb$ at a and a that of $y \mapsto ay$ at b.

Problems

6.2.1 Evaluate the 1-form $5x\,\mathrm{d}x$ at $p = 2$ on $\mathbf{v} = 3$ and the 1-form $\mathrm{d}x$ at $p = -7$ on $\mathbf{v} = 0.4$.

⋆ Hint · Answer · Solution

6.2.2 Which of the following maps $T_p\mathbb{R} \to \mathbb{R}$ are linear functionals? (a) $\mathbf{v} \mapsto -4\mathbf{v}$; (b) $\mathbf{v} \mapsto \mathbf{v}^2$; (c) $\mathbf{v} \mapsto 0$; (d) $\mathbf{v} \mapsto 2\mathbf{v} - 1$.

⋆ Hint · Answer · Solution

6.2.3 For $f(x) = x^2$, compute $(\mathrm{d}f)_p(\mathbf{v})$ and the increment $f(p+\mathbf{v}) - f(p)$. Show that their difference is $o(\mathbf{v})$. ⋆⋆ Hint · Answer · Solution

6.2.4 A 1-form ω on $\mathbb{R}$ satisfies $\omega_p(2) = 6p^2$ for every p. Write ω as $f(x)\,\mathrm{d}x$ and find a function F with $\omega = \mathrm{d}F$. ⋆⋆ Hint · Answer · Solution

6.2.5 Compute the pullback of $\omega = u\,\mathrm{d}u$ by $g(x) = x^2 + 1$ and evaluate it at $p = 1$ on $\mathbf{v} = 3$. ⋆⋆ Hint · Answer · Solution

6.2.6 On $(0,\infty)$ let $y = x^3$ and $z = x^2$. Express $\mathrm{d}y$ and $\mathrm{d}z$ as multiples of $\mathrm{d}x$ and compute the quotient of 1-forms $\mathrm{d}y/\mathrm{d}z$. ⋆⋆ Hint · Answer · Solution

6.2.7 Let f be a function on $\mathbb{R}$. For $c \in \mathbb{R}$ and $k \neq 0$ let $\tau(x) = x + c$ and $\sigma(x) = kx$. Show that $\tau^*(f(u)\,\mathrm{d}u) = f(x+c)\,\mathrm{d}x$ and $\sigma^*(f(u)\,\mathrm{d}u) = k\,f(kx)\,\mathrm{d}x$. Explain why a translation leaves the coefficient of $\mathrm{d}x$ unscaled while a dilation multiplies it by k. ⋆⋆⋆ Hint · Answer · Solution

6.2.8 Let ω be a 1-form on I and $p \in I$. Prove that if $\omega_p(\mathbf{v}) \geq 0$ for every displacement $\mathbf{v}$, then $\omega_p = 0$. Deduce that the rule assigning to each p the map $\mathbf{v} \mapsto |\mathbf{v}|$ is not a 1-form. ⋆⋆⋆ Hint · Answer · Solution

6.2.9 Give $C([0,1])$ the sup norm and let $G(f) = f^3$. Show that G is differentiable at every f with $\mathrm{d}G_f h = 3f^2 h$. For $f(x) = x$ and $h(x) = 1 - x$ evaluate the function $\mathrm{d}G_f h$ at $x = \tfrac{1}{2}$. Find the operator norm of $\mathrm{d}G_f$ for this f. ⋆⋆ Hint · Answer · Solution

6.2.10 Let F with values in W be differentiable at $\mathbf{a}$ in the sense of Definition 6.2.14. Let $T\colon W \to U$ be a bounded linear map. Prove that $T \circ F$ is differentiable at $\mathbf{a}$ with $\mathrm{d}(T\circ F)_{\mathbf{a}} = T \circ \mathrm{d}F_{\mathbf{a}}$. Deduce the rule $(cf)'(a) = c\,f'(a)$ for a real function f and a constant c. ⋆⋆⋆ Hint · Answer · Solution

6.3 Rules of Differentiation

Overview

Objectives: Prove and apply the linearity, product, quotient and power rules; state each rule for the derivative f' and for the differential d; prove the chain rule from the linear-approximation form of the derivative and apply it to nested functions; state the chain rule for differentials as the commutation of d with pullback; differentiate an inverse function.
Prerequisites: The derivative and Theorems 6.1.9 and 6.1.11; the differential and the pullback (Section 6.2); the limit laws and continuity (Sections 5.2 and 5.3); induction (Theorem 1.4.2); the intermediate value theorem (Theorem 5.4.2).
Key results: The basic rules Theorem 6.3.1 and their forms for d Eq. (6.3.6); the chain rule Eq. (6.3.9) and its form for differentials Eq. (6.3.15); the derivative of an inverse (Theorem 6.3.9).

Computing every derivative from Eq. (6.1.1) would make even a cubic polynomial a page of algebra. The rules of this section reduce the derivative of any function built from simpler ones by sums, products, quotients and composition to the derivatives of the pieces. Each rule is proved once from the definition and then stated twice, for the derivative f' and for the differential $\mathrm{d}f = f'(x)\,\mathrm{d}x$ of Eq. (6.2.6). Chapter 9 reverses two of the rules: integration by parts reverses the product rule and substitution reverses the chain rule.

Theorem 6.3.1 (Basic differentiation rules)**.** *If f, g are differentiable at x and $k \in \mathbb{R}$, then $kf + g$, fg and (when $g(x) \neq 0$) f/g are differentiable at x, with*

$$(kf+g)' = kf' + g', \tag{6.3.1}$$

$$(fg)' = f'g + fg', \tag{6.3.2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (g \neq 0). \tag{6.3.3}$$

For every integer n (and $x \neq 0$ when $n \leq 0$) the power rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x^n = n\, x^{n-1} \tag{6.3.4}$$

is true.

Proof. **(i)** Linearity. The difference quotient of $kf + g$ at x is

$$\frac{(kf+g)(x+h) - (kf+g)(x)}{h} = k\,\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}.$$

The right side tends to $kf'(x) + g'(x)$ by the limit laws.
(ii) Product. Add and subtract $f(x+h)\,g(x)$ in the difference quotient of fg:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x+h)\,\frac{g(x+h)-g(x)}{h} + g(x)\,\frac{f(x+h)-f(x)}{h}.$$

As $h \to 0$ the two quotients tend to $g'(x)$ and $f'(x)$. The factor $f(x+h)$ tends to $f(x)$ because f is continuous at x (Theorem 6.1.11). The limit laws give Eq. (6.3.2).

(iii) Reciprocal. Let $g(x) \neq 0$. Since g is continuous at x there is $\delta > 0$ such that $|h| < \delta$ implies $|g(x+h) - g(x)| < \frac{1}{2}|g(x)|$ and hence $g(x+h) \neq 0$. For $0 < |h| < \delta$,

$$\frac{1}{h}\Big(\frac{1}{g(x+h)} - \frac{1}{g(x)}\Big) = -\frac{g(x+h)-g(x)}{h} \cdot \frac{1}{g(x+h)\,g(x)} \longrightarrow -\frac{g'(x)}{g(x)^2}. \tag{6.3.5}$$

(iv) Quotient. Apply the product rule (6.3.2) and (6.3.5) to $f \cdot (1/g)$:

$$\Big(\frac{f}{g}\Big)' = f' \cdot \frac{1}{g} + f \cdot \Big(-\frac{g'}{g^2}\Big) = \frac{f'g - fg'}{g^2}.$$

(v) Power. For $n = 0$ the constant function $x^0 = 1$ has difference quotient 0. For $n = 1$ the derivative is 1 (Section 6.2). If $(x^n)' = nx^{n-1}$ for some $n \geq 1$, the product rule gives

$$(x^{n+1})' = (x \cdot x^n)' = 1 \cdot x^n + x \cdot nx^{n-1} = (n+1)x^n.$$

By induction (Theorem 1.4.2), Eq. (6.3.4) is true for every $n \geq 0$. For $n = -m$ with $m \geq 1$ and $x \neq 0$ the reciprocal rule (6.3.5) gives

$$(x^{-m})' = \Big(\frac{1}{x^m}\Big)' = -\frac{mx^{m-1}}{x^{2m}} = -m\,x^{-m-1}.$$

This is Eq. (6.3.4) with $n = -m$. ∎

Applying Eq. (6.2.6) to each rule for the derivative turns an identity between coefficient functions into the corresponding identity between 1-forms. For the product,

$$\mathrm{d}(fg) = (fg)'\,\mathrm{d}x = (f'g + fg')\,\mathrm{d}x = g\,f'\,\mathrm{d}x + f\,g'\,\mathrm{d}x = g\,\mathrm{d}f + f\,\mathrm{d}g.$$

Together with linearity this gives the rules for d,

$$\mathrm{d}(kf+g) = k\,\mathrm{d}f + \mathrm{d}g, \qquad \mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f, \tag{6.3.6}$$

and in the same way

$$\mathrm{d}\Big(\frac{f}{g}\Big) = \frac{g\,\mathrm{d}f - f\,\mathrm{d}g}{g^2}, \qquad \mathrm{d}(x^n) = n\,x^{n-1}\,\mathrm{d}x. \tag{6.3.7}$$

A linear map that satisfies the product rule Eq. (6.3.6) is called a *derivation*. The operator d is a derivation from functions to 1-forms with $\mathrm{d}c = 0$ for every constant c.

Example 6.3.2 (Polynomial and rational function)**.** By linearity and the power rule, $(4x^5 - 3x^2 + 7)' = 20x^4 - 6x$ and hence $\mathrm{d}(4x^5 - 3x^2 + 7) = (20x^4 - 6x)\,\mathrm{d}x$. By the quotient rule, for $x \neq -1$,

$$\Big(\frac{x-1}{x+1}\Big)' = \frac{1\cdot(x+1) - (x-1)\cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

Common misconception

The derivative of a product differs from the product of the derivatives. For $f(x) = g(x) = x$ the product x^2 has derivative $2x$ and the product of the derivatives is $f'g' = 1$. The correct rule Eq. (6.3.2) has two terms, one for each factor differentiated while the other stays fixed.

6.3.3 Chain rule

Differentiating a composite $f \circ g$ needs the linear-approximation form of the derivative. The difference quotient of $f \circ g$ factors as a product of two difference quotients:

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}. \tag{6.3.8}$$

The factorisation (6.3.8) breaks down whenever $g(x+h) = g(x)$ for values of h arbitrarily close to 0, as it does for $g(x) = x^2 \sin(1/x)$ at 0. The proof below avoids the division altogether.

Theorem 6.3.4 (Chain rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and*

$$(f \circ g)'(x) = f'(g(x))\, g'(x). \tag{6.3.9}$$

Proof. Put $u = g(x)$ and let f be defined on $(u - \eta, u + \eta)$ for some $\eta > 0$. Define a function ε on $|k| < \eta$ by

$$\varepsilon(k) = \frac{f(u+k) - f(u)}{k} - f'(u) \quad (k \neq 0), \qquad \varepsilon(0) = 0. \tag{6.3.10}$$

Because f is differentiable at u, $\varepsilon(k) \to 0 = \varepsilon(0)$ as $k \to 0$. Hence ε is continuous at 0. Multiplying out the definition (6.3.10) gives

$$f(u+k) - f(u) = \bigl(f'(u) + \varepsilon(k)\bigr)\, k \tag{6.3.11}$$

for every $k \neq 0$. It is also true for $k = 0$ because both sides are then 0. Now let $k = k(h) = g(x+h) - g(x)$. Since g is continuous at x (Theorem 6.1.11), there is $\rho > 0$ with $|k(h)| < \eta$ for $|h| < \rho$. For such h the composite $f \circ g$ is defined at $x + h$ and $\varepsilon(k(h))$ is defined. Substituting into Eq. (6.3.11),

$$f(g(x+h)) - f(g(x)) = \bigl(f'(u) + \varepsilon(k(h))\bigr)\, \bigl(g(x+h) - g(x)\bigr), \tag{6.3.12}$$

with no division by k anywhere. Dividing (6.3.12) by h with $0 < |h| < \rho$,

$$\frac{f(g(x+h)) - f(g(x))}{h} = \bigl(f'(u) + \varepsilon(k(h))\bigr) \cdot \frac{g(x+h) - g(x)}{h}. \tag{6.3.13}$$

As $h \to 0$ the second factor of (6.3.13) tends to $g'(x)$. By continuity of g at x, $k(h) \to 0$. Since ε is continuous at 0 with $\varepsilon(0) = 0$, the composite $\varepsilon(k(h))$ tends to 0 (Theorem 5.3.7). The first factor therefore tends to $f'(u)$. The limit laws give Eq. (6.3.9). ∎

Because Eq. (6.3.11) is true for $k = 0$ as well, the case $g(x+h) = g(x)$ that defeats the naive argument causes no trouble here.

In Leibniz notation, with $y = f(u)$ and $u = g(x)$, the chain rule reads

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{6.3.14}$$

Here $\mathrm{d}y/\mathrm{d}u$ is evaluated at $u = g(x)$. The identity (6.3.14) resembles a cancellation of $\mathrm{d}u$. It is a theorem about 1-forms, proved above and restated below.

Theorem 6.3.5 (Chain rule for differentials)**.** *If $g : I \to J$ and $F : J \to \mathbb{R}$ are differentiable on open intervals, then*

$$\mathrm{d}(F \circ g) = g^{*}(\mathrm{d}F). \tag{6.3.15}$$

Proof. By Eq. (6.2.6) and the chain rule Eq. (6.3.9), $\mathrm{d}(F \circ g) = F'(g(x))\, g'(x)\, \mathrm{d}x$. The pullback Eq. (6.2.8) of $\mathrm{d}F = F'(u)\, \mathrm{d}u$ by g is the same 1-form. ∎

Theorem 6.3.5 says that the operator d commutes with pullback: differentiating after the change of variable gives the same 1-form as differentiating first and then pulling back. Every substitution step in Chapters 8 and 9 is Eq. (6.3.15) or Eq. (6.2.8).

Example 6.3.6 (Chain rule on a power)**.** Let $y = (x^2 + 1)^5$. With $u = x^2 + 1$ and $y = u^5$, Eq. (6.3.9) gives $y' = 5u^4 \cdot 2x = 10x(x^2 + 1)^4$. As 1-forms, the pullback of $\mathrm{d}(u^5) = 5u^4\, \mathrm{d}u$ by $u = x^2 + 1$ is $5(x^2 + 1)^4 \cdot 2x\, \mathrm{d}x$, equal to $\mathrm{d}y$ by Eq. (6.3.15).

Caution

The most frequent error with the chain rule is to omit the inner derivative. The derivative of $(x^2 + 1)^5$ is not $5(x^2 + 1)^4$. In the language of forms the omission replaces the pullback $g^*(\mathrm{d}F)$ by the 1-form $F'(g(x))\, \mathrm{d}x$ that ignores the stretching $g'(x)$ of displacements.

6.3.7 Inverse functions

If f has an inverse, the graph of f^{-1} is the graph of f reflected in the line $y = x$. Because the reflection exchanges the two coordinates of every displacement along a tangent line, it turns a tangent of slope $m \neq 0$ into one of slope $1/m$. The theorem below makes this exact. Its proof needs the continuity of the inverse, proved in Chapter 5 and restated here.

Lemma 6.3.8 (Continuity of an inverse)**.** *Let f be continuous and strictly monotone on an interval I. Then $J = f(I)$ is an interval, f is a bijection from I to J, and the inverse $f^{-1} : J \to I$ is continuous.*

Proof. A strictly monotone function is injective (Theorem 4.1.19). Hence $f : I \to J$ is a bijection (Definition 1.5.15). That J is an interval and f^{-1} is continuous is Theorem 5.4.9. ∎

Theorem 6.3.9 (Derivative of an inverse function)**.** *Let f be continuous and strictly monotone on an open interval I, and let f be differentiable at $a \in I$ with $f'(a) \neq 0$. Then f^{-1} is differentiable at $b = f(a)$ and*

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}. \tag{6.3.16}$$

Proof. By Lemma 6.3.8, $J = f(I)$ is an interval and f^{-1} is continuous on it. Since I is open, it contains points $a - \eta$ and $a + \eta$ for some $\eta > 0$. Because b lies strictly between the images of these points, it is an interior point of J. For $y \in J$ with $y \neq b$ put $x = f^{-1}(y)$. Then $x \neq a$ because f^{-1} is injective. The difference quotient of f^{-1} at b is

$$\frac{f^{-1}(y) - f^{-1}(b)}{y - b} = \frac{x - a}{f(x) - f(a)} = \frac{1}{Q(x - a)}. \tag{6.3.17}$$

Here Q denotes the difference quotient of f at a. It is non-zero because f is injective. Extend Q by $Q(0) = f'(a)$. The extended function is continuous at 0 because $Q(t) \to f'(a)$ as $t \to 0$. As $y \to b$, continuity of f^{-1} gives $x - a = f^{-1}(y) - a \to 0$, and Theorem 5.3.7 gives $Q(x - a) \to f'(a)$. For $y \neq b$ the point $x - a$ is non-zero and the extension changes no value used in (6.3.17). Since $f'(a) \neq 0$ the quotient (6.3.17) tends to $1/f'(a)$. ∎

In Leibniz notation, with $y = f(x)$ and $x = f^{-1}(y)$,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\mathrm{d}y/\mathrm{d}x}. \tag{6.3.18}$$

If f is differentiable with $f' \neq 0$ at every point of I, then Eq. (6.2.6) and Eq. (6.3.16) give the identity of 1-forms on J $\mathrm{d}(f^{-1}) = \dfrac{\mathrm{d}y}{f'(f^{-1}(y))}$. The hypothesis $f'(a) \neq 0$ cannot be dropped: the strictly increasing function $f(x) = x^3$ has $f'(0) = 0$ and an inverse $y^{1/3}$ that is not differentiable at 0 (Example 6.1.17).

Example 6.3.10 (Square root as an inverse)**.** On $I = (0, \infty)$ the function $f(x) = x^2$ is continuous and strictly increasing with inverse $f^{-1}(y) = \sqrt{y}$ and $f'(x) = 2x \neq 0$. By Eq. (6.3.16),

$$\left(\sqrt{y}\right)' = \frac{1}{2\sqrt{y}} \qquad (y > 0),$$

in agreement with Worked Example 6.1.1. In the same way the inverse of x^n on $(0, \infty)$ gives $\left(y^{1/n}\right)' = \dfrac{1}{n}\, y^{1/n-1}$.

Memory aid

Each integration technique in Chapter 9 is one differentiation rule run backwards: integration by parts (Theorem 9.2.1) reverses the product rule Eq. (6.3.6) and substitution (Theorem 8.4.2) reverses the chain rule Eq. (6.3.15).

Study guide

Differentiate from the outside in. Identify the last operation used to build the function (a sum, a product, a quotient or a composition) and apply the matching rule. Then differentiate each piece by the same method. For a composition, write the inner function $u = g(x)$ explicitly on a line of its own and include its derivative $g'(x)$.
Simplify only at the end. When in doubt, check the result at one point against a numerical difference quotient. The form version of each rule is the same computation with $\mathrm{d}x$ attached: the product rule reads $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$ and the chain rule reads $\mathrm{d}(F \circ g) = g^*(\mathrm{d}F)$.

Worked Example 6.3.1

Differentiate $y = \dfrac{x^2}{x^3 + 1}$ and write $\mathrm{d}y$ as a multiple of $\mathrm{d}x$.
The function is a quotient with $f = x^2$ and $g = x^3 + 1$, defined for $x \neq -1$. With $f' = 2x$ and $g' = 3x^2$, Eq. (6.3.3) gives

$$y' = \frac{2x(x^3 + 1) - x^2 \cdot 3x^2}{(x^3 + 1)^2} = \frac{2x^4 + 2x - 3x^4}{(x^3 + 1)^2} = \frac{2x - x^4}{(x^3 + 1)^2}.$$

By Eq. (6.2.6), $\mathrm{d}y = \dfrac{2x - x^4}{(x^3 + 1)^2}\,\mathrm{d}x$. The quotient rule for d gives the same 1-form directly: $\mathrm{d}y = \left((x^3 + 1)\,2x\,\mathrm{d}x - x^2\,3x^2\,\mathrm{d}x\right)/(x^3 + 1)^2$.

Worked Example 6.3.2

Differentiate $y = \left(1 + (2x-1)^3\right)^{-2}$.
The outermost operation is the power -2. Write

$$y = v^{-2}, \qquad v = 1 + u^3, \qquad u = 2x - 1.$$

The chain rule applied twice gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = (-2v^{-3}) \cdot (3u^2) \cdot 2 = -\frac{12(2x-1)^2}{\left(1 + (2x-1)^3\right)^3},$$

each factor evaluated at the point produced by the inner functions. The formula is valid for $x \neq 0$, where $1 + (2x-1)^3 \neq 0$.

Worked Example 6.3.3

Let $f(x) = x^3 + x$. Show that f has a differentiable inverse and find $(f^{-1})'(2)$.
If $x_1 < x_2$, then $x_1^3 < x_2^3$ and so $f(x_1) < f(x_2)$. The function is therefore strictly increasing and continuous on $\mathbb{R}$ with a derivative $f'(x) = 3x^2 + 1$ that is never zero. Theorem 6.3.9 applies at every point. To use it at $b = 2$, find a with $f(a) = 2$: by inspection $a = 1$. Then

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{4}.$$

No formula for f^{-1} was needed. The inverse rule needs the point $a = f^{-1}(b)$ and the derivative of f there.

Problems

6.3.1 Differentiate $4x^5 - 3x^2 + 7$ and $(x^2+1)(x^3-2x)$. ⋆ Hint · Answer · Solution

6.3.2 Differentiate $\dfrac{x-1}{x+1}$ and $\dfrac{3}{x^4}$. ⋆ Hint · Answer · Solution

6.3.3 Differentiate $(x^2 - 3x + 1)^7$ and $\dfrac{1}{(1+x^2)^3}$. ⋆⋆ Hint · Answer · Solution

6.3.4 Write $\mathrm{d}y$ as a multiple of $\mathrm{d}x$ for $y = x^2(1-x)^3$, using the product rule for d.
⋆⋆ Hint · Answer · Solution

6.3.5 Let $f(x) = x^5 + 2x + 1$. Find $(f^{-1})'(4)$. ⋆⋆ Hint · Answer · Solution

6.3.6 Compute the pullback of $\omega = u^3\,\mathrm{d}u$ by $g(x) = x^2 + 1$ and check that it equals $\mathrm{d}(\frac{1}{4}(x^2+1)^4)$ as Eq. (6.3.15) gives for $F(u) = \frac{1}{4}u^4$. ⋆⋆ Hint · Answer · Solution

6.3.7 Let $f(x) = x|x|$. Show that f is differentiable at every x, with $f'(x) = 2|x|$.
⋆⋆ Hint · Answer · Solution

6.3.8 Let f be differentiable on an open interval. Prove by induction that $(f^n)' = nf^{n-1}f'$ for every integer $n \geq 1$, using the product rule alone.
⋆⋆⋆ Hint · Answer · Solution

6.3.9 Let f and g have derivatives f', g' that are themselves differentiable. Prove that $(fg)'' = f''g + 2f'g' + fg''$. ⋆⋆⋆ Hint · Answer · Solution

6.3.10 Let $g(x) = x^2$ and let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ with $f(0) = 0$. Compute $(g \circ f)'(0)$ by the chain rule and directly. Explain why this example defeats the product-of-quotients argument at the start of this subsection. ⋆⋆⋆ Hint · Answer · Solution

6.4 Derivatives of the Elementary Functions

Overview

Objectives: Prove $(e^x)' = e^x$ from the definition of the exponential; differentiate $\ln x$, a^x and x^r for real r; deduce from $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$ the derivatives of the six trigonometric functions; differentiate the inverse trigonometric functions by the inverse rule; use logarithmic differentiation.

Prerequisites: The exponential and logarithm (Definitions 4.4.19 and 4.4.22) and the existence of e (Theorem 5.1.20); radian measure and the trigonometric functions (Definition 4.5.2 and Section 4.5); Bernoulli's inequality (Theorem 2.3.11); the rules of Section 6.3.

Key results: $(e^x)' = e^x$ (Theorem 6.4.4); $(\ln x)' = 1/x$; $(x^r)' = rx^{r-1}$ (Theorem 6.4.6); $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$ (Theorem 5.2.18, from Chapter 5); $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ (Theorem 6.4.8); the derivative table.

Chapter 4 defined the exponential, the logarithm and the trigonometric functions and established their algebra. Their derivatives rest on two limits that concern the functions themselves at a single point and that the rules of Section 6.3 cannot supply:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1, \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1. \tag{6.4.1}$$

Each limit in (6.4.1) is the derivative at 0 of the function concerned. The first is proved below from the definition of the exponential. The second is Theorem 5.2.18 of Chapter 5, proved there from the inequality $\sin x < x < \tan x$ of Eq. (5.2.4) comparing a radian measure with its sine and tangent. That inequality is part (3) of Theorem 8.6.6 in Chapter 8, proved from an arc-length integral of an algebraic function with no trigonometric derivative. Every other derivative in this section then follows from the rules.

6.4.1 Exponential and logarithm

The proof uses the following facts about the exponential function of Definition 4.4.19. For all real x and y, $e^x > 0$, $e^{x+y} = e^x e^y$ and $e^0 = 1$ (Theorem 4.4.16). For every real x,

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \tag{6.4.2}$$

by Theorem 5.4.15. The exponential maps $\mathbb{R}$ onto $(0, \infty)$ (Theorem 5.4.12) with inverse the natural logarithm ln (Definition 4.4.22).

Lemma 6.4.2 (Linear and reciprocal bounds for the exponential)**.** *For every real x, $e^x \geq 1 + x$. For every $x < 1$, $e^x \leq \dfrac{1}{1-x}$.*

Proof. Fix x. For every integer $n > |x|$ the number $t = x/n$ satisfies $t > -1$. Bernoulli's inequality (Theorem 2.3.11) then gives $(1 + x/n)^n \geq 1 + n \cdot \frac{x}{n} = 1 + x$. Since a limit of terms that are all at least $1 + x$ is at least $1 + x$ (Theorem 5.1.10) and e^x is the limit (6.4.2), it follows that $e^x \geq 1 + x$. For $x < 1$, apply this to $-x$: $e^{-x} \geq 1 - x > 0$. Since $e^x e^{-x} = e^0 = 1$, it follows that $e^x = 1/e^{-x} \leq 1/(1-x)$. ∎

Theorem 6.4.3 (Exponential limit)**.** $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

Proof. Let $0 < |h| < 1$. By Lemma 6.4.2, $h \le e^h - 1 \le \dfrac{1}{1-h} - 1 = \dfrac{h}{1-h}$. Dividing by h keeps the inequalities for $0 < h < 1$ and reverses them for $-1 < h < 0$. In both cases $\dfrac{e^h - 1}{h}$ lies between 1 and $\dfrac{1}{1-h}$. Since both bounds tend to 1 as $h \to 0$, the squeeze theorem (Section 5.2) gives the limit. ∎

Theorem 6.4.4 (Derivative of the exponential)**.** *The function e^x is differentiable on $\mathbb{R}$ with*

$$\frac{\mathrm{d}}{\mathrm{d}x}\, e^x = e^x, \qquad \mathrm{d}(e^x) = e^x\, \mathrm{d}x. \tag{6.4.3}$$

Proof. By $e^{x+h} = e^x e^h$,

$$\frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h} \longrightarrow e^x \cdot 1$$

by Theorem 6.4.3. The form statement is Eq. (6.2.6). ∎

The exponential is equal to its own derivative. Every function f with $f' = f$ on $\mathbb{R}$ is a constant multiple of it. For such f the product rule gives $(fe^{-x})' = f'e^{-x} - fe^{-x} = 0$. A function with zero derivative on an interval is constant by Corollary 7.2.2. Its proof in Chapter 7 uses the Mean Value Theorem and nothing about the exponential. The function fe^{-x} is therefore constant. The exponential is strictly increasing (Theorem 4.4.16) and continuous by Theorem 6.1.11. Theorem 6.3.9 applies to it on $\mathbb{R}$ with $(e^x)' = e^x \neq 0$ everywhere.

Theorem 6.4.5 (Derivative of the logarithm)**.** *For $x > 0$, $(\ln x)' = \dfrac{1}{x}$. For $x \neq 0$, $\big(\ln|x|\big)' = \dfrac{1}{x}$. Equivalently, $\mathrm{d}\ln|x| = \dfrac{\mathrm{d}x}{x}$ on each of $(-\infty, 0)$ and $(0, \infty)$.*

Proof. Apply Eq. (6.3.16) to $f(t) = e^t$ at $a = \ln x$, $b = x$:

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

For $x < 0$, $\ln|x| = \ln(-x)$ and the chain rule gives $\dfrac{1}{-x} \cdot (-1) = \dfrac{1}{x}$. ∎

For $a > 0$ and real x, $a^x = e^{x \ln a}$ (Theorem 4.4.24). For $x > 0$ and real r, $x^r = e^{r \ln x}$ by the same theorem with base x, and trivially for $x = 1$, where both sides equal 1. The chain rule now differentiates every power and every exponential.

Theorem 6.4.6 (Exponentials and real powers)**.** *For $a > 0$, $(a^x)' = a^x \ln a$ on $\mathbb{R}$. For every real r,*

$$(x^r)' = r\, x^{r-1} \qquad (x > 0), \qquad \mathrm{d}(x^r) = r\, x^{r-1}\, \mathrm{d}x. \tag{6.4.4}$$

Proof. By Eq. (6.3.9) and Theorem 6.4.4, $(e^{x \ln a})' = e^{x \ln a} \cdot \ln a = a^x \ln a$. For the power,

$$(e^{r \ln x})' = e^{r \ln x} \cdot \frac{r}{x} = x^r \cdot r\, x^{-1} = r\, x^{r-1},$$

by $(\ln x)' = 1/x$ and $x^r x^{-1} = x^{r-1}$. ∎

Theorem 6.4.6 extends the power rule Eq. (6.3.4) from integers to every real exponent on $(0, \infty)$. Roots are powers with fractional exponents: $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2}$, in agreement with Worked Example 6.1.1. Before differentiating a root or a reciprocal, rewrite it as a power.

Caution

The power rule $(x^r)' = rx^{r-1}$ needs a constant exponent and the exponential rule $(a^x)' = a^x \ln a$ needs a constant base. Neither applies to x^x with its varying base and exponent. Write it as $e^{x \ln x}$ and follow Worked Example 6.4.1.

6.4.7 Sine and cosine

The sine and cosine of Section 4.5 are defined from the unit circle with angles measured in radians (Definition 4.5.2). The radian measure is what makes the derivative of sin equal to cos with no constant factor. Chapter 5 derives from the inequality Eq. (5.2.4) the two limits

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0. \tag{6.4.5}$$

The first is Theorem 5.2.18. The second follows from $\lim_{h\to 0}(1-\cos h)/h^2 = \frac{1}{2}$ of Worked Example 5.2.2 on multiplying by $h \to 0$. The proof below uses the angle-sum formulas and no other property of sin and cos.

Common misconception

The limit $\sin h/h \to 1$ is false in degrees. If h is measured in degrees, then $\sin h$ means $\sin(\pi h/180)$ in radians and the limit is $\pi/180$. Every derivative formula for the trigonometric functions assumes radians.

Theorem 6.4.8 (Derivatives of sine and cosine)**.** *The sine and cosine are differentiable on* $\mathbb{R}$ *with*

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \tag{6.4.6}$$

that is, $\mathrm{d}\sin x = \cos x\,\mathrm{d}x$ *and* $\mathrm{d}\cos x = -\sin x\,\mathrm{d}x$.

Proof. By the angle-sum formulas of Section 4.5,

$$\frac{\sin(x+h)-\sin x}{h} = \sin x\,\frac{\cos h - 1}{h} + \cos x\,\frac{\sin h}{h} \longrightarrow \sin x \cdot 0 + \cos x \cdot 1,$$

$$\frac{\cos(x+h)-\cos x}{h} = \cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h} \longrightarrow \cos x \cdot 0 - \sin x \cdot 1,$$

by (6.4.5) and the limit laws. ∎

Since the remaining four trigonometric functions are quotients of sine and cosine, the quotient rule Eq. (6.3.3) with $\cos^2 x + \sin^2 x = 1$ gives their derivatives wherever the denominator is nonzero:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x, \tag{6.4.7}$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \sec x \tan x, \tag{6.4.8}$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x, \tag{6.4.9}$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x. \tag{6.4.10}$$

Each "co-" function is its partner at the complementary angle, as in $\cos x = \sin(\frac{\pi}{2} - x)$. Differentiating by the chain rule brings down an inner derivative -1 that is the source of every minus sign: $(\cos x)' = \cos(\frac{\pi}{2} - x) \cdot (-1) = -\sin x$.

6.4.9 Inverse trigonometric functions

Section 4.5 defines arcsin, arccos and arctan as the inverses of sin on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, of cos on $[0, \pi]$ and of tan on $(-\frac{\pi}{2}, \frac{\pi}{2})$, on each of which the function is continuous and strictly monotone.

Theorem 6.4.10 (Derivatives of the inverse trigonometric functions)**.**

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1), \tag{6.4.11}$$

$$(\arctan x)' = \frac{1}{1+x^2} \qquad (x \in \mathbb{R}). \tag{6.4.12}$$

Proof. The sine is continuous and strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by Theorem 4.5.13. Its derivative there is $\cos y > 0$ by Theorem 4.5.7(5). For $-1 < x < 1$ the number $y = \arcsin x$ lies in that interval. By Eq. (6.3.16), $(\arcsin x)' = 1/\cos y$. Since $\cos y > 0$ and $\cos^2 y = 1 - \sin^2 y = 1 - x^2$, it follows that $\cos y = \sqrt{1-x^2}$. For $y = \arccos x \in (0, \pi)$ the derivative of cos is $-\sin y < 0$ by Theorem 4.5.7(5), with $\sin y = \sqrt{1-x^2}$ for the same reason. The tangent maps $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$ (Corollary 5.4.13). Hence arctan is defined at every real x. For $y = \arctan x$ the derivative of tan is $\sec^2 y = 1 + \tan^2 y = 1 + x^2 \neq 0$. Hence Eq. (6.3.16) gives $1/(1+x^2)$. ∎

The same computation can be organised as implicit differentiation (Section 6.5): for $y = \arctan x$ the equation $\tan y = x$ gives $\sec^2 y \cdot y' = 1$. That route presupposes the differentiability of y that Theorem 6.3.9 proves.

6.4.11 Derivative and antiderivative tables

The exponential, logarithmic, trigonometric and inverse trigonometric functions appear from the start of Chapter 8. Each antiderivative below reverses a derivative rule. The two tables state the same facts read in opposite directions. In the tables a is positive with $a \neq 1$ and r is real.

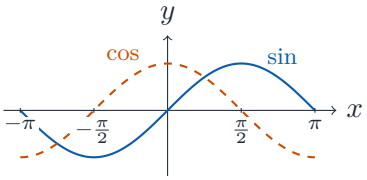
Derivatives.

$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$	$\frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$	$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$
$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x$	$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$
$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x$	$\frac{\mathrm{d}}{\mathrm{d}x} \ln\lvert x\rvert = \frac{1}{x}$
$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$	$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1+x^2}$	$\frac{\mathrm{d}}{\mathrm{d}x} x^r = r\, x^{r-1}$

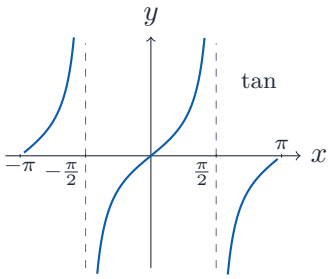
Antiderivatives (each with $+C$).

$$\int \sin x\,dx = -\cos x \qquad \int \csc^2 x\,dx = -\cot x$$
$$\int \cos x\,dx = \sin x \qquad \int \csc x\cot x\,dx = -\csc x$$
$$\int \sec^2 x\,dx = \tan x \qquad \int e^x\,dx = e^x$$
$$\int \sec x\tan x\,dx = \sec x \qquad \int \frac{1}{x}\,dx = \ln|x|$$
$$\int a^x\,dx = \frac{a^x}{\ln a} \qquad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$
$$\int \frac{dx}{1+x^2} = \arctan x \qquad \int x^r\,dx = \frac{x^{r+1}}{r+1}\ (r \neq -1)$$

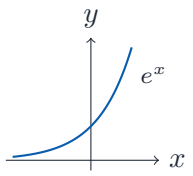
Each line of the second table states that the 1-form on the left is the differential of the function on the right: $\mathrm{d}(-\cos x) = \sin x\,dx$ for the first entry. Chapter 8 defines the indefinite integral (Definition 8.1.4) and shows that two antiderivatives on an interval differ by a constant (Theorem 8.1.3). The entry $\int \frac{1}{x}\,dx = \ln|x|$ is valid on any interval avoiding 0. Section 4.3 explains why every root and reciprocal should be rewritten as a power before it is used. The graphs of these functions are collected in Fig. 6.4.1.



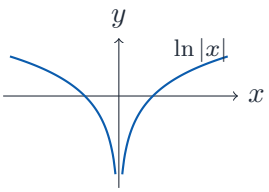
(a) $\sin x$ and $\cos x$ on $[-\pi, \pi]$.



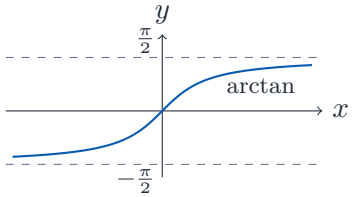
(b) $\tan x$ on $(-\pi, \pi)$, asymptotes at $\pm\frac{\pi}{2}$.



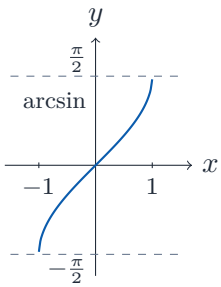
(c) e^x: positive, increasing, $y \to 0$ as $x \to -\infty$.



(d) $\ln|x|$: even with a vertical asymptote at $x = 0$.



(e) $\arctan x$: range $(-\frac{\pi}{2}, \frac{\pi}{2})$.



(f) $\arcsin x$ on $[-1, 1]$ with range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Figure 6.4.1: Graphs of the elementary transcendental functions.

Caution

The "co-" functions pick up a minus sign: $(\cos x)' = -\sin x$, $(\cot x)' = -\csc^2 x$, $(\csc x)' = -\csc x\cot x$ and $(\arccos x)' = -1/\sqrt{1-x^2}$. Equivalently $\int \sin x\,dx = -\cos x + C$. Misplaced signs here are the most common error.

Study guide

Learn the derivative table. The antiderivative table is the same table read backwards. Every derivative of an elementary function is then a finite application of the rules of Section 6.3 to entries of the table. Rewrite roots and reciprocals as powers and a variable base raised to a variable exponent as an exponential before differentiating.
The two limits Theorems 5.2.18 and 6.4.3 are the only places where the definitions of e^x and $\sin x$ enter. Know how each is proved: the exponential limit by squeezing between the bounds of Lemma 6.4.2, the sine limit by squeezing with the inequality Eq. (5.2.4).

Worked Example 6.4.1

Differentiate $y = x^x$ for $x > 0$.
Neither the power rule nor the exponential rule applies. Write $x^x = e^{x \ln x}$. By the product rule the exponent $u = x \ln x$ has derivative $1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$. The chain rule then gives

$$y' = e^{x \ln x}\,(\ln x + 1) = x^x(\ln x + 1).$$

The same computation is often organised as *logarithmic differentiation*. Since $y > 0$ the logarithm $\ln y = x \ln x$ is defined. Differentiating both sides by the chain rule gives $y'/y = \ln x + 1$. At $x = 1/e$ the derivative vanishes and the graph of x^x has a horizontal tangent.

Worked Example 6.4.2

Differentiate $y = \sin^2(3x)$ and $z = e^{-x}\cos 2x$.
For y the outermost operation is squaring. With $v = \sin u$ and $u = 3x$,

$$y' = 2v \cdot \cos u \cdot 3 = 6 \sin 3x \cos 3x = 3 \sin 6x,$$

the last form by the double-angle formula. For z the product rule and the chain rule give

$$z' = -e^{-x}\cos 2x + e^{-x} \cdot (-2 \sin 2x) = -e^{-x}(\cos 2x + 2 \sin 2x).$$

In the language of forms, $\mathrm{d}z = -e^{-x}(\cos 2x + 2 \sin 2x)\,\mathrm{d}x$.

Worked Example 6.4.3

Show that $y = \arctan \dfrac{1+x}{1-x}$ and $\arctan x$ have the same derivative wherever $x \neq 1$.
Let $u = (1+x)/(1-x)$. By the quotient rule, $u' = \dfrac{(1-x) + (1+x)}{(1-x)^2} = \dfrac{2}{(1-x)^2}$. Also

$$1 + u^2 = \frac{(1-x)^2 + (1+x)^2}{(1-x)^2} = \frac{2(1+x^2)}{(1-x)^2}.$$

The chain rule and Theorem 6.4.10 give

$$y' = \frac{u'}{1+u^2} = \frac{2}{(1-x)^2} \cdot \frac{(1-x)^2}{2(1+x^2)} = \frac{1}{1+x^2}.$$

The two functions differ by a constant on each of $(-\infty, 1)$ and $(1, \infty)$, by Corollary 7.2.2. Evaluating at $x = 0$ shows that the constant on $(-\infty, 1)$ is $\pi/4$.

Problems

6.4.1 Differentiate $e^{3x} \sin x$ and $\ln(x^2 + 1)$. ⋆ Hint · Answer · Solution

6.4.2 Differentiate $\tan 5x$ and $x^2 e^{-x}$. ⋆ Hint · Answer · Solution

6.4.3 Differentiate 2^{x^2} and $\sqrt{1 + \sin^2 x}$. ⋆⋆ Hint · Answer · Solution

6.4.4 Differentiate $\arctan \sqrt{x}$ for $x > 0$ and $\arcsin(x/2)$ for $|x| < 2$. ⋆⋆ Hint · Answer · Solution

6.4.5 Differentiate $x^{\sin x}$ for $x > 0$. ⋆⋆ Hint · Answer · Solution

6.4.6 Write $\mathrm{d}(\ln|\cos x|)$ and $\mathrm{d}(\sec x \tan x)$ as multiples of $\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

6.4.7 Find the tangent line to $y = e^x$ at $x = 0$ and to $y = \ln x$ at $x = 1$. Explain why the two lines are reflections of each other in $y = x$. ⋆⋆ Hint · Answer · Solution

6.4.8 Derive $(\arccos x)' = -1/\sqrt{1 - x^2}$ from Theorem 6.3.9 in full and show that $\arcsin x + \arccos x$ has derivative 0 on $(-1, 1)$. Find the value of this sum. ⋆⋆⋆ Hint · Answer · Solution

6.4.9 Use the derivative of ln at 1 to prove that $\lim_{h \to 0} \dfrac{\ln(1+h)}{h} = 1$ and deduce that $\lim_{h \to 0^+} (1 + h)^{1/h} = e$. ⋆⋆⋆ Hint · Answer · Solution

6.4.10 Prove that $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$ for every integer $n \geq 0$, with $^{(n)}$ denoting the nth derivative. ⋆⋆⋆ Hint · Answer · Solution

6.5 Implicit Differentiation and Related Rates

Overview

Objectives: Find the slope of a curve given by an equation in x and y without solving for y; compute second derivatives implicitly; set up and solve related-rates problems in which several quantities depend on time.
Prerequisites: The chain rule Eq. (6.3.9) and its form for differentials Eq. (6.3.15); the uniqueness statement of Theorem 6.2.8; the derivative table of Section 6.4.
Key results: Implicit differentiation (Proposition 6.5.2); the related-rates procedure.

The circle $x^2 + y^2 = 25$ is not the graph of a single function, yet near the point $(3, 4)$ it coincides with the graph of $y = \sqrt{25 - x^2}$ and has a tangent line there. Many curves are given only by an equation that cannot be solved for y in usable form, such as $x^3 + y^3 = 6xy$. Implicit differentiation finds the slope of such a curve directly from its equation.

6.5.1 Implicit differentiation

Suppose that on an open interval I a differentiable function $y = \varphi(x)$ satisfies an equation $F(x, \varphi(x)) = c$ for every $x \in I$. Both sides are then the same function of x on I with the same derivative and the same differential. Differentiating the left side by the rules of Section 6.3, with y treated as the function φ of x, gives an equation that can be solved for $y' = \varphi'(x)$.

Proposition 6.5.2 (Implicit differentiation)**.** *Let φ be differentiable on an open interval I and let $y = \varphi(x)$. Suppose that applying* d *to an identity between functions of x on I gives*

$$P(x, y)\,\mathrm{d}x + R(x, y)\,\mathrm{d}y = 0, \tag{6.5.1}$$

where $P(x, y)$ and $R(x, y)$ stand for the functions $P(x, \varphi(x))$ and $R(x, \varphi(x))$ of x on I. Then $(P + R\,\varphi')\,\mathrm{d}x = 0$. At every point where $R(x, y) \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{P(x, y)}{R(x, y)}. \tag{6.5.2}$$

Proof. By Eq. (6.2.6), $\mathrm{d}y = \varphi'(x)\,\mathrm{d}x$. The given 1-form (6.5.1) therefore equals $(P + R\,\varphi')\,\mathrm{d}x$. By the uniqueness half of Theorem 6.2.8, a 1-form $g(x)\,\mathrm{d}x$ is zero exactly when g is zero at every point. Hence $P + R\varphi' = 0$ on I. Dividing by $R \neq 0$ gives (6.5.2). ∎

Example 6.5.3 (Circle)**.** On the upper half of the circle $x^2 + y^2 = 25$, applying d gives $2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$. Where $y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}. \tag{6.5.3}$$

At $(3, 4)$ the slope is $-\frac{3}{4}$ and the tangent line is $y = 4 - \frac{3}{4}(x - 3)$. Differentiating $\sqrt{25 - x^2}$ directly gives $-x/\sqrt{25 - x^2} = -x/y$ as well.

Caution

Implicit differentiation assumes that the curve is, near the point, the graph of a differentiable function $y = \varphi(x)$. It does not prove this. The formula $-x/y$ that the same algebra produces from $x^2 + y^2 = 0$ means nothing at the only solution $(0, 0)$ of that equation. The existence of φ near a point where $R \neq 0$ is the content of the implicit function theorem of multivariable calculus [1]. In this book the hypothesis is stated and then used.

The second derivative follows by differentiating the formula (6.5.3) for y' again with y a function of x and then substituting for y'. On the circle,

$$y'' = \left(-\frac{x}{y}\right)' = -\frac{y - xy'}{y^2} = -\frac{y + x^2/y}{y^2} = -\frac{x^2 + y^2}{y^3} = -\frac{25}{y^3}. \tag{6.5.4}$$

On the upper half the second derivative is negative and the circle bends downwards.

6.5.4 Related rates

When several quantities change with time and are tied by an equation, their rates of change are tied by the derivative of that equation. With each quantity a function of the time t, the chain rule turns the equation into a relation between the derivatives with respect to t.

Related-rates procedure

(1) Draw a diagram and name every varying quantity as a function of t. (2) Write the equation that relates them, valid at every time. (3) Differentiate the equation with respect to t and apply the chain rule to every varying quantity. (4) Only then substitute the values at the instant of interest and solve for the unknown rate. Substituting values before differentiating turns varying quantities into constants and loses their rates.

Example 6.5.5 (Expanding circle)**.** A circular ripple has radius $r(t)$ and area $A = \pi r^2$. Differentiating with respect to t gives $A'(t) = 2\pi r\, r'(t)$. If the radius grows at 0.5 metres per second, the area grows at $2\pi \cdot 4 \cdot 0.5 = 4\pi$ square metres per second at the instant when $r = 4$.

Study guide

Implicit differentiation is the chain rule applied to an equation. Every term containing y produces a factor y': y^3 differentiates to $3y^2y'$ by the chain rule and xy to $y + xy'$ by the product rule. Collect the terms with y' on one side and solve. In the language of forms, apply d to both sides, write $\mathrm{d}y = y'\,\mathrm{d}x$, and equate the coefficient of $\mathrm{d}x$ to zero.
In a related-rates problem, keep the quantities as functions of t until after differentiation. Include units with every rate and check its sign against the direction of change: a decreasing quantity has a negative rate.

Worked Example 6.5.1

Find the slope of the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$ and the equation of the tangent line there.
The point lies on the curve because $27 + 27 = 54 = 6 \cdot 3 \cdot 3$. Applying d to both sides, with the product rule Eq. (6.3.6) on the right,

$$3x^2\,\mathrm{d}x + 3y^2\,\mathrm{d}y = 6y\,\mathrm{d}x + 6x\,\mathrm{d}y.$$

Collecting terms gives $(3x^2 - 6y)\,\mathrm{d}x + (3y^2 - 6x)\,\mathrm{d}y = 0$. Where $3y^2 - 6x \neq 0$, Proposition 6.5.2 gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 - 6y}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}.$$

At $(3, 3)$ this is $(6-9)/(9-6) = -1$. The tangent line is $y = 3 - (x - 3)$, that is $x + y = 6$. The curve is symmetric in the line $y = x$ and its tangent at $(3, 3)$ is perpendicular to that line.

Worked Example 6.5.2

A rigid pole 10 metres long leans against a vertical wall. Its foot slides away from the wall at 1 metre per second. How fast is the top sliding down the wall when the foot is 6 metres from the wall?

Let $x(t)$ be the distance from the foot to the wall and $y(t)$ the height of the top, both in metres. At every time, $x^2 + y^2 = 100$. Differentiating with respect to t,

$$2x\,x' + 2y\,y' = 0, \qquad y' = -\frac{x\,x'}{y}.$$

At the instant of interest $x = 6$, $y = \sqrt{100 - 36} = 8$ and $x' = 1$. Hence $y' = -6 \cdot 1/8 = -\frac{3}{4}$. The negative sign records that y decreases: the top slides down at $\frac{3}{4}$ metre per second.

Worked Example 6.5.3

Water flows at 2 cubic metres per minute into a tank in the shape of an inverted cone of height 4 metres and top radius 2 metres. How fast is the water level rising when the water is 3 metres deep?

Let $h(t)$ be the depth, $r(t)$ the radius of the water surface and $V(t)$ the volume. By similar triangles $r/h = 2/4$. Hence $r = h/2$ at every time and

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{12}\pi h^3.$$

Differentiating with respect to t gives $V' = \frac{1}{4}\pi h^2 h'$. With $V' = 2$ and $h = 3$,

$$h' = \frac{4V'}{\pi h^2} = \frac{8}{9\pi} \approx 0.283$$

metres per minute.

Problems

6.5.1 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $(-3, 4)$ on the circle $x^2 + y^2 = 25$ and at $(2, 3)$ on the hyperbola $xy = 6$.
⋆ Hint · Answer · Solution

6.5.2 The curve $e^y + xy = e$ passes through $(0, 1)$. Find the slope there.
⋆⋆ Hint · Answer · Solution

6.5.3 On the hyperbola $x^2 - y^2 = 1$, find y' and y'' in terms of x and y where $y \neq 0$.
⋆⋆ Hint · Answer · Solution

6.5.4 Find the tangent line to $\sin(xy) = x$ at the point $(\frac{1}{2}, \frac{\pi}{3})$.
⋆⋆ Hint · Answer · Solution

6.5.5 The radius of a sphere increases at 2 centimetres per second. How fast is its volume increasing when the radius is 5 centimetres?
⋆⋆ Hint · Answer · Solution

6.5.6 Two cars leave the same junction, one travelling north at 60 km/h and the other east at 80 km/h. How fast is the distance between them increasing when the first is 30 km and the second 40 km from the junction?
⋆⋆ Hint · Answer · Solution

6.5.7 A person 1.8 metres tall walks away from a lamp post 6 metres tall at 1.5 metres per second. How fast is the length of the person's shadow increasing?

⋆⋆ Hint · Answer · Solution

6.5.8 A rocket rises vertically from a point 4 km from an observer. When the rocket is 3 km high it is rising at 0.5 km/s. How fast is the angle of elevation θ seen by the observer increasing?

⋆⋆⋆ Hint · Answer · Solution

6.5.9 Let (x_0, y_0) lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $y_0 \neq 0$. Prove that the tangent line there is $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = 1$.

⋆⋆⋆ Hint · Answer · Solution

Chapter 7

Differentiation as a Linear Operator

Chapter 6 defined the derivative of a function at a single point as the limit of a difference quotient. A derivative at one point describes the function only near that point. Most questions in calculus are global: whether a function increases on an interval, whether two functions with the same derivative differ by a constant, how far a function strays from its tangent line over a whole interval. This chapter answers those questions.

The bridge from local to global information is the Mean Value Theorem of Section 7.1. Its proof rests on the Extreme Value Theorem and hence on the completeness of the real numbers. Section 7.2 draws its consequences: the monotonicity test, the first and second derivative tests, convexity, inequalities such as $e^x \geq 1 + x$, and L'Hôpital's rule for indeterminate limits together with the ranking of logarithms, powers and exponentials by growth.

Section 7.3 then changes the point of view. The sum and constant-multiple rules state that the assignment $f \mapsto f'$ is a linear map between vector spaces of functions. The language of linear algebra (kernel, image, matrix, eigenvector) then organises much of what the first two sections proved. On polynomials of bounded degree the map has an explicit nilpotent matrix, the exponential functions are its eigenvectors, and the sine and cosine span the solutions of $f'' = -f$. Section 7.4 closes the chapter with Taylor's theorem, the quantitative form of the statement that a function with $n+1$ derivatives agrees with a polynomial of degree n up to an error of order $n+1$.

The book writes a capital D for differentiation in this chapter alone and only in operator statements such as D^2, $\ker D$ or e^{hD}. Everywhere else a derivative is written f' or as the Leibniz quotient $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ of Chapter 6.

7.1 Mean Value Theorem

Overview

Objectives: Define local extrema and prove Fermat's theorem that the derivative vanishes at an interior local extremum; prove Rolle's theorem (Theorem 7.1.7) from the Extreme Value Theorem; prove the Mean Value Theorem (Theorem 7.1.9) and its Cauchy form (Theorem 7.1.13); locate the point the theorems promise in simple cases and use Rolle's theorem to count roots.

Prerequisites: The Extreme Value Theorem (Theorem 5.4.6); the Intermediate Value Theorem (Theorem 5.4.2); the definition of the derivative (Definition 6.1.2 and Eq. (6.1.2)) and the fact that a differentiable function is continuous (Theorem 6.1.11).

Key results: If f is continuous on $[a,b]$ and differentiable on (a,b), then $f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a,b)$.

A car that covers 120 kilometres in two hours has an average speed of 60 kilometres per hour. Common sense says that at some instant of the journey the speedometer read exactly 60. The Mean Value Theorem is the precise form of that statement: the average rate of change of a differentiable function over an interval equals its instantaneous rate of change at some interior point.

The statement looks obvious and its proof is short, yet it depends on the completeness of the real numbers. Over the rational numbers it fails. The function on $\mathbb{Q}$ that equals 0 at rationals below $\sqrt{2}$ and 1 at rationals above $\sqrt{2}$ has derivative 0 at every rational point because each rational point has a rational neighbourhood on which the function is constant. Its values nevertheless satisfy $f(2) - f(1) = 1$ with no rational c between 1 and 2 at which $f'(c) = 1$. Completeness of the real numbers excludes such gaps and enters the proof through the Extreme Value Theorem.

7.1.1 Local extrema and Fermat's theorem

Definition 7.1.2 (Local extremum)**.** Let f be defined on a set containing an open interval around c. The function f has a *local maximum* at c if there is $\delta > 0$ with $f(x) \le f(c)$ for every x satisfying $|x - c| < \delta$. It has a *local minimum* at c if there is $\delta > 0$ with $f(x) \ge f(c)$ for every such x. A *local extremum* is a local maximum or a local minimum. A point c in the domain at which $f'(c) = 0$ or $f'(c)$ does not exist is a *critical point* of f.

Example 7.1.3 (Local extrema of a quartic)**.** The function $f(x) = x^4 - 2x^2$ has a local maximum at 0 because $f(x) = x^2(x^2 - 2) \le 0 = f(0)$ whenever $|x| < \sqrt{2}$. It has local minima at ± 1 because $f(x) + 1 = (x^2 - 1)^2 \ge 0$ for every x. The local maximum at 0 is not a maximum on all of $\mathbb{R}$ because $f(2) = 8$.

Example 7.1.4 (Endpoint maximum (non-example))**.** On $[0, 1]$ the function $f(x) = x$ attains its largest value at the endpoint 1. That point is not a local maximum in the sense of Definition 7.1.2 because the definition requires f to be defined on an open interval around it. Fermat's theorem below excludes such endpoint extrema. The one-sided derivative at 1 is $1 \neq 0$.

Theorem 7.1.5 (Fermat's theorem on local extrema)**.** *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c and choose $\delta > 0$ with $f(c + h) \le f(c)$ whenever $|h| < \delta$. For $0 < h < \delta$ the difference quotient $(f(c + h) - f(c))/h$ has a numerator ≤ 0 and a positive denominator and is therefore ≤ 0. Letting $h \to 0^+$ gives $f'(c) \le 0$ because a limit of quantities that are ≤ 0 is ≤ 0. For $-\delta < h < 0$ the numerator is ≤ 0 and the denominator negative. The quotient is then ≥ 0 and has limit $f'(c) \ge 0$ as $h \to 0^-$. Since f is differentiable at c, both one-sided limits equal $f'(c)$ (Proposition 6.1.14) and the two inequalities give $f'(c) = 0$. For a local minimum, apply this case to the local maximum of $-f$ at c and use $(-f)'(c) = -f'(c)$ (Theorem 6.3.1). ∎

Caution

The converse of Fermat's theorem is false. The function $f(x) = x^3$ has $f'(0) = 0$ and no local extremum at 0 because $f(x) < 0 < f(x')$ for every $x < 0 < x'$. A local extremum may also occur where the derivative does not exist: $|x|$ has a local minimum at 0 and no derivative there. Fermat's theorem narrows the search for interior extrema to the critical points and says nothing more.

7.1.6 Rolle's theorem

Theorem 7.1.7 (Rolle's theorem)**.** *Let f be continuous on $[a,b]$ and differentiable on (a,b), with $f(a)=f(b)$. Then there is a point $c\in(a,b)$ with $f'(c)=0$.*

Proof. By the Extreme Value Theorem (Theorem 5.4.6) the continuous function f attains a largest value M and a smallest value m on $[a,b]$. If both are attained only at the endpoints, then $M=m=f(a)$ because $f(a)=f(b)$. In that case f is constant on $[a,b]$ and every $c\in(a,b)$ has $f'(c)=0$. Otherwise one of the two values is attained at some c with $a<c<b$. Since the open interval (a,b) contains a neighbourhood of c on which $f(x)\le f(c)$ (or $f(x)\ge f(c)$), f has a local extremum at c. Since f is differentiable at c, Fermat's theorem (Theorem 7.1.5) gives $f'(c)=0$. ∎

Figure 7.1.1 shows the geometry: a smooth graph that leaves the height $f(a)$ and returns to it must turn somewhere in between at a point where the tangent line is horizontal.

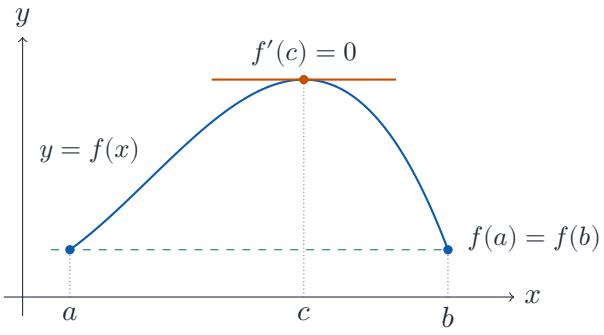


Figure 7.1.1: Rolle's theorem. A graph with equal heights at a and b has a horizontal tangent at some interior point c, here the point at which f attains its maximum on $[a,b]$.

7.1.8 Secant and tangent slopes

The Mean Value Theorem removes the condition $f(a)=f(b)$ from Rolle's theorem. Subtracting from f the secant line through $(a,f(a))$ and $(b,f(b))$ produces a function with equal endpoint values and makes Rolle's theorem applicable.

Theorem 7.1.9 (Mean Value Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point $c\in(a,b)$ with*

$$f'(c)=\frac{f(b)-f(a)}{b-a},\qquad \textit{equivalently}\qquad f(b)-f(a)=f'(c)\,(b-a). \tag{7.1.1}$$

Proof. Let $m=(f(b)-f(a))/(b-a)$ be the slope of the secant and define

$$h(x)=f(x)-f(a)-m\,(x-a),\qquad a\le x\le b.$$

As the difference of f and a polynomial, h is continuous on $[a,b]$ and differentiable on (a,b) with $h'(x)=f'(x)-m$ (Theorem 6.3.1). It satisfies $h(a)=0$ and $h(b)=f(b)-f(a)-m(b-a)=0$. Rolle's theorem (Theorem 7.1.7) gives a point $c\in(a,b)$ with $0=h'(c)=f'(c)-m$, the equation Eq. (7.1.1). ∎

Geometrically some tangent line is parallel to the secant line (Fig. 7.1.2). The theorem asserts that such a point exists and gives no procedure for finding it. The figure shows two such points.

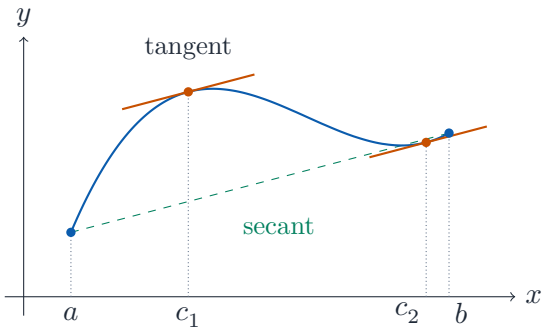


Figure 7.1.2: Mean Value Theorem. At c_1 and at c_2 the tangent line (solid) is parallel to the secant through the endpoints of the graph (dashed). Either point satisfies Eq. (7.1.1).

Example 7.1.10 (Mean value point of a parabola)**.** For $f(x) = x^2$ on $[0, 2]$ the secant slope is $(4 - 0)/2 = 2$ and $f'(c) = 2c$. The equation $2c = 2$ gives the midpoint $c = 1$. For every parabola $f(x) = px^2 + qx + r$ with $p \neq 0$ on any interval $[a, b]$ the same computation gives $2pc + q = p(a + b) + q$ and hence $c = (a + b)/2$.

Example 7.1.11 (Corner (non-example))**.** The function $f(x) = |x|$ is continuous on $[-1, 2]$ with secant slope $(2-1)/3 = \frac{1}{3}$. Its derivative takes only the values -1 and 1 and does not exist at 0. No point therefore has $f'(c) = \frac{1}{3}$. The Mean Value Theorem does not apply because f fails to be differentiable at the interior point 0.

Common misconception

The point c of the Mean Value Theorem depends on f, a and b. The theorem gives no formula for it. Arguments that use the theorem need only that c lies in (a, b). Any bound on f' valid on the whole interval then also bounds $f'(c)$. Each hypothesis is also needed. Without continuity at an endpoint the conclusion fails for the function equal to x on $[0, 1)$ and to 0 at 1. Without differentiability it fails for $|x|$ as in the non-example above.

7.1.12 Cauchy Mean Value Theorem

Two functions f and g on $[a, b]$ describe the curve in the plane formed by the points $(g(t), f(t))$ for $a \leq t \leq b$. Its chord joins $(g(a), f(a))$ to $(g(b), f(b))$. Its velocity at time t is the vector $\mathbf{v}(t) = (g'(t), f'(t))$. The Cauchy form of the theorem states that at some time the velocity is parallel to the chord. It is the tool behind L'Hôpital's rule (Theorem 7.2.16) and Taylor's theorem (Theorem 7.4.8).

Theorem 7.1.13 (Cauchy Mean Value Theorem)**.** *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ with*

$$\big(f(b) - f(a)\big)\, g'(c) = \big(g(b) - g(a)\big)\, f'(c). \tag{7.1.2}$$

If in addition $g'(x) \neq 0$ for every $x \in (a, b)$, then $g(b) \neq g(a)$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{7.1.3}$$

Proof. Define $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. It is continuous on $[a, b]$ and differentiable on (a, b). Expanding gives

$$h(a) = f(b)g(a) - g(b)f(a) = h(b).$$

Rolle's theorem gives a point $c \in (a, b)$ with $h'(c) = 0$, the equation Eq. (7.1.2). If g' never vanishes on (a, b), then $g(b) \neq g(a)$ because otherwise Rolle's theorem applied to g would produce a zero of g'. Dividing Eq. (7.1.2) by the nonzero number $(g(b) - g(a))g'(c)$ gives the quotient form (7.1.3). ∎

Taking $g(x) = x$ recovers Theorem 7.1.9. Applying the Mean Value Theorem to f and g separately produces two points c_1 and c_2 that need not coincide. The Cauchy form therefore needs its own proof.

Worked Example 7.1.1

Find every number c that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - x$ on $[0, 2]$.
The theorem applies because the polynomial f is continuous and differentiable everywhere. The secant slope is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2} = 3.$$

Since $f'(c) = 3c^2 - 1$, the equation $f'(c) = 3$ becomes $c^2 = \frac{4}{3}$, with solutions $c = \pm 2/\sqrt{3}$. Only the positive root $c = 2/\sqrt{3} \approx 1.155$ lies in $(0, 2)$.

Worked Example 7.1.2

Prove that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.
Let $f(x) = x^3 + 3x + 1$. Since $f(-1) = -3 < 0 < 1 = f(0)$ and f is continuous, the Intermediate Value Theorem (Theorem 5.4.2) gives a root in $(-1, 0)$. For uniqueness, suppose $f(r) = f(s) = 0$ with $r < s$. Rolle's theorem (Theorem 7.1.7) on $[r, s]$ gives c with $f'(c) = 0$. This contradicts $f'(x) = 3x^2 + 3 \geq 3$ for every x. Hence the root is unique. The Intermediate Value Theorem finds a root and Rolle's theorem forbids a second.

Worked Example 7.1.3

Find the point c of the Cauchy Mean Value Theorem for $f(x) = x^2$ and $g(x) = x^3$ on $[0, 1]$. Interpret it.
Since $f(1) - f(0) = 1$ and $g(1) - g(0) = 1$, Eq. (7.1.2) reads $1 \cdot 3c^2 = 1 \cdot 2c$, that is $c(3c - 2) = 0$. The root in $(0, 1)$ is $c = \frac{2}{3}$. The curve $t \mapsto (t^3, t^2)$ runs from $(0, 0)$ to $(1, 1)$. At $t = \frac{2}{3}$ its velocity $(3t^2, 2t) = (\frac{4}{3}, \frac{4}{3})$ is parallel to the chord direction $(1, 1)$. Applying Theorem 7.1.9 to f and g separately gives $c_1 = \frac{1}{2}$ and $c_2 = 1/\sqrt{3}$, two different points.

Study guide

Learn the three theorems as one chain. Fermat's theorem says a derivative vanishes at an interior extremum. Rolle's theorem combines Fermat's theorem with the Extreme Value Theorem: equal endpoint values force a horizontal tangent. The Mean Value Theorem reduces to Rolle's theorem by subtracting the secant line. The Cauchy form does the same for a pair of functions.

To apply any of them, first check the hypotheses: continuity on the closed interval and differentiability on the open interval. To count roots of an equation, use the Intermediate Value Theorem for existence and Rolle's theorem for uniqueness. To bound a difference $f(b)-f(a)$, write it as $f'(c)(b-a)$ and bound f' on the whole interval.

Problems

7.1.1 Find the number c promised by the Mean Value Theorem for $f(x)=x^2+2x$ on $[0,3]$.

⋆ Hint · Answer · Solution

7.1.2 Find the number c promised by the Mean Value Theorem for $f(x)=\sqrt{x}$ on $[0,4]$, after checking the hypotheses.

⋆ Hint · Answer · Solution

7.1.3 Find the number c promised by the Mean Value Theorem for $f(x)=1/x$ on $[1,3]$.

⋆ Hint · Answer · Solution

7.1.4 Verify the hypotheses of Rolle's theorem for $f(x)=\sin x$ on $[0,\pi]$ and find the point c.

⋆ Hint · Answer · Solution

7.1.5 For $f(x)=|x-1|$ on $[0,3]$, show that no $c\in(0,3)$ satisfies $f'(c)=(f(3)-f(0))/3$. Name the hypothesis of the Mean Value Theorem that fails.

⋆⋆ Hint · Answer · Solution

7.1.6 Find the point $c\in(0,\pi/2)$ of the Cauchy Mean Value Theorem for $f(x)=\sin x$ and $g(x)=\cos x$ on $[0,\pi/2]$.

⋆⋆ Hint · Answer · Solution

7.1.7 Prove that $x^5+x-1=0$ has exactly one real solution.

⋆⋆ Hint · Answer · Solution

7.1.8 Let $f(x)=\left(\sqrt[3]{x}\right)^2$ on $[-1,1]$, with $\sqrt[3]{x}$ the real cube root. Show that $f(-1)=f(1)$ while f' has no zero in $(-1,1)$. Explain why Rolle's theorem is not contradicted.

⋆⋆ Hint · Answer · Solution

7.1.9 Let f be differentiable on an interval I with $f'(x)\neq 0$ for every $x\in I$. Prove that f is injective on I (Definition 1.5.13).

⋆⋆ Hint · Answer · Solution

7.1.10 Prove that a polynomial of degree $n\geq 1$ has at most n real roots.

⋆⋆⋆ Hint · Answer · Solution

7.2 Consequences of the Mean Value Theorem

Overview

Objectives: Prove that a function with zero derivative on an interval is constant (Corollary 7.2.2); decide where a function increases or decreases from the sign of its derivative (Theorem 7.2.4); classify critical points by the first and second derivative tests; prove convexity and inequalities such as $e^x \geq 1+x$; evaluate indeterminate limits by L'Hôpital's rule (Theorem 7.2.16) and prove the growth ranking of logarithms, powers and exponentials by L'Hôpital's rule (Theorem 7.2.18).
Prerequisites: The Mean Value Theorem and its Cauchy form (Theorems 7.1.9 and 7.1.13); the derivatives of the elementary functions (Section 6.4); one-sided limits and limits at infinity (Definitions 5.2.10, 5.2.20 and 5.2.21).
Key results: $f' = 0$ on an interval implies f constant. $f' > 0$ on an interval implies f strictly increasing. For the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, $\lim f/g = \lim f'/g'$ whenever the second limit exists. As $x \to \infty$, $\ln x \ll x^p \ll e^x$ for every $p > 0$.

Each result of this section converts information about f' on a whole interval into information about f itself. The Mean Value Theorem is the only tool needed: it writes every difference $f(x_2) - f(x_1)$ as $f'(c)$ times $x_2 - x_1$ with a factor $f'(c)$ whose sign or size then controls the difference.

7.2.1 Zero derivative and the monotonicity test

Corollary 7.2.2 (Zero derivative implies constant)**.** *If $f'(x) = 0$ for every x in an interval I, then f is constant on I.*

Proof. For any $x_1 < x_2$ in I the function f is differentiable and hence continuous (Theorem 6.1.11) on $[x_1, x_2] \subseteq I$. Theorem 7.1.9 gives a c with $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$ and hence $f(x_2) = f(x_1)$. ∎

Corollary 7.2.3 (Functions with equal derivatives differ by a constant)**.** *If $f'(x) = g'(x)$ for every x in an interval I, then there is a constant C with $f(x) = g(x) + C$ for every $x \in I$.*

Proof. The difference $f - g$ has derivative $f' - g' = 0$ on I (Theorem 6.3.1) and is constant by Corollary 7.2.2. ∎

The hypothesis that the domain is an interval cannot be dropped. Section 7.3 returns to this point with the function that equals 0 on $(-\infty, 0)$ and 1 on $(0, \infty)$ and has derivative 0 at every point of its domain.

A function f is *nondecreasing* on a set S if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ in S. It is *strictly increasing* if $f(x_1) < f(x_2)$ for all such pairs. *Nonincreasing* and *strictly decreasing* reverse the inequalities. A point of an interval that is not an endpoint is an *interior point*.

Theorem 7.2.4 (Monotonicity test)**.** *Let f be continuous on an interval I and differentiable at every interior point of I.*

1. *If $f'(x) > 0$ at every interior point, then f is strictly increasing on I.*
2. *If $f'(x) \geq 0$ at every interior point, then f is nondecreasing on I.*
3. *If $f'(x) < 0$ at every interior point, then f is strictly decreasing on I. If $f'(x) \leq 0$ there, then f is nonincreasing on I.*

Conversely, if f is nondecreasing on I and differentiable at an interior point x, then $f'(x) \geq 0$.

Proof. Let $x_1 < x_2$ be points of I. Since every point of (x_1, x_2) is interior to I, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Theorem 7.1.9 gives $c \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1). \tag{7.2.1}$$

The factor $x_2 - x_1$ in (7.2.1) is positive. If $f' > 0$ at interior points, then the right side is positive and $f(x_1) < f(x_2)$. If $f' \geq 0$, then the right side is nonnegative and $f(x_1) \leq f(x_2)$. The third statement follows on applying the first two to $-f$.

For the converse, let f be nondecreasing and differentiable at the interior point x. For every $h \neq 0$ small enough that $x + h \in I$ the numerator $f(x + h) - f(x)$ has the sign of h or is zero. The difference quotient $(f(x + h) - f(x))/h$ is therefore ≥ 0 and so is its limit $f'(x)$. ∎

Example 7.2.5 (Strict increase with a vanishing derivative)**.** The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$ with $f'(0) = 0$. The converse in Theorem 7.2.4 gives only $f' \geq 0$ for a strictly increasing function. This example shows that $f' > 0$ cannot be expected. The strict increase follows from part (1) of the theorem applied separately on $(-\infty, 0]$ and on $[0, \infty)$ with $f' > 0$ at their interior points. For $x_1 < 0 < x_2$ the two halves give $f(x_1) < f(0) < f(x_2)$.

Caution

A positive derivative at a single point does not make a function increasing on any interval around that point. The theorem needs the sign of f' on a whole interval. Since a derivative need not even be continuous (Section 7.3), the sign of $f'(c)$ alone says nothing about the sign of f' near c.

7.2.6 First and second derivative tests

Theorem 7.2.7 (First derivative test)**.** *Let f be continuous on $(c - \delta, c + \delta)$ and differentiable there except possibly at c. If $f' > 0$ on $(c - \delta, c)$ and $f' < 0$ on $(c, c + \delta)$, then f has a local maximum at c. If $f' < 0$ on $(c - \delta, c)$ and $f' > 0$ on $(c, c + \delta)$, then f has a local minimum at c.*

Proof. In the first case Theorem 7.2.4 applies on the interval $(c - \delta, c]$ with interior points those of $(c - \delta, c)$. It shows that f is increasing there and that $f(x) \leq f(c)$ for $c - \delta < x \leq c$. The same theorem on $[c, c + \delta)$ shows that f is decreasing and that $f(x) \leq f(c)$ for $c \leq x < c + \delta$. The second case follows on replacing f by $-f$. ∎

Theorem 7.2.8 (Second derivative test)**.** *Let f be differentiable on an open interval containing c with $f'(c) = 0$ and suppose $f''(c)$ exists. If $f''(c) > 0$, then f has a local minimum at c. If $f''(c) < 0$, then f has a local maximum at c.*

Proof. Suppose $f''(c) > 0$. Because $f'(c) = 0$,

$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{x - c} > 0. \tag{7.2.2}$$

Applying Definition 5.2.1 to the limit (7.2.2) with $\varepsilon = f''(c)$ gives $\delta > 0$ such that $f'(x)/(x - c) > 0$ whenever $0 < |x - c| < \delta$. On $(c - \delta, c)$ the denominator is negative and therefore $f'(x) < 0$. On $(c, c + \delta)$ the denominator is positive and $f'(x) > 0$. Since f is differentiable and hence continuous near c, the first derivative test (Theorem 7.2.7) gives a local minimum. The case $f''(c) < 0$ follows on replacing f by $-f$. ∎

When $f''(c) = 0$ the second derivative test is silent. The functions x^4, $-x^4$ and x^3 all have $f'(0) = f''(0) = 0$ with, respectively, a local minimum, a local maximum and no extremum at 0.

Worked Example 7.2.1

Find the intervals on which $f(x) = x^3 - 3x$ increases and decreases. Classify its critical points.
The derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ vanishes at ± 1 and has the following signs: positive on $(-\infty, -1)$, negative on $(-1, 1)$, positive on $(1, \infty)$. By Theorem 7.2.4, f is strictly increasing on $(-\infty, -1]$ and on $[1, \infty)$ and strictly decreasing on $[-1, 1]$. The first derivative test gives a local maximum $f(-1) = 2$ and a local minimum $f(1) = -2$. The second derivative test agrees: $f''(x) = 6x$ has $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$. Neither extremum is a global one because $f(x) \to \pm\infty$ as $x \to \pm\infty$.

7.2.9 Convexity

Definition 7.2.10 (Convex function)**.** A function f on an interval I is *convex* if

$$f((1-t)x + ty) \le (1-t)f(x) + t\,f(y) \qquad \text{for all } x, y \in I \text{ and } t \in [0,1]. \tag{7.2.3}$$

It is *concave* if $-f$ is convex.

As t runs over $[0,1]$ the point $(1-t)x + ty$ runs over the segment from x to y. The right side of Eq. (7.2.3) is the height of the chord joining $(x, f(x))$ to $(y, f(y))$ above that point. Convexity is the statement that every chord lies on or above the graph (Fig. 7.2.1).

Example 7.2.11 (Convexity of the square)**.** For $f(x) = x^2$ a direct computation gives

$$(1-t)x^2 + ty^2 - \big((1-t)x + ty\big)^2 = t(1-t)(x-y)^2 \ge 0,$$

and Eq. (7.2.3) follows. The cube x^3 is not convex on $\mathbb{R}$: with $x = -1$, $y = 0$ and $t = \frac{1}{2}$ the left side of Eq. (7.2.3) is $-\frac{1}{8}$ and the right side is $-\frac{1}{2}$.

Theorem 7.2.12 (Convexity from the derivative)**.** *Let f be differentiable on an open interval I with f' nondecreasing on I. This is the case in particular when $f'' \ge 0$ on I. Then*

1. the graph lies on or above every tangent line:

$$f(x) \ge f(a) + f'(a)(x-a) \qquad \textit{for all } a, x \in I; \tag{7.2.4}$$

2. f is convex on I.

Proof. If $f'' \ge 0$ on I, then f' is nondecreasing by Theorem 7.2.4. (1) For $x \ne a$ the Mean Value Theorem gives c strictly between a and x with $f(x) - f(a) = f'(c)(x-a)$. If $x > a$, then $c > a$ and $f'(c) \ge f'(a)$. Multiplying by the positive number $x - a$ gives $f'(c)(x-a) \ge f'(a)(x-a)$. If $x < a$, then $f'(c) \le f'(a)$ and multiplying by the negative number $x - a$ gives the same inequality. Either way Eq. (7.2.4) follows.

(2) Let $x, y \in I$ and $t \in [0,1]$ and set $z = (1-t)x + ty \in I$. Applying Eq. (7.2.4) with $a = z$ at the points x and y gives

$$f(x) \ge f(z) + f'(z)(x-z), \qquad f(y) \ge f(z) + f'(z)(y-z). \tag{7.2.5}$$

Multiply the first inequality of (7.2.5) by $1-t \geq 0$ and the second by $t \geq 0$ and add. Since $(1-t)(x-z)+t(y-z) = (1-t)x+ty-z = 0$, the result is the inequality $(1-t)f(x)+tf(y) \geq f(z)$ of Eq. (7.2.3). ∎

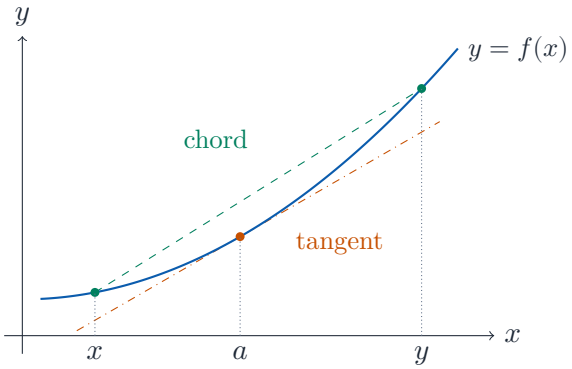


Figure 7.2.1: Convex function. Chords (dashed) lie on or above the graph (solid) and tangent lines (dash-dotted) on or below it, as Eqs. (7.2.3) and (7.2.4) state.

7.2.13 Inequalities from the Mean Value Theorem

A bound on f' over an interval becomes a bound on the change in f: if $|f'(t)| \leq M$ for every t between a and b, then

$$|f(b) - f(a)| \leq M\,|b-a|. \tag{7.2.6}$$

The Mean Value Theorem writes $f(b)-f(a) = f'(c)(b-a)$ with c between a and b and therefore with $|f'(c)| \leq M$.

Example 7.2.14 (Lipschitz bound for the sine)**.** Since $|\cos t| \leq 1$ for every t, Eq. (7.2.6) gives

$$|\sin a - \sin b| \leq |a-b| \qquad \text{for all real } a, b. \tag{7.2.7}$$

With $b = 0$ the bound (7.2.7) is $|\sin a| \leq |a|$. The same argument applied to the cosine gives $|\cos a - \cos b| \leq |a-b|$.

Worked Example 7.2.2

Prove that $e^x \geq 1+x$ for every real x, with equality only at $x = 0$.
At $x = 0$ both sides equal 1. For $x \neq 0$ the Mean Value Theorem applied to the exponential on the interval between 0 and x gives c strictly between 0 and x with

$$e^x - 1 = e^c\,x.$$

If $x > 0$, then $c > 0$ and $e^c > 1$. Hence $e^c x > x$. If $x < 0$, then $c < 0$ and $0 < e^c < 1$. Multiplying $e^c < 1$ by the negative number x gives $e^c x > x$. In both cases $e^x - 1 > x$. Geometrically, $y = 1+x$ is the tangent line at 0 to the convex function e^x. Equation (7.2.4) already gives the weak inequality.

7.2.15 L'Hôpital's rule

A limit of a quotient $f(x)/g(x)$ in which f and g both tend to 0 or both grow without bound is called an *indeterminate form* of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The limit laws of Chapter 5 do not apply because

the limit of the denominator is 0 or does not exist. The answer depends on how fast numerator and denominator move. The examples x/x, x^2/x and x/x^2 as $x \to 0^+$ give 1, 0 and ∞. The Cauchy Mean Value Theorem compares the two rates.

Theorem 7.2.16 (L'Hôpital's rule)**.** *Let f and g be differentiable on an interval (a, b) with $g'(x) \neq 0$ for every $x \in (a, b)$. Suppose that either*

(i) $\lim_{x\to a^+} f(x) = 0$ *and* $\lim_{x\to a^+} g(x) = 0$, *or*

(ii) $\lim_{x\to a^+} |g(x)| = \infty$.

If $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ *with L a real number, ∞ or $-\infty$, then*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L. \tag{7.2.8}$$

The same holds for left-hand limits at b and for limits as $x \to \infty$ or $x \to -\infty$ when f and g are differentiable on an unbounded interval. The conclusion is also true for the two-sided limit at a point a when f and g are differentiable on $(a-r, a)\cup(a, a+r)$ for some $r > 0$ with $g' \neq 0$ there and (i) or (ii) is true as $x \to a$.

Proof. Write $Q(t) = f'(t)/g'(t)$.
1. L finite.
(i) Set $f(a) = g(a) = 0$. Then f and g are continuous on $[a, x]$ for every $x \in (a, b)$ and differentiable on (a, x) with g' free of zeros there. The quotient form of Theorem 7.1.13 gives $g(x) = g(x) - g(a) \neq 0$ and a point $c_x \in (a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = Q(c_x). \tag{7.2.9}$$

Given $\varepsilon > 0$, choose $\delta \in (0, b-a]$ with $|Q(t) - L| < \varepsilon$ for $a < t < a + \delta$. For $a < x < a + \delta$ the point c_x lies in $(a, a + \delta)$ and therefore $|f(x)/g(x) - L| = |Q(c_x) - L| < \varepsilon$.
(ii) Let $\varepsilon \in (0, 1)$ and choose $\delta \in (0, b - a)$ with $|Q(t) - L| < \varepsilon$ for $a < t < a + \delta$. Fix $y = a + \delta$. For $a < x < y$, Theorem 7.1.13 on $[x, y]$ gives $g(x) \neq g(y)$ and a point $c \in (x, y)$ with $f(x) - f(y) = Q(c)(g(x) - g(y))$. Because $|g(x)| \to \infty$, there is $\delta_1 \in (0, \delta)$ with $g(x) \neq 0$ for $a < x < a + \delta_1$. Dividing by $g(x)$ for such x and rearranging gives

$$\frac{f(x)}{g(x)} - L = (Q(c) - L) + \frac{f(y) - Q(c)\, g(y)}{g(x)}. \tag{7.2.10}$$

The first term of (7.2.10) has absolute value less than ε. In the second the numerator has absolute value at most $K = |f(y)| + (|L| + 1)|g(y)|$, a number that does not depend on x. Since $|g(x)| \to \infty$, there is $\delta_2 \in (0, \delta_1]$ with $K/|g(x)| < \varepsilon$ for $a < x < a + \delta_2$. For those x, $|f(x)/g(x) - L| < 2\varepsilon$.
2. L infinite. Suppose $L = \infty$ and let $M > 0$. Under hypothesis (i) choose $\delta \in (0, b - a]$ with $Q(t) > M$ on $(a, a + \delta)$. The point c_x of (7.2.9) then gives $f(x)/g(x) = Q(c_x) > M$ for $a < x < a + \delta$. Under hypothesis (ii) choose $\delta \in (0, b - a)$ with $Q(t) > 4M$ on $(a, a + \delta)$. Fix $y = a + \delta$ as before and write

$$\frac{f(x)}{g(x)} = Q(c)\Big(1 - \frac{g(y)}{g(x)}\Big) + \frac{f(y)}{g(x)}. \tag{7.2.11}$$

For x close enough to a the ratio $g(y)/g(x)$ has absolute value below $\frac{1}{2}$ and $|f(y)/g(x)| < M$. For such x, $f(x)/g(x) > 4M \cdot \frac{1}{2} - M = M$. The value $L = -\infty$ follows on replacing f by $-f$.
3. Other limits. For a left-hand limit at b, apply the result to $F(t) = f(-t)$ and $G(t) = g(-t)$ on $(-b, -a)$ as $t \to (-b)^+$. The chain rule (Theorem 6.3.4) gives $F'(t)/G'(t) = f'(-t)/g'(-t)$. The hypotheses and the conclusion therefore transfer. A two-sided limit is the common value of the two one-sided limits (Theorem 5.2.11). For $L = \pm\infty$ the same argument applies: a δ that works on each side in Definition 5.2.21 gives one that works on both: the smaller of the two. For $x \to \infty$, with f and g differentiable on (A, ∞) and $A > 0$, set $F(t) = f(1/t)$ and $G(t) = g(1/t)$ for $0 < t < 1/A$. The chain rule gives

$$\frac{F'(t)}{G'(t)} = \frac{-t^{-2} f'(1/t)}{-t^{-2} g'(1/t)} = \frac{f'(1/t)}{g'(1/t)},$$

with limit L as $t \to 0^+$ because $1/t \to \infty$. The first part applied to F and G at 0 gives $F(t)/G(t) \to L$ as $t \to 0^+$. This is the statement $f(x)/g(x) \to L$ as $x \to \infty$. The case $x \to -\infty$ is the same with $t \to 0^-$. ∎

Common misconception

L'Hôpital's rule divides the derivative of the numerator by the derivative of the denominator. It does not differentiate the quotient. Before each application, check that the limit has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The limit $\lim_{x \to 0} \dfrac{x+1}{x+2}$ equals $\frac{1}{2}$ by continuity. The rule applied to it would give the wrong value 1. The rule is also one-directional: when $\lim f'/g'$ fails to exist, the limit of f/g may still exist. For example $(x + \sin x)/x \to 1$ as $x \to \infty$ because $|\sin x/x| \le 1/x$. The quotient of derivatives $(1 + \cos x)/1$ oscillates.

Worked Example 7.2.3

Evaluate $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$ and $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.
Both are of type $\frac{0}{0}$ with $g(x) = x^2$ and $g'(x) = 2x \neq 0$ for $x \neq 0$. One application of Theorem 7.2.16 produces a new quotient

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x},$$

provided the right side exists. The right side is again of type $\frac{0}{0}$. A second application gives $\lim_{x \to 0} e^x/2 = \frac{1}{2}$. Reading the chain from right to left each limit exists and equals $\frac{1}{2}$. In the same way

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Other indeterminate forms reduce to these two. A product $f \cdot g$ of type $0 \cdot \infty$ is the quotient $f/(1/g)$. A power f^g of type 1^∞, 0^0 or ∞^0 equals $e^{g \ln f}$. Since the exponential is continuous, it suffices to find the limit of the exponent $g \ln f$ of type $0 \cdot \infty$.

Worked Example 7.2.4

Evaluate $\lim_{x \to 0^+} x \ln x$ and $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$.

The first is of type $0 \cdot (-\infty)$. Written as $\ln x/(1/x)$ it has type $\frac{\infty}{\infty}$ with $g(x) = 1/x$ and $g'(x) = -1/x^2 \neq 0$. Hence

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

The second is of type 1^∞. Its logarithm is the quotient $x\ln(1+1/x) = \ln(1+1/x)/(1/x)$ of type $\frac{0}{0}$ as $x \to \infty$, with

$$\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty} \frac{\dfrac{-1/x^2}{1+1/x}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1.$$

By continuity of the exponential, $(1+1/x)^x = e^{x\ln(1+1/x)} \to e^1 = e$.

7.2.17 Growth ranking

Theorem 5.2.26 proved the first two limits below from the definition of the limit, the second for every real p. L'Hôpital's rule reproves them in a few lines and adds the third, a limit at 0^+.

Theorem 7.2.18 (Growth ranking by L'Hôpital's rule)**.** *For every $p > 0$,*

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0, \qquad \lim_{x\to\infty} \frac{x^p}{e^x} = 0, \qquad \lim_{x\to 0^+} x^p \ln x = 0. \tag{7.2.12}$$

Proof. The first limit is of type $\frac{\infty}{\infty}$ with $g(x) = x^p$ and $g'(x) = px^{p-1} \neq 0$ for $x > 0$. Since

$$\frac{(\ln x)'}{(x^p)'} = \frac{1/x}{px^{p-1}} = \frac{1}{px^p} \to 0 \qquad (x \to \infty),$$

Theorem 7.2.16 gives $\ln x/x^p \to 0$.

For the second, first take the quotient $u(x) = x/e^{x/p}$ of type $\frac{\infty}{\infty}$. Its derivative quotient $1/(p^{-1}e^{x/p}) = pe^{-x/p}$ tends to 0. By Theorem 7.2.16 the quotient $u(x)$ tends to 0 with $u(x) > 0$ for $x > 0$. Since $x^p/e^x = u(x)^p = e^{p\ln u(x)}$ and $p\ln u(x) \to -\infty$ as $u(x) \to 0^+$, the quotient tends to 0.

For the third, substitute $x = 1/t$. As $x \to 0^+$ the new variable t tends to ∞ and $x^p \ln x = -\ln t/t^p \to 0$ by the first limit of (7.2.12). ∎

The first two limits of (7.2.12) are the ranking Eq. (5.2.9): logarithms grow more slowly than every positive power and every power more slowly than the exponential. The comparisons of improper integrals in Section 9.8 and many convergence tests for series in Chapter 10 use these rates.

Study guide

Every result here follows one pattern: write a difference of values of f as $f'(c)$ times a difference of inputs and use what is known about f' on the whole interval. Zero derivative gives a constant. A sign of f' gives monotonicity. A bound on f' gives Eq. (7.2.6). A nondecreasing f' gives convexity.

To classify a critical point, use the sign change of f' (first derivative test) or the sign of f'' (second derivative test). When $f''(c) = 0$ the second derivative test gives no conclusion. For an indeterminate limit, confirm the form before differentiating numerator and denominator

separately. Reduce $0 \cdot \infty$ to a quotient and a power to the exponential of a product. Memorise the ranking $\ln x \ll x^p \ll e^x$.

Problems

7.2.1 Find the intervals on which $f(x) = x^4 - 2x^2$ increases and decreases together with its local extreme values. ⋆ Hint · Answer · Solution

7.2.2 Use the second derivative test to classify the critical points of $f(x) = x^3 - 6x^2 + 9x + 1$. ⋆ Hint · Answer · Solution

7.2.3 Evaluate $\displaystyle\lim_{x \to 1} \frac{\ln x}{x - 1}$. ⋆ Hint · Answer · Solution

7.2.4 Evaluate $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x^3}$. ⋆⋆ Hint · Answer · Solution

7.2.5 Evaluate $\displaystyle\lim_{x \to 0^+} x^x$. ⋆⋆ Hint · Answer · Solution

7.2.6 Evaluate $\displaystyle\lim_{x \to \infty} \frac{(\ln x)^2}{\sqrt{x}}$. ⋆⋆ Hint · Answer · Solution

7.2.7 Evaluate $\displaystyle\lim_{x \to \infty} x^3 e^{-x}$ and $\displaystyle\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$. ⋆⋆ Hint · Answer · Solution

7.2.8 Prove that $|\arctan a - \arctan b| \le |a - b|$ for all real a and b. ⋆⋆ Hint · Answer · Solution

7.2.9 Prove that $\ln(1 + x) \le x$ for every $x > -1$, with equality only at $x = 0$. ⋆⋆ Hint · Answer · Solution

7.2.10 Show that $\displaystyle\lim_{x \to \infty} \frac{x + \sin x}{x} = 1$ although the quotient of derivatives has no limit. Does this contradict Theorem 7.2.16? ⋆⋆ Hint · Answer · Solution

7.2.11 Prove that $1 + x \le e^x \le \dfrac{1}{1 - x}$ for every $x < 1$. ⋆⋆ Hint · Answer · Solution

7.2.12 Let f be differentiable on $\mathbb{R}$ with $f'(x) = f(x)$ for every x. Prove that $f(x) = f(0)\, e^x$ for every x. ⋆⋆⋆ Hint · Answer · Solution

7.3 Differentiation as a Linear Operator

Overview

Objectives: Recognise the spaces $C(I)$, $C^k(I)$ and $C^\infty(I)$ as vector spaces (Definition 7.3.2); state that differentiation is a linear map $D\colon C^1(I) \to C(I)$; identify its kernel on an interval and on a union of intervals; read the Leibniz rule as the statement that D is a derivation; write the matrix of D on polynomials of degree at most n and prove that it is nilpotent; describe the integral operator as a right inverse; find the eigenfunctions of D and the solutions of $f'' = -f$; compute $D^n(fg)$ by the general Leibniz formula; decide for which norm on $C^1([a,b])$ the map D is bounded.
Prerequisites: Vector spaces, linear maps, kernels, images and matrices (Sections 3.2 and 3.3); normed spaces and bounded linear maps (Section 3.6); the differentiation rules (Theorems 6.3.1 and 6.3.4); Corollary 7.2.2; induction (Theorem 1.4.2); binomial coefficients (Eq. (4.6.15)).
Key results: D is linear, $\ker D$ on an interval is the constants, $D(fg) = (Df)g + f(Dg)$, $D^{n+1} = 0$ on polynomials of degree at most n, $De^{\lambda x} = \lambda e^{\lambda x}$, the solutions of $f'' = -f$ are $a\cos x + b\sin x$, and $D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$; D is unbounded for the sup norm and bounded for the C^1 norm (Proposition 7.3.16).

Notation

$C(I)$	the continuous real functions on the interval I
$C^k(I)$	functions on I with k derivatives, the kth continuous
$C^\infty(I)$	functions on I with derivatives of every order
D, D^n	differentiation $f \mapsto f'$ and its nth power $f \mapsto f^{(n)}$ (this chapter only)
$\ker T$, $\operatorname{im} T$	kernel and image of a linear map T (Definitions 3.3.7 and 3.3.8)
$\lVert f \rVert_{C^1}$	the C^1 norm $\lVert f \rVert_\infty + \lVert f' \rVert_\infty$ (Eq. (7.3.7))
$\lVert f \rVert_{C^k}$	the C^k norm $\sum_{j=0}^{k} \lVert f^{(j)} \rVert_\infty$
id	the identity map $f \mapsto f$
$\mathcal{P}_n$	the real polynomials of degree at most n
J	the integral operator, a right inverse of D

The sum and constant-multiple rules of Theorem 6.3.1 state that $(f+g)' = f' + g'$ and $(cf)' = cf'$. In the language of linear algebra these two rules state that differentiation is a *linear map*. In this language Corollary 7.2.2 describes the kernel of that map, the uniqueness of antiderivatives up to a constant describes the solution set of a linear equation, and the exponential, sine and cosine appear as eigenvectors. The vocabulary is that of Chapter 3: vector spaces and subspaces (Definitions 3.2.2 and 3.2.12), linear maps with their kernels and images (Definitions 3.3.2, 3.3.7 and 3.3.8), matrices (Definition 3.3.14) and, for the size of a function, the norms of Section 3.6.

7.3.1 Vector spaces of functions

The real functions on an interval I form the vector space $\mathcal{F}(I)$ of Section 3.2 under the pointwise operations $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$. The vector spaces of this section are

subspaces of it (Definition 3.2.12). Each is a vector space in its own right by Theorem 3.2.13. The set of functions f on $[0,1]$ with $f(0)=1$ is not a subspace. It omits the zero function and is not closed under addition: the sum of two of its members takes the value 2 at 0.

Definition 7.3.2 (Spaces of differentiable functions)**.** Let I be an interval. Write $C(I)$ for the space of continuous functions $I \to \mathbb{R}$ of Section 3.2. For an integer $k \geq 1$, write $C^k(I)$ for the set of functions f on I whose derivatives $f', f'', \dots, f^{(k)}$ exist at every point of I with $f^{(k)}$ continuous on I. Set $C^0(I) = C(I)$. The set $C^\infty(I)$ consists of the functions with derivatives of every order on I. At an endpoint that belongs to I, derivatives are one-sided.

Each of these sets is a subspace of the space of all functions on I. Sums and constant multiples of continuous functions are continuous (Chapter 5). Sums and constant multiples of k times differentiable functions are k times differentiable with $(f+g)^{(k)} = f^{(k)} + g^{(k)}$ and $(cf)^{(k)} = cf^{(k)}$ by Theorem 6.3.1. A differentiable function is continuous (Theorem 6.1.11). If $f \in C^{k+1}(I)$, then $f^{(k)}$ is differentiable and hence continuous. Hence $f \in C^k(I)$. The result is the chain of subspaces

$$C^\infty(I) \subseteq \cdots \subseteq C^2(I) \subseteq C^1(I) \subseteq C(I). \tag{7.3.1}$$

Polynomials, the exponential, the sine and the cosine lie in $C^\infty(\mathbb{R})$ because each derivative is again a function of the same kind.

Example 7.3.3 (Functions on the boundaries of the chain)**.** The absolute value $|x|$ lies in $C(\mathbb{R})$ and not in $C^1(\mathbb{R})$ because it has no derivative at 0. The function $f(x) = x|x|$ equals x^2 for $x \geq 0$ and $-x^2$ for $x < 0$. Away from 0 its derivative is $2|x|$. At 0 the difference quotient $f(h)/h = |h|$ tends to 0. Hence $f'(x) = 2|x|$ everywhere. This derivative is continuous and $f \in C^1(\mathbb{R})$. Since $f' = 2|x|$ has no derivative at 0, $f \notin C^2(\mathbb{R})$.

Example 7.3.4 (Discontinuous derivative (non-example))**.** Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. At 0, $|f(h)/h| = |h \sin(1/h)| \leq |h|$ and therefore $f'(0) = 0$. For $x \neq 0$ the product and chain rules give $f'(x) = 2x \sin(1/x) - \cos(1/x)$. At the points $x_k = 1/(2k\pi)$ this equals -1. At $y_k = 1/((2k+1)\pi)$ it equals 1. Since both sequences tend to 0, f' has no limit at 0 and is not continuous there. The function is differentiable on $\mathbb{R}$ and yet does not lie in $C^1(\mathbb{R})$. Adding $x/2$ gives $g(x) = x/2 + f(x)$ with $g'(0) = \frac{1}{2} > 0$ and $g'(x_k) = -\frac{1}{2} < 0$ at points arbitrarily close to 0. By Theorem 7.2.4, applied in reverse, g is increasing on no interval around 0.

7.3.5 Kernel and image of a linear map

A linear map $T\colon V \to W$ (Definition 3.3.2) has a kernel and an image (Definitions 3.3.7 and 3.3.8), subspaces of V and of W by Proposition 3.3.9. Multiplication by a fixed function m, the map $f \mapsto mf$, is linear on any space of functions on I that it maps into itself. The map $f \mapsto f^2$ is not linear. By Theorem 3.3.10 a linear map is injective exactly when its kernel is $\{0\}$. If $Tv_0 = w$, the same theorem gives the solutions of $Tv = w$ as the vectors $v_0 + k$ with $k \in \ker T$:

$$\{v \in V : Tv = w\} = v_0 + \ker T. \tag{7.3.2}$$

Theorem 7.3.6 (Differentiation is linear)**.** *For every interval I the map $D\colon C^1(I) \to C(I)$, $Df = f'$, is linear. For each $k \geq 0$ it maps $C^{k+1}(I)$ into $C^k(I)$ and $C^\infty(I)$ into itself.*

Proof. For $f \in C^1(I)$ the derivative f' is continuous by Definition 7.3.2. The map D therefore sends $C^1(I)$ into $C(I)$. Linearity is the sum and constant-multiple rules of Theorem 6.3.1. If $f \in C^{k+1}(I)$, then f' has k derivatives with a continuous kth derivative $f^{(k+1)}$. Hence $f' \in C^k(I)$. The statement for $C^\infty(I)$ follows. ∎

Theorem 7.3.7 (Kernel of differentiation)**.** *On an interval I, $\ker D$ is the set of constant functions. Hence if $F' = g$ on I then the functions f with $f' = g$ on I are exactly $f = F + C$ with C constant.*

Proof. A constant function has derivative 0. Conversely, $f' = 0$ on I makes f constant by Corollary 7.2.2. The second statement is Corollary 7.2.3 applied to f and F. When g is continuous it is also Eq. (7.3.2) with $T = D$, $v_0 = F$ and $w = g$. ∎

Caution

Theorem 7.3.7 needs the domain to be an interval. On the domain $U = (-\infty, 0) \cup (0, \infty)$ the function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

is differentiable at every point of U with $H' = 0$ because every point of U has a neighbourhood in U on which H is constant. Yet H is not constant (Fig. 7.3.1). The proof of Corollary 7.2.2 breaks down because the segment $[-1, 1]$ joining two points of U does not lie in U. The Mean Value Theorem cannot be applied on that segment.

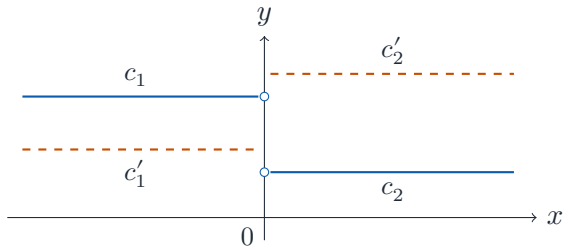


Figure 7.3.1: Two elements of $\ker D$ on $(-\infty, 0) \cup (0, \infty)$. Each is constant on each of the two intervals (solid and dashed), with the two constants chosen independently.

Theorem 7.3.8 (Kernel on a union of intervals)**.** *Let $U = I_1 \cup \cdots \cup I_m$ be a union of pairwise disjoint open intervals. A differentiable function f on U has $f' = 0$ on U exactly when there are constants $c_1, \ldots, c_m$ with $f = c_j$ on I_j for each j. Hence the kernel of differentiation $f \mapsto f'$ on the differentiable functions on U is an m-parameter family.*

Proof. If $f' = 0$ on U, then $f' = 0$ on each interval I_j and Corollary 7.2.2 makes f constant there. Conversely, a function equal to c_j on the open interval I_j is constant on a neighbourhood of each point of I_j and has derivative 0 there. ∎

7.3.9 Leibniz rule and derivations

The product rule in operator form reads

$$D(fg) = (Df)\, g + f\, (Dg). \tag{7.3.3}$$

A linear map on a space of functions that is closed under products and satisfies Eq. (7.3.3) is called a *derivation*, the notion Section 6.3 met for the operator d. The rule determines much of the behaviour of D without any further limit. Taking $f = g = 1$ gives $D(1) = 2D(1)$ and hence $D(1) = 0$. Induction then computes D on every polynomial from its single value $D(x) = 1$.

Proposition 7.3.10 (Derivations on polynomials)**.** *Let δ be a linear map from the polynomials to the polynomials that satisfies $\delta(pq) = \delta(p)\,q + p\,\delta(q)$. Then $\delta(p) = p'\,\delta(x)$ for every polynomial p. In particular $\delta = D$ when $\delta(x) = 1$.*

Proof. As above, $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$ gives $\delta(1) = 0$. We prove $\delta(x^n) = nx^{n-1}\delta(x)$ by induction on $n \geq 1$. The case $n = 1$ is immediate. If it holds for n, then

$$\delta(x^{n+1}) = \delta(x^n \cdot x) = nx^{n-1}\delta(x)\,x + x^n\delta(x) = (n+1)x^n\delta(x).$$

Hence $\delta(x^n) = (x^n)'\delta(x)$ for every $n \geq 0$. Since both sides of $\delta(p) = p'\,\delta(x)$ are linear in p and every polynomial is a linear combination of powers of x, the identity holds for all p. ∎

7.3.11 Polynomials and the matrix of differentiation

The space $\mathcal{P}_n$ of real polynomials of degree at most n has the monomial basis $1, x, \ldots, x^n$ (Definition 3.2.22 and Example 3.2.19) and dimension $n+1$ (Definition 3.2.26). The coordinate vector of $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is $(a_0, \ldots, a_n)$ (Theorem 3.2.23). Differentiation maps $\mathcal{P}_n$ into itself. The matrix of a linear map $T\colon \mathcal{P}_n \to \mathcal{P}_n$ in the monomial basis (Definition 3.3.14) has as column k (counting from 0) the coordinate vector of Tx^k. Applying T to p multiplies the coordinate vector of p by this matrix (Theorem 3.3.15). Section 3.3 found the matrix below for $n = 3$ from the coefficients alone (Worked Example 3.3.2). The theorem identifies that coefficient map with the derivative of Chapter 6.

Theorem 7.3.12 (Differentiation on polynomials is nilpotent)**.** *Let $n \geq 1$. In the basis $1, x, \ldots, x^n$ of $\mathcal{P}_n$ the matrix A of D has entries $A_{k-1,k} = k$ for $k = 1, \ldots, n$ and 0 elsewhere, with rows and columns numbered from 0. For $n = 3$,*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.3.4}$$

On $\mathcal{P}_n$ the map D satisfies $D^{n+1} = 0$ and $D^n \neq 0$. Its kernel is the constants and its image is $\mathcal{P}_{n-1}$.

Proof. Since $Dx^k = kx^{k-1}$, column k of A has the single nonzero entry k in row $k-1$. Each application of D lowers the degree of a nonconstant polynomial by one or sends a constant to 0. Hence $D^{n+1}p = 0$ for $p \in \mathcal{P}_n$. On the other hand $D^nx^n = n! \neq 0$. The kernel of D on $\mathcal{P}_n$ is the constants by Theorem 7.3.7. Every polynomial of degree at most n has a derivative of degree at most $n-1$. Conversely $q(x) = \sum_{k=0}^{n-1} b_kx^k$ is the derivative of $\sum_{k=0}^{n-1} b_kx^{k+1}/(k+1) \in \mathcal{P}_n$. The image is therefore $\mathcal{P}_{n-1}$. ∎

A linear map with some power equal to 0 is called *nilpotent*. The dimensions in Theorem 7.3.12 add up as the rank–nullity theorem (Theorem 3.3.11) requires: $\ker D$ has dimension 1 and $\operatorname{im} D = \mathcal{P}_{n-1}$ has dimension n, with sum $n + 1 = \dim \mathcal{P}_n$.

Worked Example 7.3.1

Use the matrix of D on $\mathcal{P}_3$ to differentiate $p(x) = 2 - x + 5x^3$. Compute D^3p and D^4p.

The coordinate vector of p is $(2,-1,0,5)$. Multiplying by the matrix A of Theorem 7.3.12 gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 15 \\ 0 \end{pmatrix},$$

the coordinate vector of $p'(x) = -1 + 15x^2$. Two more multiplications give $(0,30,0,0)$ and $(30,0,0,0)$, that is $D^2p = 30x$ and $D^3p = 30$. A fourth gives the zero vector: $D^4p = 0$ because $D^4 = 0$ on $\mathcal{P}_3$. Each further power of A moves the nonzero diagonal one step further from the main diagonal until A^4 is the zero matrix.

7.3.13 Image of differentiation and right inverses

Theorem 7.3.12 shows that on polynomials every $q \in \mathcal{P}_{n-1}$ is a derivative. The polynomial antiderivative used there defines a linear map J on polynomials with $J(x^k) = x^{k+1}/(k+1)$:

$$J\Big(\sum_k b_k x^k\Big) = \sum_k \frac{b_k}{k+1}\, x^{k+1}. \tag{7.3.5}$$

Proposition 7.3.14 (Right inverse on polynomials)**.** *For every polynomial p, $DJp = p$ and $JDp = p - p(0)$. In operator form, $DJ = \mathrm{id}$ and $JD = \mathrm{id} - E_0$ with $E_0 p$ the constant function of value $p(0)$, the evaluation of Definition 3.4.10 read as a constant polynomial.*

Proof. Since both identities are linear in p, it suffices to check them on the monomials. For $k \geq 0$, $DJx^k = D(x^{k+1}/(k+1)) = x^k$. For $k \geq 1$, $JDx^k = J(kx^{k-1}) = x^k = x^k - 0^k$. For $k = 0$, $JD1 = J0 = 0 = 1 - 1$. ∎

A map J with $DJ = \mathrm{id}$ is a *right inverse* of D. Since $\ker D \neq \{0\}$, D is not injective and has no left inverse: no map L satisfies $LD = \mathrm{id}$ because LD sends every constant to 0. The identity $JD = \mathrm{id} - E_0$ measures the failure by the value at 0 that JD discards. The map D has many right inverses. Replacing the base point 0 by a gives another right inverse with values that differ from those of J by constants (elements of $\ker D$).

For continuous functions on a closed interval $[a,b]$ the right inverse is the integral operator

$$(Jg)(x) = \int_a^x g(t)\,\mathrm{d}t. \tag{7.3.6}$$

Chapter 8 defines the integral and proves (Theorem 8.3.10) that $DJ = \mathrm{id}$ on $C([a,b])$ and $JD = \mathrm{id} - E_a$ on $C^1([a,b])$, with $E_a f$ the constant function $f(a)$. The first identity states that $D\colon C^1([a,b]) \to C([a,b])$ is surjective: every continuous function is a derivative. On polynomials Proposition 7.3.14 has already proved both identities with $a = 0$.

7.3.15 Differentiation and norms

Give $C([a,b])$ the sup norm of Definition 3.6.9. A function in $C^1([a,b])$ has a continuous derivative, bounded on $[a,b]$ by Theorem 5.4.6. The space $C^1([a,b])$ then has two natural norms: the sup norm $\|f\|_\infty$ and the C^1 *norm*

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty. \tag{7.3.7}$$

The C^1 norm satisfies (N1) because its first term does (Proposition 3.6.10). It satisfies (N2) and (N3) because both terms do: the second by the linearity of D and the same axioms for the sup norm. In the language of Section 3.6 the question whether D is continuous becomes the question whether D is bounded (Theorem 3.6.21). The answer depends on the norm on the domain.

Proposition 7.3.16 (Norms for differentiation)**.** *Let $a < b$ and give $C([a,b])$ the sup norm.*

1. *With the sup norm on $C^1([a,b])$ the map $D\colon C^1([a,b]) \to C([a,b])$ is unbounded and continuous at no point.*
2. *With the C^1 norm on $C^1([a,b])$ the map D is bounded with operator norm 1.*

Proof. (1) For $n \geq 1$ let $f_n(x) = ((x-a)/(b-a))^n$. On $[a,b]$ the base lies in $[0,1]$ and $f_n(b) = 1$. Hence $\|f_n\|_\infty = 1$. The derivative $Df_n(x) = \dfrac{n}{b-a}((x-a)/(b-a))^{n-1}$ has sup norm $n/(b-a)$, attained at b. A bound $\|Df\|_\infty \leq M\|f\|_\infty$ would give $n \leq M(b-a)$ for every n, against Theorem 2.4.10. By Theorem 3.6.21 the map D is not continuous at 0. Continuity at a point f_0 would give continuity at 0 because $Df - D0 = D(f+f_0) - Df_0$ and $\|(f+f_0) - f_0\| = \|f\|$.

(2) For every f, $\|Df\|_\infty = \|f'\|_\infty \leq \|f\|_{C^1}$. Hence D is bounded with $\|D\| \leq 1$. For the functions of (1),

$$\frac{\|Df_n\|_\infty}{\|f_n\|_{C^1}} = \frac{n/(b-a)}{1+n/(b-a)} = \frac{n}{n+b-a}.$$

Given $\varepsilon > 0$, this ratio exceeds $1-\varepsilon$ once $n > (b-a)/\varepsilon$. Hence $\|D\| = 1$. ∎

The functions $f_n/\sqrt{n}$ have sup norm $1/\sqrt{n}$ and derivatives of sup norm $\sqrt{n}/(b-a)$. A function uniformly close to 0 can therefore have a derivative far from 0. The C^1 norm counts a function as small only when its derivative is small as well. With this norm D is continuous. Section 3.6 met the same behaviour for $p \mapsto p'$ on polynomials. The operator norm 1 in (2) is not attained because $\|f'\|_\infty < \|f\|_\infty + \|f'\|_\infty$ for every $f \neq 0$.

The same construction gives a norm for each order of differentiability. On $C^k([a,b])$ the *C^k norm* $\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty$ satisfies (N1) through its term $j = 0$ and (N2) and (N3) term by term. The C^0 norm is the sup norm and the C^1 norm is Eq. (7.3.7). For $f \in C^{k+1}([a,b])$,

$$\|Df\|_{C^k} = \sum_{j=0}^{k} \|f^{(j+1)}\|_\infty = \sum_{j=1}^{k+1} \|f^{(j)}\|_\infty \leq \|f\|_{C^{k+1}}. \tag{7.3.8}$$

The map D from $C^{k+1}([a,b])$ with the C^{k+1} norm to $C^k([a,b])$ with the C^k norm is therefore bounded with $\|D\| \leq 1$ by (7.3.8) and continuous by Theorem 3.6.21. As a bounded linear map it is its own derivative in the sense of Definition 6.2.14: $\mathrm{d}D_f = D$ at every f. Statements about the continuity of D are therefore statements about a pair of norms as Section 3.6 found for $p \mapsto p'$ on polynomials.

7.3.17 Eigenfunctions

A nonzero vector v with $Tv = \lambda v$ for a real number λ is an *eigenvector* of T with *eigenvalue* λ. For a map on functions an eigenvector is also called an *eigenfunction*. The chain rule gives

$$De^{\lambda x} = \lambda e^{\lambda x}.$$

Every real number is therefore an eigenvalue of D on $C^\infty(\mathbb{R})$. The next theorem finds all the eigenfunctions.

Theorem 7.3.18 (Eigenfunctions of differentiation)**.** *Let $\lambda \in \mathbb{R}$ and let f be differentiable on an interval I containing 0 with $f' = \lambda f$ on I. Then $f(x) = f(0)\, e^{\lambda x}$ for every $x \in I$. The solutions of $Df = \lambda f$ form the one-dimensional space of multiples of $e^{\lambda x}$.*

Proof. By the product and chain rules,

$$\left(f(x)e^{-\lambda x}\right)' = f'(x)e^{-\lambda x} - \lambda f(x)e^{-\lambda x} = 0$$

on I. Corollary 7.2.2 makes $f(x)e^{-\lambda x}$ constant on I with the value $f(0)$ it takes at 0. For $\lambda = 1$ this is the argument given after Theorem 6.4.4. ∎

On $\mathcal{P}_n$ the only eigenvalue of D is 0: if $Dp = \lambda p$ with $\lambda \neq 0$, then $0 = D^{n+1}p = \lambda^{n+1}p$ forces $p = 0$. A nilpotent map has no nonzero eigenvalue.

The second derivative D^2 is the composite of D with itself. Its eigenfunctions include those of D because $D^2 e^{\lambda x} = \lambda^2 e^{\lambda x}$. They also include the sine and the cosine with $D^2 \sin = -\sin$ and $D^2 \cos = -\cos$. The next theorem identifies the solution set $\ker(D^2 + \mathrm{id})$ of $f'' = -f$, the equation of simple harmonic motion.

Theorem 7.3.19 (Solutions of $f'' = -f$)**.** *Let f be twice differentiable on $\mathbb{R}$ with $f'' = -f$. Then*

$$f(x) = f(0)\cos x + f'(0)\sin x \qquad \text{for every } x. \tag{7.3.9}$$

Hence $\ker(D^2 + \mathrm{id})$ consists of the functions $a\cos x + b\sin x$ with basis $\cos, \sin$.

Proof. Let $g(x) = f(x) - f(0)\cos x - f'(0)\sin x$. By linearity $g'' = -g$ with $g(0) = g'(0) = 0$. Consider the *energy*

$$E(x) = g(x)^2 + g'(x)^2. \tag{7.3.10}$$

The derivative of (7.3.10) is $E' = 2gg' + 2g'g'' = 2g'(g + g'') = 0$. By Corollary 7.2.2 the energy is constant on $\mathbb{R}$ with value $E(0) = 0$. A sum of two squares that is 0 has both terms 0. Hence $g(x) = 0$ for every x and (7.3.9) follows. Conversely every $a\cos x + b\sin x$ satisfies $f'' = -f$. If $a\cos x + b\sin x = 0$ for every x, then $x = 0$ gives $a = 0$ and $x = \pi/2$ gives $b = 0$. The two functions are therefore linearly independent and form a basis. ∎

The energy argument proves uniqueness: a solution is determined by its value and its derivative at one point. On the two-dimensional space spanned by cos and sin, differentiation acts by $D\cos = -\sin$ and $D\sin = \cos$. In the basis $\cos, \sin$ its matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.3.11}$$

The square of (7.3.11) is -1 times the identity matrix, as $D^2 = -\mathrm{id}$ there requires. Since no real number λ satisfies $\lambda^2 = -1$, D has no real eigenvector in this space.

Worked Example 7.3.2

Find the function f on $\mathbb{R}$ with $f'' = -f$, $f(0) = 2$ and $f'(0) = 3$. Write it in the form $R\sin(x + \varphi)$.

By Theorem 7.3.19 the function $f(x) = 2\cos x + 3\sin x$ is the only one that satisfies the three conditions. Expanding $R\sin(x + \varphi) = R\cos\varphi \sin x + R\sin\varphi \cos x$ and comparing coefficients in the basis $\cos, \sin$ gives $R\sin\varphi = 2$ and $R\cos\varphi = 3$. Hence $R = \sqrt{13}$ and

$\varphi = \arctan\frac{2}{3}$. Then $f(x) = \sqrt{13}\,\sin(x + \arctan\frac{2}{3})$. The energy $f^2 + f'^2$ of this solution is constant and equal to $2^2 + 3^2 = 13 = R^2$.

7.3.20 Higher derivatives and the general Leibniz formula

Powers of D give higher derivatives: $D^n f = f^{(n)}$ for $f \in C^n(I)$, with $D^0 = \mathrm{id}$. The Leibniz rule Eq. (7.3.3) extends to D^n with binomial coefficients by the same induction that proves the binomial theorem.

Theorem 7.3.21 (General Leibniz formula)**.** *If f and g are n times differentiable on an interval, then so is fg with*

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)\,(D^{n-k} g). \tag{7.3.12}$$

Proof. For $n = 0$ both sides equal fg. Suppose Eq. (7.3.12) holds for n and let f and g be $n+1$ times differentiable. Each term on the right of Eq. (7.3.12) is differentiable. By Eq. (7.3.3)

$$D^{n+1}(fg) = \sum_{k=0}^{n} \binom{n}{k} \Big((D^{k+1} f)(D^{n-k} g) + (D^k f)(D^{n+1-k} g) \Big). \tag{7.3.13}$$

Shift the index in the first group of (7.3.13) by writing $j = k+1$ and collect the terms containing $(D^j f)(D^{n+1-j} g)$:

$$D^{n+1}(fg) = \sum_{j=0}^{n+1} \left(\binom{n}{j-1} + \binom{n}{j} \right) (D^j f)(D^{n+1-j} g), \tag{7.3.14}$$

with $\binom{n}{-1} = \binom{n}{n+1} = 0$. Pascal's rule $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ follows from the factorial formula Eq. (4.6.15) on a common denominator and completes the induction (Theorem 1.4.2). ∎

Worked Example 7.3.3

Compute $D^{10}(x^2 e^{2x})$.
Take $f(x) = x^2$ and $g(x) = e^{2x}$. Since $D^k x^2 = 0$ for $k \geq 3$, only three terms of Eq. (7.3.12) remain. With $D^m e^{2x} = 2^m e^{2x}$,

$$\begin{aligned} D^{10}(x^2 e^{2x}) &= \binom{10}{0} x^2\, 2^{10} e^{2x} + \binom{10}{1} 2x\, 2^9 e^{2x} + \binom{10}{2} 2 \cdot 2^8 e^{2x} \\ &= e^{2x}(1024x^2 + 10240x + 23040). \end{aligned}$$

Common misconception

On a domain that is not an interval, Theorem 7.3.7 is false and "$+C$" does not describe every antiderivative. The function $\ln|x|$ has derivative $1/x$ on $(-\infty, 0) \cup (0, \infty)$. By Theorem 7.3.8 the general solution of $f' = 1/x$ there is $\ln|x| + c_1$ for $x < 0$ and $\ln|x| + c_2$ for $x > 0$ with two independent constants.

Study guide

Translate between the two languages. "$f' = 0$ on an interval implies f constant" is "$\ker D$ is the constants". "Antiderivatives differ by a constant" is Eq. (7.3.2). The product rule is "D is a derivation". On $\mathcal{P}_n$, D has the matrix with $1, 2, \ldots, n$ immediately above the diagonal and satisfies $D^{n+1} = 0$.

To solve $f' = \lambda f$, multiply by $e^{-\lambda x}$ and use Corollary 7.2.2. To prove uniqueness for $f'' = -f$, use the energy $g^2 + g'^2$. For a high derivative of a product in which one factor is a polynomial, use Eq. (7.3.12): the polynomial truncates the sum.

Problems

7.3.1 Write the matrix of D on $\mathcal{P}_2$ in the basis $1, x, x^2$ and compute the matrices of D^2 and D^3. ⋆ Hint · Answer · Solution

7.3.2 Find every function f on $(-\infty, 0) \cup (0, \infty)$ with $f'(x) = -1/x^2$. ⋆ Hint · Answer · Solution

7.3.3 Find f on $\mathbb{R}$ with $f'' = -f$, $f(0) = 1$ and $f'(0) = -2$. ⋆ Hint · Answer · Solution

7.3.4 Show that D maps the space spanned by e^x and xe^x into itself. Find its matrix in the basis e^x, xe^x and deduce $D^n(xe^x)$. ⋆⋆ Hint · Answer · Solution

7.3.5 Compute $D^5(x^3 e^x)$. ⋆⋆ Hint · Answer · Solution

7.3.6 With J as in Eq. (7.3.5), compute $J(3x^2 - 4x + 1)$. Verify $JDp = p - p(0)$ for $p(x) = x^2 + 5$. ⋆⋆ Hint · Answer · Solution

7.3.7 Prove that $x|x|$ lies in $C^1(\mathbb{R})$ and that $|x|^3$ lies in $C^2(\mathbb{R})$ but not in $C^3(\mathbb{R})$. ⋆⋆ Hint · Answer · Solution

7.3.8 Let $\lambda \neq \mu$. Prove that $e^{\lambda x}$ and $e^{\mu x}$ are linearly independent: if $ae^{\lambda x} + be^{\mu x} = 0$ for every x, then $a = b = 0$. ⋆⋆ Hint · Answer · Solution

7.3.9 Let f be differentiable on $\mathbb{R}$ and $n \geq 1$. Use Eq. (7.3.3) and induction to prove $D(f^n) = nf^{n-1}Df$. ⋆⋆ Hint · Answer · Solution

7.3.10 Let f be twice differentiable on $\mathbb{R}$ with $f'' = f$. Prove that $f(x) = Ae^x + Be^{-x}$ for constants A and B. Find them in terms of $f(0)$ and $f'(0)$. ⋆⋆⋆ Hint · Answer · Solution

7.3.11 On $[0, 1]$ let $g_n(x) = x^n/\sqrt{n}$. Compute $\|g_n\|_\infty$, $\|Dg_n\|_\infty$ and $\|g_n\|_{C^1}$. Conclude that the sup norm and the C^1 norm on $C^1([0, 1])$ are not equivalent (Definition 3.6.14). ⋆⋆ Hint · Answer · Solution

7.4 Taylor's Theorem

Overview

Objectives: Define the Taylor polynomial of order n and characterise it as the best polynomial approximation of order n (Theorem 7.4.6); prove Taylor's theorem with the Lagrange remainder (Theorem 7.4.8 and Eq. (7.4.6)) from the Cauchy Mean Value Theorem; bound the error of a Taylor approximation; prove that the operator e^{hD} shifts polynomials.
Prerequisites: The Cauchy Mean Value Theorem (Theorem 7.1.13); L'Hôpital's rule (Theorem 7.2.16); the operator D and the space $\mathcal{P}_n$ (Section 7.3).
Key results: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x. If $|f^{(n+1)}| \le M$, the error is at most $M|x-a|^{n+1}/(n+1)!$. On polynomials, $e^{hD}p(x) = p(x+h)$.

Notation

$P_n(x)$	the Taylor polynomial of order n of f at a
$R_n(x)$	the remainder $f(x) - P_n(x)$
$o((x-a)^n)$	a function r with $r(x)/(x-a)^n \to 0$ as $x \to a$
e^{hD}	the operator $\sum_{k\ge 0} h^k D^k/k!$, a finite sum on polynomials

The tangent line $f(a) + f'(a)(x-a)$ is the polynomial of degree at most 1 that agrees with f at a to first order: its value and its derivative at a equal those of f. A function with more derivatives can be matched to higher order by a polynomial of higher degree. Two questions arise: in what sense the matching polynomial is the best approximation and how large the error is at a given point x. Theorem 7.4.6 answers the first and Taylor's theorem (Theorem 7.4.8) the second. Taylor's theorem extends the Mean Value Theorem (the case $n = 0$) to estimates of every order.

7.4.1 Taylor polynomials

Definition 7.4.2 (Taylor polynomial)**.** Let f be n times differentiable at a. The *Taylor polynomial of order n* of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{7.4.1}$$

The *remainder* is $R_n(x) = f(x) - P_n(x)$. At $a = 0$ the polynomial is also called the *Maclaurin polynomial*.

Example 7.4.3 (Taylor polynomials of the exponential)**.** Every derivative of e^x equals e^x, with value 1 at 0. The Maclaurin polynomials are

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}. \tag{7.4.2}$$

Figure 7.4.1 shows P_1, P_2 and P_3 against e^x.

Example 7.4.4 (Function without a second derivative (non-example))**.** The function $f(x) = x|x|$ of Section 7.3 has $f(0) = f'(0) = 0$ and no second derivative at 0. Its Taylor polynomial

of order 1 at 0 is 0 and P_2 is undefined. Since $f(x)/x^2 = \pm 1$, no polynomial of degree at most 2 approximates f near 0 with an error of order $o(x^2)$, in agreement with the hypotheses of Theorem 7.4.6 below.

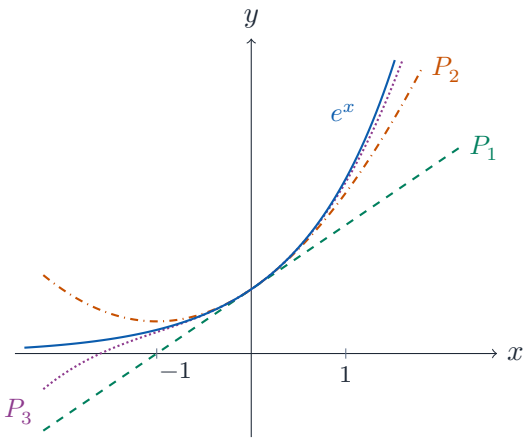


Figure 7.4.1: Maclaurin polynomials of e^x. The exponential (solid) is shown with P_1 (dashed), P_2 (dash-dotted) and P_3 (dotted). Each polynomial agrees with e^x to higher order at 0 and stays close over a wider interval than the one before.

The coefficients in Eq. (7.4.1) are forced by matching derivatives. Write a polynomial of degree at most n in powers of $x-a$ as $p(x) = \sum_{k=0}^{n} c_k (x-a)^k$. Applying D^j and evaluating at a removes every term except $k = j$. For $k > j$ the derivative $D^j (x-a)^k$ is a multiple of $(x-a)^{k-j}$ and vanishes at a. For $k < j$ it is 0. Hence

$$p^{(j)}(a) = j!\, c_j, \qquad 0 \le j \le n. \tag{7.4.3}$$

Hence P_n is the only polynomial of degree at most n whose derivatives of orders $0, 1, \dots, n$ at a equal those of f.

7.4.5 Best approximation of order n

Theorem 7.4.6 (Taylor polynomial as best approximation)**.** *Let $n \ge 1$ and let $f \in C^n(I)$ on an open interval I containing a. Then*

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0, \tag{7.4.4}$$

that is, $f(x) = P_n(x) + o((x-a)^n)$ in the notation of Definition 6.1.7 with $(x-a)^n$ in place of h. The polynomial P_n is the only polynomial of degree at most n with this property.

Proof. Let $R = f - P_n$. Then $R \in C^n(I)$ and, by Eq. (7.4.3), $R^{(k)}(a) = 0$ for $k = 0, 1, \dots, n$. For $0 \le j \le n-1$ consider the quotient

$$\frac{R^{(j)}(x)}{\dfrac{n!}{(n-j)!}(x-a)^{n-j}}, \tag{7.4.5}$$

with denominator the jth derivative of $(x-a)^n$. Both numerator and denominator tend to 0 as $x \to a$ because $R^{(j)}$ is continuous with $R^{(j)}(a) = 0$ and $n - j \geq 1$. The derivative $\frac{n!}{(n-j-1)!}(x-a)^{n-j-1}$ of the denominator is nonzero for $x \neq a$. For the last quotient, $j = n-1$,

$$\frac{R^{(n-1)}(x)}{n!\,(x-a)} = \frac{1}{n!} \cdot \frac{R^{(n-1)}(x) - R^{(n-1)}(a)}{x-a} \to \frac{R^{(n)}(a)}{n!} = 0$$

by the definition of the derivative of $R^{(n-1)}$ at a. L'Hôpital's rule (Theorem 7.2.16) applied with $j = n-2, n-3, \dots, 0$ in turn shows that each quotient (7.4.5) tends to 0. The quotient for $j = 0$ is the one in Eq. (7.4.4).

For uniqueness, let Q be a polynomial of degree at most n that also satisfies Eq. (7.4.4) and set $S = P_n - Q = \sum_{k=0}^{n} s_k(x-a)^k$. Subtracting the two limits gives $S(x)/(x-a)^n \to 0$. If some s_k were nonzero, let m be the smallest such index. Then $S(x)/(x-a)^m = s_m + s_{m+1}(x-a) + \cdots \to s_m \neq 0$. This contradicts $S(x)/(x-a)^m = (x-a)^{n-m} \cdot S(x)/(x-a)^n \to 0$. Hence $S = 0$ and $Q = P_n$. ∎

For $n = 1$ the limit Eq. (7.4.4) is Proposition 6.1.10 with $m = f'(a)$. That proposition needs only differentiability at a. In general the proof uses only the derivatives $f', \dots, f^{(n-1)}$ on I and the single value $f^{(n)}(a)$. The continuity of $f^{(n)}$ in the hypothesis keeps the statement short.

7.4.7 Taylor's theorem with Lagrange remainder

Theorem 7.4.6 says that the error is small compared with $(x-a)^n$ as $x \to a$ and gives no number for a fixed x. Taylor's theorem gives an exact formula for the error in terms of the next derivative at an unknown point.

Theorem 7.4.8 (Taylor's theorem)**.** *Let f be $n+1$ times differentiable on an open interval I containing a. For every $x \in I$ with $x \neq a$ there is a point c strictly between a and x with*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}. \tag{7.4.6}$$

The expression for R_n is the Lagrange form of the remainder.

Proof. Suppose $x > a$. The case $x < a$ is the same with every interval $[a,t]$ replaced by $[t,a]$. Let $R = f - P_n$ and $G(t) = (t-a)^{n+1}$. By Eq. (7.4.3),

$$R^{(k)}(a) = 0 \quad \text{and} \quad G^{(k)}(a) = 0 \qquad \text{for } 0 \leq k \leq n, \tag{7.4.7}$$

while $R^{(n+1)} = f^{(n+1)}$ because $P_n^{(n+1)} = 0$. Also $G^{(n+1)} = (n+1)!$. For $1 \leq k \leq n+1$, $G^{(k)}(t) = \dfrac{(n+1)!}{(n+1-k)!}(t-a)^{n+1-k}$ has no zero for $t > a$.

Set $c_0 = x$. Suppose $c_k \in (a, x]$ has been found for some $k \leq n$. The functions $R^{(k)}$ and $G^{(k)}$ are differentiable on I and hence continuous on $[a, c_k]$. The derivative $G^{(k+1)}$ has no zero on (a, c_k). By (7.4.7) the quotient form of the Cauchy Mean Value Theorem (Theorem 7.1.13) gives $c_{k+1} \in (a, c_k)$ with

$$\frac{R^{(k)}(c_k)}{G^{(k)}(c_k)} = \frac{R^{(k)}(c_k) - R^{(k)}(a)}{G^{(k)}(c_k) - G^{(k)}(a)} = \frac{R^{(k+1)}(c_{k+1})}{G^{(k+1)}(c_{k+1})}. \tag{7.4.8}$$

Chaining the $n+1$ equalities (7.4.8) from $k=0$ to $k=n$ gives

$$\frac{R(x)}{(x-a)^{n+1}} = \frac{R^{(n+1)}(c_{n+1})}{G^{(n+1)}(c_{n+1})} = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}, \tag{7.4.9}$$

with $a < c_{n+1} < x$. Taking $c = c_{n+1}$ and multiplying (7.4.9) by $(x-a)^{n+1}$ gives Eq. (7.4.6). ∎

For $n=0$, Eq. (7.4.6) reads $f(x) = f(a) + f'(c)(x-a)$, the Mean Value Theorem. Like the Mean Value Theorem, Taylor's theorem does not locate c. Applications replace $f^{(n+1)}(c)$ by a bound valid between a and x.

For $f \in C^{n+1}(I)$ the remainder also has an *integral form*

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t.$$

Problem 9.2.5 in Chapter 9 proves it by integration by parts and derives the bound of Corollary 7.4.9 from it. The Lagrange form needs only $n+1$ derivatives and no integral. It suffices for every use of the remainder in this book.

Corollary 7.4.9 (Taylor remainder bound)**.** *If in addition $|f^{(n+1)}(t)| \le M$ for every t between a and x, then*

$$|f(x) - P_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!}. \tag{7.4.10}$$

Proof. Take absolute values in Eq. (7.4.6) and use $|f^{(n+1)}(c)| \le M$. ∎

Caution

The point c in Eq. (7.4.6) depends on x and on n. A bound M must hold on the whole interval between a and x. For a decreasing derivative such as $t^{-3/2}$ on $[4, 4.1]$ the maximum sits at the left endpoint. For e^t on $[0, x]$ it sits at the right endpoint x.

Worked Example 7.4.1

Approximate $\sqrt{4.1}$ by the Taylor polynomials of orders 1 and 2 of $\sqrt{x}$ at $a=4$. Bound each error.

With $f(x) = x^{1/2}$,

$$f'(x) = \tfrac{1}{2}x^{-1/2}, \quad f''(x) = -\tfrac{1}{4}x^{-3/2}, \quad f'''(x) = \tfrac{3}{8}x^{-5/2},$$

so $f(4)=2$, $f'(4)=\tfrac{1}{4}$ and $f''(4) = -\tfrac{1}{32}$. At $x=4.1$, with $x-a=0.1$,

$$P_1(4.1) = 2 + \tfrac{1}{4}(0.1) = 2.025, \qquad P_2(4.1) = 2.025 - \tfrac{1}{64}(0.01) = 2.02484375.$$

For P_1 the remainder is $R_1 = \tfrac{1}{2}f''(c)(0.1)^2$ with $4 < c < 4.1$. Since $|f''(t)| = \tfrac{1}{4}t^{-3/2} \le \tfrac{1}{4}\cdot 4^{-3/2} = \tfrac{1}{32}$ on $[4, 4.1]$, Corollary 7.4.9 gives $|R_1| \le \tfrac{1}{32}\cdot\tfrac{0.01}{2} = \tfrac{1}{6400} \approx 1.6\times 10^{-4}$. Because $f'' < 0$, the remainder R_1 is negative and 2.025 is an overestimate. For P_2, $|f'''(t)| \le \tfrac{3}{8}\cdot 4^{-5/2} = \tfrac{3}{256}$ and

$$|R_2| \le \frac{3}{256}\cdot\frac{(0.1)^3}{6} = \frac{1}{512000} \approx 1.95\times 10^{-6}.$$

The value is $\sqrt{4.1} = 2.0248456\ldots$ with an actual error of P_2 of about 1.92×10^{-6}, within the bound.

Worked Example 7.4.2

Prove that $e < 3$. Find n such that $P_n(1) = \sum_{k=0}^{n} 1/k!$ approximates e with error less than 10^{-6}.
For $f(x) = e^x$ at $a = 0$ and $x = 1$, Eq. (7.4.6) gives

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^c}{(n+1)!}, \qquad 0 < c < 1.$$

Since $1 < e^c < e$, the remainder lies strictly between $1/(n+1)!$ and $e/(n+1)!$. With $n = 2$ this gives $e < 2.5 + e/6$, that is $\frac{5}{6}e < 2.5$ and $e < 3$. For general n the error therefore satisfies $0 < e - P_n(1) < 3/(n+1)!$. Since $9! = 362880$ and $10! = 3628800$, the bound $3/(n+1)!$ first drops below 10^{-6} at $n = 9$ with the value $3/10! \approx 8.3 \times 10^{-7}$. Hence

$$e \approx \sum_{k=0}^{9} \frac{1}{k!} = 2.7182815\ldots,$$

with error below 10^{-6}. The true error is about 3.0×10^{-7}.

Worked Example 7.4.3

Estimate $\sin 0.1$ by $0.1 - \frac{(0.1)^3}{6}$ and bound the error.
The Maclaurin polynomials of $\sin x$ have only odd powers because the even derivatives $\pm\sin x$ vanish at 0. Hence $P_3(x) = P_4(x) = x - x^3/6$. Using order 4 in Eq. (7.4.6) gives the smaller remainder $R_4(x) = \sin^{(5)}(c)\, x^5/5! = \cos(c)\, x^5/120$ with $|\cos c| \le 1$. At $x = 0.1$,

$$\sin 0.1 \approx 0.1 - \frac{0.001}{6} = 0.0998333\ldots, \qquad |R_4(0.1)| \le \frac{(0.1)^5}{120} \approx 8.3 \times 10^{-8}.$$

Since $0 < c < 0.1$ makes $\cos c > 0$, the error is positive and the estimate is slightly low. The value is $\sin 0.1 = 0.09983341\ldots$.

7.4.10 Exponential of the differentiation operator

Apply Taylor's theorem to a polynomial p of degree at most n with centre x at the point $x + h$. Since $p^{(n+1)} = 0$, the remainder vanishes and

$$p(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!}\, p^{(k)}(x) = \Big(\sum_{k=0}^{n} \frac{h^k D^k}{k!}\, p\Big)(x). \tag{7.4.11}$$

The operator in parentheses is the exponential series $\sum h^k D^k/k!$ with hD in place of a number. On $\mathcal{P}_n$ the series stops because $D^{n+1} = 0$ (Theorem 7.3.12).

Definition 7.4.11 (Exponential of hD on polynomials)**.** For a real number h and a polynomial p define

$$e^{hD} p = \sum_{k \ge 0} \frac{h^k}{k!} D^k p, \tag{7.4.12}$$

a finite sum because $D^k p = 0$ once k exceeds the degree of p. The shift operator T_h is defined by $(T_h p)(x) = p(x+h)$.

Theorem 7.4.12 (Shift theorem for polynomials)**.** *On polynomials, $e^{hD} = T_h$ for every real h. Consequently $e^{hD}e^{sD} = e^{(h+s)D}$ for all real h and s.*

Proof. The identity $e^{hD}p = T_h p$ is Eq. (7.4.11). It can also be checked on monomials without Taylor's theorem. Since $D^k x^m = \dfrac{m!}{(m-k)!}x^{m-k}$ for $k \le m$ and 0 for $k > m$,

$$e^{hD}x^m = \sum_{k=0}^{m} \frac{h^k}{k!} \cdot \frac{m!}{(m-k)!}x^{m-k} = \sum_{k=0}^{m} \binom{m}{k} h^k x^{m-k} = (x+h)^m$$

by the binomial theorem. Since both sides are linear in p, they agree on every polynomial. Finally $T_h T_s p(x) = p(x+s+h) = T_{h+s}p(x)$ and therefore $e^{hD}e^{sD} = e^{(h+s)D}$. ∎

For $f \in C^\infty(I)$ with $f^{(k)}(x) \neq 0$ for infinitely many k the sum $\sum_{k\ge0} h^k f^{(k)}(x)/k!$ is an infinite series. It is the Taylor series of f at x, and Definition 7.4.11 does not apply to it. Whether it converges to $f(x+h)$ is a question about power series. Chapter 11 proves convergence of the Maclaurin series of the exponential, the sine and the cosine with the remainder estimate of Theorem 7.4.8 as the main tool. The same estimate at any centre x gives $\sum_{k\ge0} h^k f^{(k)}(x)/k! = f(x+h)$ for these three functions. For the eigenfunctions of Theorem 7.3.18 each $f^{(k)}$ is $\lambda^k f$ and the series reads

$$\sum_{k\ge0} \frac{h^k}{k!}\left(e^{\lambda x}\right)^{(k)} = \sum_{k\ge0} \frac{(h\lambda)^k}{k!}e^{\lambda x} = e^{\lambda h}e^{\lambda x} = e^{\lambda(x+h)}, \tag{7.4.13}$$

in which the middle equality is the power series of the exponential. Section 11.5 extends e^{hD} to the sums of power series and proves the shift identity for them (Theorem 11.5.7). Problem 7.4.11 gives a function in $C^\infty(\mathbb{R})$ whose Taylor series at 0 sums to 0 at h while $f(h) \neq 0$ for every $h \neq 0$.

Common misconception

Taylor's theorem gives an exact formula for the error at each n. It does not say that $P_n(x)$ gets closer to $f(x)$ as n grows for a fixed x. Whether that error tends to 0 depends on how fast $f^{(n+1)}$ grows compared with $(n+1)!/|x-a|^{n+1}$. Chapter 11 takes up that question. A small error near a also does not imply a small error far from a: Fig. 7.4.1 shows each polynomial leaving e^x once $|x|$ is large.

Study guide

To build P_n, compute $f(a), f'(a), \dots, f^{(n)}(a)$ and divide the kth by $k!$. To bound the error at x, write Eq. (7.4.6), bound $|f^{(n+1)}|$ on the whole interval between a and x and apply Corollary 7.4.9. When the next Taylor coefficient is 0, as for the sine and the cosine at 0, use the higher remainder for a sharper bound. The sign of $f^{(n+1)}$ on that interval gives the sign of the error.

For a limit as $x \to a$, Theorem 7.4.6 replaces f by P_n plus a term $o((x-a)^n)$. The substitution often settles in one line a limit that needs several applications of L'Hôpital's rule. On polynomials the shift $p(x) \mapsto p(x+h)$ is the finite sum e^{hD}.

Problems

7.4.1 Find the Maclaurin polynomial of order 3 of $\ln(1+x)$. ⋆ Hint · Answer · Solution

7.4.2 Find the Maclaurin polynomial of order 3 of $\tan x$. ⋆ Hint · Answer · Solution

7.4.3 Use the Taylor polynomial of order 2 of $\sqrt{x}$ at 9 to approximate $\sqrt{9.2}$. Bound the error. ⋆⋆ Hint · Answer · Solution

7.4.4 How large must n be for the Maclaurin polynomial P_n of e^x to give $e^{0.5}$ with an error below 10^{-4}? Use $e^{0.5} < 2$. ⋆⋆ Hint · Answer · Solution

7.4.5 Prove that $|\sin x - x| \leq \dfrac{|x|^3}{6}$ for every real x. ⋆⋆ Hint · Answer · Solution

7.4.6 Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$ using Theorem 7.4.6. ⋆⋆ Hint · Answer · Solution

7.4.7 Prove that $\cos x \geq 1 - \dfrac{x^2}{2}$ for every real x. ⋆⋆ Hint · Answer · Solution

7.4.8 Write the matrix of e^{hD} on $\mathcal{P}_2$ in the basis $1, x, x^2$ and check that it sends x^2 to $(x+h)^2$. ⋆⋆ Hint · Answer · Solution

7.4.9 Let f be twice differentiable on an open interval containing a with f'' continuous at a. Prove that

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

⋆⋆⋆ Hint · Answer · Solution

7.4.10 Prove that e is irrational. ⋆⋆⋆ Hint · Answer · Solution

7.4.11 Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f \in C^\infty(\mathbb{R})$ with $f^{(k)}(0) = 0$ for every k. Conclude that every Taylor polynomial of f at 0 is 0 although $f(h) > 0$ for $h \neq 0$. ⋆⋆⋆ Hint · Answer · Solution

Part III

Integral Calculus

Chapter 8

Integral

Differential calculus starts from a function F and produces its differential $\mathrm{d}F = F'(x)\,\mathrm{d}x$ (Eq. (6.2.6)). Integral calculus runs the other way. Given a 1-form $f(x)\,\mathrm{d}x$, two questions arise: which functions F have $\mathrm{d}F = f(x)\,\mathrm{d}x$, and what number results from adding the values of $f(x)\,\mathrm{d}x$ along an oriented interval. The first question leads to antiderivatives and the second to the definite integral. The Fundamental Theorem of Calculus states that the two answers determine each other.

This chapter develops the integral one theorem at a time. Antiderivatives come first, with the Mean Value Theorem of Chapter 7 as the source of the constant of integration. The definite integral follows as a limit of Riemann sums. Uniform continuity (Theorem 5.5.10) proves that every continuous function on a closed bounded interval is integrable. With the sup norm of Definition 3.6.9 the integral is a bounded linear functional on the continuous functions (Theorem 8.2.17). After the Fundamental Theorem reduces integration to antidifferentiation, substitution integrates the chain rule for differentials and the last section measures the length of a graph. That last section also completes the radian measure of Section 4.5 by computing arc length on the unit circle from an algebraic integrand (Theorem 8.6.6). The chapter uses the derivatives of Chapter 6 and the Mean Value Theorem of Chapter 7.

8.1 Antiderivatives

Overview

Objectives: Reverse differentiation: given f, find F with $F' = f$, that is, with $\mathrm{d}F = f(x)\,\mathrm{d}x$ (Definition 8.1.4 and Eq. (6.2.6)). Prove that two antiderivatives on an interval differ by a constant (Theorem 8.1.3). Apply the power rule for antiderivatives, include the constant of integration, and determine that constant from an initial condition.

Prerequisites: The derivative rules for powers, sin, cos, $\sec^2$, e^x and arctan (Theorem 6.3.1 and Section 6.4); differentials and 1-forms (Section 6.2); the Mean Value Theorem and its corollary on functions with zero derivative (Theorem 7.1.9 and Corollary 7.2.2).

Key results: $\displaystyle\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$. Any two antiderivatives of f on an interval differ by a constant, with one independent constant on each interval of the domain.

Notation

$F'(x) = f(x)$	F is an antiderivative of f (Definition 8.1.1)
$\int f(x)\,\mathrm{d}x$	all antiderivatives of the 1-form $f(x)\,\mathrm{d}x$ (Definition 8.1.4)
$+\,C$	constant of integration, one per interval

A physical law often specifies a rate of change and leaves the quantity itself to be found. A falling stone has velocity $v(t) = -9.8\,t$ metres per second, and its height $s(t)$ is a function with $s'(t) = v(t)$. When a population grows at a measured rate, the population is a function with that derivative. Each case asks for a function with a prescribed derivative.

Definition 8.1.1 (Antiderivative)**.** A function F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for every $x \in I$.

Example 8.1.2 (Antiderivative and non-antiderivative)**.** On $I = \mathbb{R}$ the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ because $F'(x) = x^2$ for every x. The function $G(x) = x^3$ is not an antiderivative of x^2 because $G'(x) = 3x^2$ differs from x^2 at every $x \neq 0$. Since $|x|$ has no derivative at 0 it is not an antiderivative on $\mathbb{R}$ of any function. On the interval $(0, \infty)$ it agrees with x and is an antiderivative of the constant 1.

Antidifferentiation has no unique answer. Adding any constant to an antiderivative gives another because the derivative of a constant is 0. On an interval nothing else is possible.

Theorem 8.1.3 (General antiderivative)**.** *If F is an antiderivative of f on an interval I, then every antiderivative of f on I has the form*

$$F(x) + C, \qquad C \in \mathbb{R}, \tag{8.1.1}$$

and conversely every such $F + C$ is an antiderivative of f on I.

Proof. A second antiderivative G has $G' = F'$ on I and is $F + C$ by Corollary 7.2.3 (Theorem 7.3.7 in operator language). Conversely $(F + C)' = F' + 0 = f$ on I for every constant C. ∎

One constant per interval

On a union of finitely many disjoint open intervals $I_1, \dots, I_m$ the kernel of differentiation has one constant for each interval (Theorem 7.3.8), and the general antiderivative is $F + C_i$ on I_i with a possibly different constant C_i on each I_i. These constants are independent of one another and are neither added together nor merged into a single C. A single "$+C$" is correct on one interval only. For example $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$ on any interval that avoids 0. On all of $\mathbb{R} \setminus \{0\}$ the antiderivatives are $\ln(-x) + C_1$ for $x < 0$ and $\ln x + C_2$ for $x > 0$ with independent C_1 and C_2.

An antiderivative is a statement about 1-forms. By Eq. (6.2.6) the condition $F' = f$ says $\mathrm{d}F = f(x)\,\mathrm{d}x$. The integral sign defined next acts on the 1-form $f(x)\,\mathrm{d}x$ as a whole.

Definition 8.1.4 (Indefinite integral)**.** An *antiderivative* of the 1-form $f(x)\,\mathrm{d}x$ on an interval I is a function F with $\mathrm{d}F = f(x)\,\mathrm{d}x$ on I. By Eq. (6.2.6) and the uniqueness in Theorem 6.2.8 this condition means $F' = f$. The collection of all antiderivatives of $f(x)\,\mathrm{d}x$ on I is its *indefinite integral*, written $\int f(x)\,\mathrm{d}x$. When F is one antiderivative, Theorem 8.1.3 gives

$$\int f(x)\,\mathrm{d}x = F(x) + C, \tag{8.1.2}$$

where the right side names the family of functions $F + C$ with C ranging over $\mathbb{R}$.

Every differentiation formula read from right to left is an antiderivative. Differentiating the right-hand side of each entry below returns the integrand. Theorem 8.1.3 then supplies every other antiderivative.

Proposition 8.1.5 (Basic antiderivatives)**.** *On every interval on which the integrand is defined,*

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1), \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C,$$
$$\int \cos x\,\mathrm{d}x = \sin x + C, \qquad \int \sin x\,\mathrm{d}x = -\cos x + C, \qquad \int \sec^2 x\,\mathrm{d}x = \tan x + C, \tag{8.1.3}$$
$$\int e^{kx}\,\mathrm{d}x = \frac{1}{k}\,e^{kx} + C \quad (k \neq 0), \qquad \int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C.$$

In the power rule the interval lies in $(0,\infty)$ when n is not an integer and avoids 0 when $n \leq -2$ is an integer. For an integer $n \geq 0$ it is arbitrary.

Proof. Each right-hand side is differentiable on the stated intervals with derivative equal to the integrand by Theorems 6.3.1, 6.4.4 to 6.4.6, 6.4.8 and 6.4.10. For the power, $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^{n+1}}{n+1} = \dfrac{(n+1)x^n}{n+1} = x^n$ for $n+1 \neq 0$. The power rule of Theorem 6.3.1 gives this for integer n on the stated intervals and Theorem 6.4.6 gives it for real n on $(0,\infty)$. For the logarithm, $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln x = \dfrac{1}{x}$ for $x > 0$ while the chain rule gives $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \dfrac{-1}{-x} = \dfrac{1}{x}$ for $x < 0$. Each formula then follows from Theorem 8.1.3. ∎

The power rule reads "raise the power by one and divide by the new power". The one exception $n = -1$ is the exponent at which the new power would be 0. The logarithm fills that gap. The rule applies to every real exponent, for instance $\int x^{1/2}\,\mathrm{d}x = \frac{2}{3}x^{3/2} + C$ on $(0,\infty)$.

Proposition 8.1.6 (Linearity of antiderivatives)**.** *If F and G are antiderivatives of f and g on an interval I and k is a constant, then $kF + G$ is an antiderivative of $kf + g$ on I. For $k \neq 0$,*

$$\int \big(f(x) + g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x, \qquad \int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x. \tag{8.1.4}$$

Proof. By linearity of d (Eq. (6.3.6)), $\mathrm{d}(kF + G) = k\,\mathrm{d}F + \mathrm{d}G = \big(kf(x) + g(x)\big)\,\mathrm{d}x$. The two identities of families follow because $F + G + C$ and $kF + C$ run over all antiderivatives of $f + g$ and of kf as C runs over $\mathbb{R}$, by Theorem 8.1.3. For $kF + C$ with $k \neq 0$ write $C = kC'$ with $C' = C/k$. ∎

A problem that prescribes F' together with one value $F(x_0) = y_0$ is an *initial value problem*. By Theorem 8.1.3 its solutions on an interval containing x_0 are the functions $F + C$ whose constant satisfies $F(x_0) + C = y_0$. This determines C uniquely. So an initial value problem on an interval has exactly one solution.

Worked Example 8.1.1

Evaluate $\displaystyle\int \Big(3x^2 - \frac{4}{x^2} + 2\cos x\Big)\,\mathrm{d}x$ on $(0,\infty)$.

Write $\frac{4}{x^2} = 4x^{-2}$ and apply Proposition 8.1.6 with the entries of Proposition 8.1.5 term by term:

$$\int \big(3x^2 - 4x^{-2} + 2\cos x\big)\,\mathrm{d}x = 3\cdot\frac{x^3}{3} - 4\cdot\frac{x^{-1}}{-1} + 2\sin x + C = x^3 + \frac{4}{x} + 2\sin x + C.$$

Differentiating the result gives the integrand $3x^2 - 4x^{-2} + 2\cos x$. A single constant suffices because the domain $(0, \infty)$ is one interval.

Worked Example 8.1.2

Find F with $F'(x) = 6x - \sin x$ for all x and $F(0) = 3$.
By Propositions 8.1.5 and 8.1.6 one antiderivative of $6x - \sin x$ is $3x^2 + \cos x$. By Theorem 8.1.3 every solution on $\mathbb{R}$ has the form $F(x) = 3x^2 + \cos x + C$. The initial condition gives $F(0) = 0 + 1 + C = 3$ and hence $C = 2$. The unique solution is

$$F(x) = 3x^2 + \cos x + 2.$$

Worked Example 8.1.3

Find every function F on $\mathbb{R} \setminus \{0\}$ with $F'(x) = \dfrac{1}{x^2}$, $F(-1) = 0$ and $F(1) = 2$.
The domain is the union of the two intervals $(-\infty, 0)$ and $(0, \infty)$. On each of them $-\frac{1}{x}$ is an antiderivative of x^{-2}. By Theorem 8.1.3 applied on each interval separately,

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1, & x < 0, \\ -\dfrac{1}{x} + C_2, & x > 0. \end{cases}$$

The condition $F(-1) = 1 + C_1 = 0$ gives $C_1 = -1$ and the condition $F(1) = -1 + C_2 = 2$ gives $C_2 = 3$. Two conditions are needed because there are two independent constants. A single formula $-\dfrac{1}{x} + C$ would have to satisfy $C = -1$ and $C = 3$ at once and cannot meet both conditions.

Common misconception

An antiderivative of a product is not the product of antiderivatives. For example $\int x \cdot x \, dx = \frac{1}{3}x^3 + C$, whereas the product $\frac{1}{2}x^2 \cdot \frac{1}{2}x^2 = \frac{1}{4}x^4$ has derivative x^3. Differentiating a proposed answer detects this error at once.

Caution

The power rule does not apply to $n = -1$: the formula $\dfrac{x^0}{0}$ has no meaning. The antiderivative of x^{-1} is $\ln|x|$, with one constant on each side of 0.

Study guide

An antiderivative of f on an interval is a function F with $F' = f$, equivalently $dF = f(x)\,dx$. The indefinite integral $\int f(x)\,dx$ is the family of all of them. On an interval it is $F(x) + C$ by Theorem 8.1.3. The proof uses only the corollary that zero derivative on an interval forces a constant.
To find an antiderivative split the integrand by linearity, rewrite roots and reciprocals as powers, and read each term off the table of Proposition 8.1.5. Check every answer by differentiating it. In an initial value problem find the general antiderivative first and then

fix the constant from the given value. On a domain made of several intervals use one constant per interval.

Problems

8.1.1 Evaluate $\displaystyle\int (2x^3 - 5x + 7)\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.1.2 Evaluate $\displaystyle\int \frac{1}{1+4x^2}\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.1.3 Solve the initial value problem: find F with $F'(x) = 3x^2$ and $F(1) = -1$. ⋆ Hint · Answer · Solution

8.1.4 Let F and G be differentiable on an interval I with $F' = G'$ on I, and suppose $F(x_0) = G(x_0)$ for one point $x_0 \in I$. Prove that $F = G$ on I. ⋆⋆ Hint · Answer · Solution

8.1.5 Find the function F on $\mathbb{R} \setminus \{0\}$ with $F'(x) = \dfrac{2}{x}$, $F(1) = 0$ and $F(-1) = 1$. ⋆⋆ Hint · Answer · Solution

8.1.6 Prove that $F(x) = \tfrac{1}{2}x|x|$ is an antiderivative of $f(x) = |x|$ on $\mathbb{R}$. ⋆⋆ Hint · Answer · Solution

8.1.7 Let $\operatorname{sgn} x$ be 1 for $x > 0$, 0 for $x = 0$ and -1 for $x < 0$. Prove that no function F on $(-1, 1)$ satisfies $F'(x) = \operatorname{sgn} x$ for every $x \in (-1, 1)$. ⋆⋆ Hint · Answer · Solution

8.1.8 Show that $F(x) = \arctan x$ and $G(x) = -\arctan(1/x)$ both have derivative $\dfrac{1}{1+x^2}$ at every $x \neq 0$. Find $G - F$ on $(0, \infty)$ and on $(-\infty, 0)$ and explain why the answer is consistent with Theorem 8.1.3. ⋆⋆ Hint · Answer · Solution

8.1.9 Let $F(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $F(0) = 0$. Prove that F is differentiable on $\mathbb{R}$, compute $f = F'$ and prove that f is not continuous at 0. Conclude that a function with an antiderivative on an interval need not be continuous. ⋆⋆ Hint · Answer · Solution

8.2 Definite Integral

Overview

Objectives: Read the definite integral as the integral of the 1-form $f(x)\,\mathrm{d}x$ over an oriented interval (Definition 8.2.2), as a limit of Riemann sums (Definition 8.2.4) and as signed area. Prove that continuous functions are integrable (Theorem 8.2.13) with upper and lower sums. Evaluate integrals using linearity, the sign-reversal and additivity rules (Theorem 8.2.15), the basic power formulas and geometry. Prove the sup-norm estimate $\left|\int_a^b f\right| \le (b-a)\|f\|_\infty$ and read the integral as a bounded linear functional on $C([a,b])$ (Theorem 8.2.17). Compute an average value.

Prerequisites: Completeness and the supremum (Assumption 2.4.5 and Definition 2.4.3); the Extreme Value Theorem (Theorem 5.4.6); uniform continuity and the Heine–Cantor theorem (Definition 5.5.1 and Theorem 5.5.10); 1-forms and Eq. (6.2.5); the sup norm and the operator norm (Definitions 3.6.9 and 3.6.19); the sums $\sum k$ and $\sum k^2$ (Section 4.6).

Key results: $\displaystyle\int_a^b x\,\mathrm{d}x = \frac{b^2-a^2}{2}$ and $\displaystyle\int_a^b x^2\,\mathrm{d}x = \frac{b^3-a^3}{3}$. Every continuous function on $[a,b]$ is integrable. Reversing the orientation reverses the sign. The integral is a linear functional of norm $b-a$ on $(C([a,b]),\|\cdot\|_\infty)$. The average value of f on $[a,b]$ is $\dfrac{1}{b-a}\int_a^b f\,\mathrm{d}x$.

Notation

P, $\|P\|$	a partition $a = x_0 < \cdots < x_n = b$ and its mesh $\max_k \Delta x_k$
Δx_k, c_k	displacement of and sample point in the kth step
$\int_{[a,b]} \omega$	integral of the 1-form ω over $[a,b]$ oriented from a to b
$\int_a^b f(x)\,\mathrm{d}x$	definite integral of $f(x)\,\mathrm{d}x$ (Definition 8.2.4)
$U(f,P)$, $L(f,P)$	upper and lower sums of f on P (Definition 8.2.8)
$\|f\|_\infty$	sup norm $\max_{a\le x\le b}\|f(x)\|$ of $f \in C([a,b])$ (Definition 3.6.9)
$\mathrm{avg}(f)$	average value $\frac{1}{b-a}\int_a^b f\,\mathrm{d}x$

A car whose speedometer reads a constant 60 kilometres per hour for two hours travels 120 kilometres. When the speed varies the trip can be cut into short stretches of time on each of which the speed is nearly constant. Speed times duration on each stretch, summed over the stretches, approximates the distance with an error that decreases as the stretches shorten. The same sum of height times width approximates the area under a graph. The definite integral is the exact value that these sums approach.

8.2.1 Integral of a 1-form

A *partition* P of $[a,b]$ with $a < b$ is a finite list of points $a = x_0 < x_1 < \cdots < x_n = b$. The interval $[a,b]$ is *oriented* from a to b when the steps $[x_{k-1}, x_k]$ are traversed in this order. The step from x_{k-1} to x_k is then the displacement $\Delta x_k = x_k - x_{k-1} > 0$. The *mesh* $\|P\|$ is the largest of the Δx_k. A *tagged* partition also chooses a sample point $c_k \in [x_{k-1}, x_k]$ in each step. A 1-form assigns a number to each step by acting on its displacement at the sample point. The integral is the limit of their sum.

Definition 8.2.2 (Integral of a 1-form)**.** Let ω be a 1-form on $[a,b]$ with $a<b$. A number J is the *integral of ω over $[a,b]$ oriented from a to b*, written

$$\int_{[a,b]} \omega = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \omega_{c_k}(\Delta x_k), \tag{8.2.1}$$

if for every $\varepsilon>0$ there is a $\delta>0$ such that $\left|\sum_k \omega_{c_k}(\Delta x_k) - J\right| < \varepsilon$ for every tagged partition with $\|P\|<\delta$. Traversing $[a,b]$ from b to a replaces every displacement Δx_k by $-\Delta x_k$. Because each ω_{c_k} is linear, every term changes sign. The integral over $[a,b]$ oriented from b to a is therefore $-J$.

At most one number J meets the condition. If J and J' both do, a tagged partition with mesh below both values of δ for a given ε has a sum within ε of each, whence $|J-J'|<2\varepsilon$ for every $\varepsilon>0$ and $J=J'$.

For $\omega = f(x)\,\mathrm{d}x$, Eq. (6.2.5) gives $\omega_{c_k}(\Delta x_k) = f(c_k)\,\Delta x_k$. The value of the 1-form on one step is the signed area of a rectangle of height $f(c_k)$ over that step (Fig. 8.2.1). Since every 1-form on $[a,b]$ is $f(x)\,\mathrm{d}x$ for exactly one function f (Theorem 6.2.8), the integral of 1-forms is the integral of functions defined next.

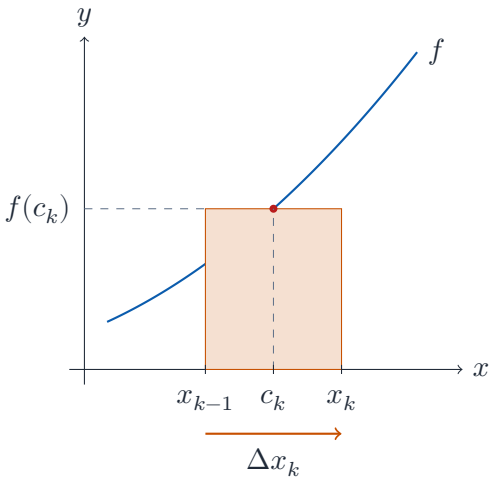


Figure 8.2.1: Value of the 1-form $f(x)\,\mathrm{d}x$ at the sample point c_k on the displacement Δx_k from x_{k-1} to x_k: the number $f(c_k)\,\Delta x_k$, the signed area of the rectangle. The integral Eq. (8.2.1) is the limit of the sum of these values over a partition.

8.2.3 Riemann sums and the definite integral

Definition 8.2.4 (Definite integral)**.** Let f be defined on $[a,b]$ with $a<b$. For a tagged partition of $[a,b]$ the *Riemann sum* of the 1-form $f(x)\,\mathrm{d}x$ adds its values $\big(f(x)\,\mathrm{d}x\big)_{c_k}(\Delta x_k)$ from Eq. (6.2.5):

$$S = \sum_{k=1}^{n} f(c_k)\,\Delta x_k. \tag{8.2.2}$$

The function f is *integrable* on $[a,b]$ when the integral of $f(x)\,\mathrm{d}x$ over $[a,b]$ oriented from a to b exists (Definition 8.2.2). Its value is the *definite integral* $\int_a^b f(x)\,\mathrm{d}x$. In terms of the sums, $J=\int_a^b f(x)\,\mathrm{d}x$ when for every $\varepsilon>0$ there is a $\delta>0$ such that every Riemann sum Eq. (8.2.2) with mesh $<\delta$ satisfies $|S-J|<\varepsilon$ (Fig. 8.2.2). The integral $\int_b^a f(x)\,\mathrm{d}x$ is the integral over the

same interval traversed from b to a. The integral $\int_a^a f(x)\,dx$ is 0 because a single point has only the zero displacement.

The letter x in $\int_a^b f(x)\,dx$ names the coordinate on the interval. Any other letter names the same coordinate and gives the same number: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

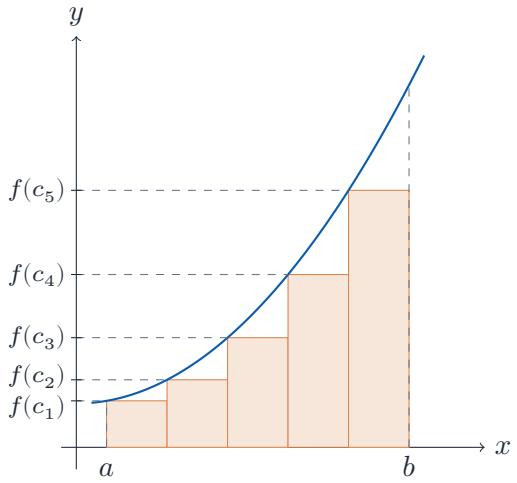


Figure 8.2.2: Left-endpoint Riemann sum $\sum_k f(c_k)\,\Delta x_k$ approximating the area under $y = f(x)$ with each rectangle's height $f(c_k)$ marked on the y-axis.

Example 8.2.5 (Constant function and Dirichlet function)**.** For a constant $f(x) = r$ every Riemann sum on $[a, b]$ equals $\sum_k r\,\Delta x_k = r(b - a)$ because the displacements add up to $b - a$. Every δ then works for every ε and $\int_a^b r\,dx = r(b - a)$.

The Dirichlet function on $[0, 1]$, equal to 1 at rational and 0 at irrational points, is not integrable. Every step of every partition contains rational and irrational points (Theorem 2.4.12). Rational sample points give the Riemann sum 1 and irrational sample points give 0 on the same partition, however small its mesh. Since no number J lies within $\frac{1}{2}$ of both 0 and 1, the definition fails for $\varepsilon = \frac{1}{2}$.

A Riemann sum samples f at one point of each step. An unbounded f has sample points at which one term of the sum is as large as we please. Only bounded functions are integrable.

Lemma 8.2.6 (Integrable functions are bounded)**.** *If f is integrable on $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Let $J = \int_a^b f(x)\,dx$ and take δ for $\varepsilon = 1$ in Definition 8.2.4. Fix a tagged partition with mesh $< \delta$, steps $[x_{k-1}, x_k]$ and sample points c_k. Let $t \in [x_{k-1}, x_k]$. Replacing c_k by t and keeping the other sample points gives a second tagged partition with the same mesh. Its Riemann sum differs from the first by $(f(t) - f(c_k))\Delta x_k$. Both sums lie within 1 of J and hence within 2 of each other. Therefore

$$|f(t)| \le |f(c_k)| + \frac{2}{\Delta x_k} \qquad (t \in [x_{k-1}, x_k]).$$

Every $t \in [a, b]$ lies in some step. The largest of the n right-hand sides bounds $|f|$ on $[a, b]$. ∎

The Riemann sum depends on the sample points. Left and right endpoints give different sums on the same partition (Fig. 8.2.3). For an integrable f every choice has the same limit.

By Theorem 8.2.13 below every continuous f is integrable. When f is known to be integrable any sequence of Riemann sums whose meshes tend to 0 converges to $\int_a^b f(x)\,\mathrm{d}x$. Equally spaced partitions with right-endpoint sample points then suffice to compute the integral (Worked Example 8.2.1).



(a) Left-endpoint sum (here an under-estimate). (b) Right-endpoint sum (here an over-estimate).

Figure 8.2.3: One f with left- and right-endpoint sample points. For an increasing f the left sum under-estimates and the right sum over-estimates. For a continuous f both sums and the midpoint sum converge to the same value $\int_a^b f$ as the mesh $\to 0$ (Theorem 8.2.13). The integral therefore does not depend on the choice of sample point.

8.2.7 Upper and lower sums

The left and right sums of Fig. 8.2.3 bracket the area for an increasing function. For a general bounded function the bracket uses the largest and smallest values of f on each step.

Definition 8.2.8 (Upper and lower sums)**.** Let f be bounded on $[a,b]$ and let P be a partition of $[a,b]$. On the kth step put $M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$ and $m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$. The *upper* and *lower sums* of f on P are

$$U(f,P) = \sum_{k=1}^{n} M_k\,\Delta x_k, \qquad L(f,P) = \sum_{k=1}^{n} m_k\,\Delta x_k. \tag{8.2.3}$$

The suprema and infima exist by completeness (Assumption 2.4.5) because f is bounded. Since $m_k \le f(c_k) \le M_k$ for every sample point, every Riemann sum S on P satisfies $L(f,P) \le S \le U(f,P)$.

Lemma 8.2.9 (Refinement of partitions)**.** *Let f be bounded on $[a,b]$. If the partition P' contains every point of P, then $L(f,P) \le L(f,P')$ and $U(f,P') \le U(f,P)$. For any two partitions P and Q of $[a,b]$, $L(f,P) \le U(f,Q)$.*

Proof. Suppose first that P' adds a single point x^* inside the step $[x_{k-1},x_k]$ of P. The supremum of f over either half $[x_{k-1},x^*]$ or $[x^*,x_k]$ is at most M_k, the supremum over the whole step. The two terms that replace $M_k\,\Delta x_k$ in the upper sum are therefore at most $M_k(x^* - x_{k-1}) + M_k(x_k - x^*) = M_k\,\Delta x_k$. Since all other terms are unchanged, $U(f,P') \le U(f,P)$. The lower sum increases by the same argument with infima. Adding finitely many points one at a time proves the first statement. For the second let $P \cup Q$ be the partition with the points of both. Then $L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,Q)$. ∎

Lemma 8.2.10 (Integrability criterion)**.** *Let f be bounded on $[a,b]$. Suppose that for every $\varepsilon > 0$ there is a $\delta > 0$ with $U(f,P) - L(f,P) < \varepsilon$ for every partition P of mesh $< \delta$. Then f is integrable on $[a,b]$, and $L(f,P) \le \int_a^b f(x)\,\mathrm{d}x \le U(f,P)$ for every partition P.*

Proof. By Lemma 8.2.9 every upper sum is at least $L(f, P_0)$ for a fixed partition P_0. The set of upper sums is therefore bounded below and has a greatest lower bound J by completeness. For every partition P we have $J \le U(f, P)$. Because every upper sum $U(f, Q)$ is at least $L(f, P)$ by Lemma 8.2.9, the number $L(f, P)$ is a lower bound of the set and $L(f, P) \le J$. Now let $\varepsilon > 0$ and take δ as in the hypothesis. For a tagged partition with mesh $< \delta$ both the Riemann sum S and the number J lie in the interval $[L(f, P), U(f, P)]$ of length less than ε. Hence $|S - J| < \varepsilon$ and $J = \int_a^b f(x)\,\mathrm{d}x$. ∎

8.2.11 Integrability of continuous functions

A function may fail to be continuous at a few points and still have an area under its graph. A step function is the simplest case. The definition below names the class of functions that the integral of this chapter handles.

Definition 8.2.12 (Piecewise continuous function)**.** A function f on $[a, b]$ is *piecewise continuous* if it is continuous at all but finitely many points of $[a, b]$ and at each exceptional point p the one-sided limits $\lim_{x \to p^-} f(x)$ and $\lim_{x \to p^+} f(x)$ exist as real numbers (only the one inside $[a, b]$ when p is a or b). An exceptional point is a *jump discontinuity* when the two one-sided limits differ and a *removable discontinuity* when they agree.

The sign function on $[-1, 1]$ is piecewise continuous with one jump at 0. The function on $[0, 1]$ equal to $\dfrac{1}{x}$ for $x > 0$ and to 0 at 0 is continuous except at 0. It is not piecewise continuous because its right-hand limit at 0 is infinite.

Sums, constant multiples and absolute values of piecewise continuous functions are piecewise continuous because the exceptional points of the result lie among those of the pieces. The restriction of a piecewise continuous function to a closed subinterval is again piecewise continuous.

Theorem 8.2.13 (Integrability of continuous functions)**.** *If f is continuous on $[a, b]$, or piecewise continuous there (Definition 8.2.12), then $\int_a^b f(x)\,\mathrm{d}x$ exists.*

Proof. **(i)** Continuous function. By the Extreme Value Theorem (Theorem 5.4.6) f is bounded on $[a, b]$. By the Heine–Cantor theorem (Theorem 5.5.10) it is uniformly continuous there (Definition 5.5.1). Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(b - a)$ for all $x, y \in [a, b]$ with $|x - y| < \delta$. Let P have mesh $< \delta$. On each step f attains its maximum $M_k = f(s_k)$ and its minimum $m_k = f(t_k)$ by Theorem 5.4.6. Because s_k and t_k lie in one step, they are less than δ apart and $M_k - m_k < \varepsilon/(b - a)$. Summing,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\,\Delta x_k < \frac{\varepsilon}{b - a} \sum_{k=1}^{n} \Delta x_k = \varepsilon. \tag{8.2.4}$$

By Lemma 8.2.10 f is integrable.

(ii) Piecewise continuous function. Let $a = p_0 < p_1 < \cdots < p_m = b$ be the endpoints together with the exceptional points of f. On each $[p_{i-1}, p_i]$ let g_i equal f at interior points, $\lim_{x \to p_{i-1}^+} f(x)$ at p_{i-1} and $\lim_{x \to p_i^-} f(x)$ at p_i. Each g_i is continuous on the closed interval $[p_{i-1}, p_i]$ and is therefore bounded (Theorem 5.4.6) and uniformly continuous (Theorem 5.5.10). Since f agrees with some g_i away from the finitely many p_j, there is a $B > 0$ with $|f| \le B$ on $[a, b]$.

Let $\varepsilon > 0$. Choose $\delta > 0$ so small that $\delta < \varepsilon/(8B(m+1))$ and that for every i, $|g_i(x) - g_i(y)| < \varepsilon/(2(b - a))$ whenever $x, y \in [p_{i-1}, p_i]$ satisfy $|x - y| < \delta$. Let P have mesh $< \delta$. Call a step *bad* if it contains one of the $m + 1$ points p_j. Since each p_j lies in at most two steps, there are at most $2(m + 1)$ bad steps. On a bad step $M_k - m_k \le 2B$ and $\Delta x_k < \delta$. A good step lies inside

some open interval (p_{i-1}, p_i) on which $f = g_i$ and therefore has $M_k - m_k \le \varepsilon/(2(b-a))$. Adding the two kinds of step,

$$U(f,P) - L(f,P) \le \frac{\varepsilon}{2(b-a)}\,(b-a) + 2(m+1)\cdot 2B\delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By Lemma 8.2.10 f is integrable. ∎

8.2.14 Properties of the integral

The integral inherits linearity from the Riemann sums. Its sign rule is the orientation reversal already built into Definition 8.2.2.

Theorem 8.2.15 (Properties of the definite integral)**.** *Let f and g be piecewise continuous on a closed interval that contains a, b and c and let $k \in \mathbb{R}$. Then*

1. *(linearity)*
$$\int_a^b (kf+g)\,\mathrm{d}x = k\int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x; \tag{8.2.5}$$
2. *(orientation)* $\int_a^a f\,\mathrm{d}x = 0$ *and* $\int_b^a f\,\mathrm{d}x = -\int_a^b f\,\mathrm{d}x$*;*
3. *(additivity)* $\int_a^b f\,\mathrm{d}x + \int_b^c f\,\mathrm{d}x = \int_a^c f\,\mathrm{d}x$*;*
4. *(order) if $a < b$ and $f \le g$ on $[a,b]$, then $\int_a^b f\,\mathrm{d}x \le \int_a^b g\,\mathrm{d}x$. In particular $m(b-a) \le \int_a^b f\,\mathrm{d}x \le M(b-a)$ when $m \le f \le M$ on $[a,b]$, and $\left|\int_a^b f\,\mathrm{d}x\right| \le \int_a^b |f|\,\mathrm{d}x$.*

Proof. Every integral in the statement exists by Theorem 8.2.13.
(i) Linearity. Take $a < b$ first. On a tagged partition let S_f and S_g be the Riemann sums of f and g. The Riemann sum of $kf+g$ is then $kS_f + S_g$. Let $\varepsilon > 0$ and choose δ so that $|S_f - \int_a^b f| < \varepsilon/(2(|k|+1))$ and $|S_g - \int_a^b g| < \varepsilon/2$ for every mesh $< \delta$. Then

$$\left|kS_f + S_g - \left(k\int_a^b f + \int_a^b g\right)\right| \le |k|\left|S_f - \int_a^b f\right| + \left|S_g - \int_a^b g\right| < \varepsilon.$$

The case $b < a$ follows from (ii) on multiplying both sides by -1. The case $a = b$ reads $0 = 0$.
(ii) Orientation. Both statements are part of Definition 8.2.4. The second is the orientation reversal of Definition 8.2.2: traversing the interval backwards negates every displacement and with it every Riemann sum.
(iii) Additivity. Take $a < b < c$ first and let $\varepsilon > 0$. Choose δ so that Riemann sums with mesh $< \delta$ are within $\varepsilon/3$ of their integrals on each of $[a,b]$, $[b,c]$ and $[a,c]$. A tagged partition of $[a,b]$ and one of $[b,c]$, each with mesh $< \delta$, together form a tagged partition of $[a,c]$ with mesh $< \delta$ whose Riemann sum is $S_1 + S_2$. Hence, for every $\varepsilon > 0$,

$$\left|\int_a^c f - \int_a^b f - \int_b^c f\right| \le \left|\int_a^c f - (S_1+S_2)\right| + \left|S_1 - \int_a^b f\right| + \left|S_2 - \int_b^c f\right| < \varepsilon.$$

Since ε is arbitrary, (iii) follows in this case. Every other ordering reduces to this one by (ii). For instance if $b < a < c$ the case of three increasing points gives $\int_b^a f + \int_a^c f = \int_b^c f$. Replacing $\int_b^a f$ by $-\int_a^b f$ and rearranging gives (iii). When two of the points coincide, (iii) follows from (ii) directly.

(iv) On each tagged partition $S_f \le S_g$. If $\int_a^b f > \int_a^b g$, put $\varepsilon = \frac{1}{2}(\int_a^b f - \int_a^b g)$ and take a tagged partition with mesh small enough for both functions. Then $S_f > \int_a^b f - \varepsilon = \int_a^b g + \varepsilon > S_g$. This is a contradiction. Comparing f with the constants m and M and using Example 8.2.5 gives the bounds $m(b-a)$ and $M(b-a)$. From $-|f| \le f \le |f|$, the order property and (i) follows $-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$. ∎

Linearity of the indefinite integral is Proposition 8.1.6. Two shortcuts are proved by substitution in Corollary 8.5.4: the integral of an odd integrand over $[-a, a]$ is 0 and that of an even integrand is $2\int_0^a f$.

8.2.16 Integral as a bounded linear functional

Linearity and the order property of Theorem 8.2.15 combine into one estimate in the language of normed spaces (Section 3.6). Write $C([a,b])$ for the vector space of continuous functions on $[a,b]$ with the sup norm $\|f\|_\infty = \max_{a\le x\le b}|f(x)|$ of Definition 3.6.9. The maximum exists by Theorem 5.4.6.

Theorem 8.2.17 (Integral as a bounded linear functional)**.** *Let $a < b$ and define $I : C([a,b]) \to \mathbb{R}$ by $I(f) = \int_a^b f(x)\,\mathrm{d}x$. Then I is linear and*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le (b-a)\,\|f\|_\infty \qquad (f \in C([a,b])). \tag{8.2.6}$$

Hence I is a bounded linear map from $(C([a,b]), \|\cdot\|_\infty)$ to $(\mathbb{R}, |\cdot|)$ (Definition 3.6.18) with operator norm $\|I\| = b - a$ (Definition 3.6.19). In particular $\int_a^b f_n\,\mathrm{d}x \to \int_a^b f\,\mathrm{d}x$ for every sequence (f_n) in $C([a,b])$ with $\|f_n - f\|_\infty \to 0$.

Proof. Linearity is Theorem 8.2.15(i). Since $|f(x)| \le \|f\|_\infty$ for every $x \in [a,b]$, the order property Theorem 8.2.15(iv) gives

$$\left|\int_a^b f\,\mathrm{d}x\right| \le \int_a^b |f|\,\mathrm{d}x \le \int_a^b \|f\|_\infty\,\mathrm{d}x = (b-a)\,\|f\|_\infty, \tag{8.2.7}$$

the last step by Example 8.2.5. Hence (8.2.7) gives $\|I\| \le b - a$. The constant function 1 has $\|1\|_\infty = 1$ and $I(1) = b - a$. The bound is therefore attained and $\|I\| = b - a$. Finally $|I(f_n) - I(f)| = |I(f_n - f)| \le (b-a)\|f_n - f\|_\infty$ by linearity and Eq. (8.2.6). The right side tends to 0. ∎

The last statement is the continuity of I that Theorem 3.6.21 guarantees for every bounded linear map. The hypothesis of convergence in the sup norm is needed. A sequence of continuous functions can converge to 0 at every point of $[a,b]$ while its integrals stay equal to 1 (Problem 8.2.5).

8.2.18 Area and average value

When $f \ge 0$ on $[a,b]$ every Riemann sum is a total area of rectangles that approximate the region under the graph. The integral $\int_a^b f(x)\,\mathrm{d}x$ is the *area* of that region. When f changes sign the rectangles below the axis contribute negative terms. The integral is then the *signed area*: the area above the axis minus the area under it.

Definition 8.2.19 (Average value)**.** The *average value* of an integrable f on $[a,b]$ with $a<b$ is

$$\operatorname{avg}(f)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \tag{8.2.8}$$

On an equally spaced partition with n steps the Riemann sum divided by $b-a$ is $\frac{1}{n}\sum_k f(c_k)$, the mean of n sampled values. The average value is the limit of these means. For continuous f Theorem 8.2.15(iv) places it between the minimum and the maximum of f.

Worked Example 8.2.1

Show that for $a<b$

$$\int_a^b x\,\mathrm{d}x=\frac{b^2-a^2}{2},\qquad \int_a^b x^2\,\mathrm{d}x=\frac{b^3-a^3}{3}.$$

Both integrands are continuous. Both integrals therefore exist by Theorem 8.2.13 and any sequence of Riemann sums with mesh tending to 0 converges to them. Divide $[a,b]$ into n equal steps of width $h=(b-a)/n$ with right-endpoint sample points $c_k=a+kh$. With the sums $\sum_{k=1}^n k=\frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^2=\frac{n(n+1)(2n+1)}{6}$ of Section 4.6,

$$\sum_{k=1}^{n}(a+kh)\,h=a(b-a)+(b-a)^2\,\frac{n(n+1)}{2n^2}\;\longrightarrow\;a(b-a)+\frac{(b-a)^2}{2}=\frac{b^2-a^2}{2}.$$

For the square write $d=b-a$ and expand $(a+kh)^2=a^2+2akh+k^2h^2$:

$$\sum_{k=1}^{n}(a+kh)^2\,h=a^2d+2ad^2\,\frac{n(n+1)}{2n^2}+d^3\,\frac{n(n+1)(2n+1)}{6n^3}\;\longrightarrow\;a^2d+ad^2+\frac{d^3}{3}.$$

The limit equals $\dfrac{(a+d)^3-a^3}{3}=\dfrac{b^3-a^3}{3}$. By the sign rule both formulas also hold for $b<a$.

Worked Example 8.2.2

Evaluate $\displaystyle\int_{-1}^{2}(3x^2-2|x|)\,\mathrm{d}x$.

By linearity Eq. (8.2.5) the integral is $3\int_{-1}^2 x^2\,\mathrm{d}x-2\int_{-1}^2|x|\,\mathrm{d}x$. The first integral is $\dfrac{2^3-(-1)^3}{3}=3$ by Worked Example 8.2.1. Additivity splits the second at the point 0 at which the formula for $|x|$ changes:

$$\int_{-1}^{2}|x|\,\mathrm{d}x=\int_{-1}^{0}(-x)\,\mathrm{d}x+\int_{0}^{2}x\,\mathrm{d}x=-\frac{0-1}{2}+\frac{4-0}{2}=\frac{5}{2}.$$

The integral is $3\cdot 3-2\cdot\frac{5}{2}=4$.

Worked Example 8.2.3

Find the average value of $f(x) = x^2$ on $[0,3]$ and a point $c \in [0,3]$ with $f(c) = \operatorname{avg}(f)$.
By Worked Example 8.2.1, $\operatorname{avg}(f) = \frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{3}\cdot\frac{27}{3} = 3$. The equation $c^2 = 3$ has the solution $c = \sqrt{3} \in [0,3]$. A continuous function always attains its average value somewhere on the interval (Lemma 8.3.2 in the next section).

Common misconception

A definite integral is not always an area. The integral $\int_{-1}^{1} x\,dx = \frac{1-1}{2} = 0$, although the region between the graph and the axis has total area 1. The integral counts area below the axis negatively. To find the geometric area, split the interval where f changes sign and integrate $|f|$.

Caution

Reversing the limits reverses the sign. The integral $\int_3^1 7\,dx$ is -14 because the interval is traversed from 3 to 1 and every displacement is negative.

Study guide

The integral $\int_a^b f(x)\,dx$ adds the values $f(c_k)\,\Delta x_k$ of the 1-form $f(x)\,dx$ over the steps of a partition of the oriented interval and takes the limit as the mesh tends to 0. The limit exists for every continuous function. The proof traps every Riemann sum between a lower and an upper sum and uses uniform continuity to make their difference small.
To evaluate an integral before the Fundamental Theorem use linearity, additivity and the sign rule of Theorem 8.2.15 to reduce it to the power formulas of Worked Example 8.2.1, to constants, or to areas of known shapes. Read $\int_b^a$ with $b > a$ as the reversed orientation and change the sign. The average value is the integral divided by the length of the interval.

Problems

8.2.1 Evaluate $\displaystyle\int_3^1 7\,dx$. ⋆ Hint · Answer · Solution

8.2.2 Show by right-endpoint Riemann sums on equally spaced partitions that $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}$.
⋆⋆ Hint · Answer · Solution

8.2.3 Let f be continuous on $[a,b]$ with $a < b$, $f \geq 0$ and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for every $x \in [a,b]$. ⋆⋆ Hint · Answer · Solution

8.2.4 Let $p \in [a,b]$ and let f on $[a,b]$ equal 1 at p and 0 elsewhere. Prove directly from Definition 8.2.4 that $\int_a^b f(x)\,dx = 0$. Deduce that changing an integrable function g at one point of $[a,b]$ leaves it integrable with the same integral.
⋆⋆ Hint · Answer · Solution

8.2.5 For $n \geq 2$ let f_n on $[0,1]$ be the continuous function with $f_n(x) = n^2 x$ on $[0, \frac{1}{n}]$, $f_n(x) = n^2(\frac{2}{n} - x)$ on $[\frac{1}{n}, \frac{2}{n}]$ and $f_n(x) = 0$ on $[\frac{2}{n}, 1]$. Show that $f_n(x) \to 0$ for every $x \in [0,1]$, that $\int_0^1 f_n\,dx = 1$ for every n, and that $\|f_n\|_\infty = n$. Explain why these facts do not contradict Theorem 8.2.17. ⋆⋆ Hint · Answer · Solution

8.2.6 Prove that every nondecreasing function f on $[a, b]$ is integrable, whether or not it is continuous. ⋆⋆⋆ Hint · Answer · Solution

8.2.7 Let f on $[0, 1]$ equal 1 at rational and -1 at irrational points. Prove that $|f|$ is integrable and that f is not. What does this example show about the inequality $\left|\int f\right| \leq \int |f|$ of Theorem 8.2.15(iv)? ⋆⋆ Hint · Answer · Solution

8.2.8 Let $f(0) = 0$ and $f(x) = \frac{1}{n}$ for $\frac{1}{n+1} < x \leq \frac{1}{n}$ and $n = 1, 2, 3, \ldots$. Show that f is defined on all of $[0, 1]$ and nondecreasing. Show that it is discontinuous at every point $\frac{1}{n}$ with $n \geq 2$ and integrable on $[0, 1]$. What does this show about the converse of Theorem 8.2.13? ⋆⋆ Hint · Answer · Solution

8.3 Fundamental Theorem of Calculus

Overview

Objectives: Prove the Mean Value Theorem for integrals (Lemma 8.3.2) and both parts of the Fundamental Theorem (Theorems 8.3.4 and 8.3.5). Use Part 2 to evaluate a definite integral as $F(b)-F(a)$ and Part 1 to differentiate an integral with a variable limit. Apply the chain rule when the limit is a function and a sign change when the variable is the lower limit. State the two parts as one pair of operator identities (Theorem 8.3.10) and compare the sup norms of integration and differentiation (Proposition 8.3.11).

Prerequisites: Antiderivatives (Section 8.1); the definite integral and its properties (Definition 8.2.4 and Theorems 8.2.13 and 8.2.15); the Extreme and Intermediate Value Theorems (Theorems 5.4.2 and 5.4.6); functions with zero derivative (Corollary 7.2.2); the chain rule (Theorem 6.3.4).

Key results: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} \mathrm{d}F = F(b)-F(a)$ for any antiderivative F (Eq. (8.3.6)); $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$.

Notation

$\big[F(x)\big]_a^b$	shorthand for $F(b)-F(a)$
$\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f$	derivative of an integral with variable upper limit
$\displaystyle\int_{u(x)}^{v(x)} f$	both limits functions of x (chain rule applies)
J	the integration map $f \mapsto \int_a^x f(t)\,\mathrm{d}t$ (Theorem 8.3.10)
$\|J\|$	operator norm of J for the sup norm (Proposition 8.3.11)

Computing $\int_0^1 x^2\,\mathrm{d}x$ from Riemann sums took a formula for $\sum k^2$ (Worked Example 8.2.1). An integrand such as $\sin x$ or $\dfrac{1}{1+x^2}$ has no such formula at hand. A moving object suggests a different route. If $s(t)$ is its position and $v(t)=s'(t)$ its velocity, the displacement between times a and b is both $\int_a^b v(t)\,\mathrm{d}t$ and $s(b)-s(a)$. The Fundamental Theorem states that this equality is true for every continuous integrand and every antiderivative.

8.3.1 Mean Value Theorem for integrals

Lemma 8.3.2 (Mean Value Theorem for integrals)**.** *If f is continuous on $[a,b]$ with $a<b$, then there is a point $c\in[a,b]$ with $\int_a^b f(x)\,\mathrm{d}x = f(c)\,(b-a)$.*

Proof. By the Extreme Value Theorem (Theorem 5.4.6) there are points $p,q\in[a,b]$ with $f(p)=m$ the minimum and $f(q)=M$ the maximum of f on $[a,b]$. The order property of Theorem 8.2.15 gives $m(b-a)\le\int_a^b f\le M(b-a)$. Dividing by $b-a>0$,

$$f(p)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le f(q).$$

If $p = q$, then f is constant and $c = p$ works. Otherwise f is continuous on the closed interval with endpoints p and q. The Intermediate Value Theorem (Theorem 5.4.2) gives a point c between them at which f takes the value of the middle term. ∎

The lemma says that a continuous function attains its average value (Definition 8.2.19) as in Worked Example 8.2.3.

8.3.3 Two parts of the Fundamental Theorem

Theorem 8.3.4 (Fundamental Theorem of Calculus (Part 1))**.** *If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a, b]$, differentiable on (a, b), and*

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x). \tag{8.3.1}$$

At the endpoints the one-sided derivatives are $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$.

Proof. The integral defining $F(x)$ exists for every $x \in [a, b]$ by Theorem 8.2.13. Let $x \in [a, b]$ and $h \neq 0$ with $x + h \in [a, b]$. By additivity (Theorem 8.2.15),

$$F(x + h) - F(x) = \int_a^{x+h} f - \int_a^x f = \int_x^{x+h} f(t)\,\mathrm{d}t. \tag{8.3.2}$$

For $h > 0$, Lemma 8.3.2 on $[x, x + h]$ gives a $c_h \in [x, x + h]$ with $\int_x^{x+h} f = f(c_h)\,h$. For $h < 0$ the lemma on $[x + h, x]$ and the sign rule give $\int_x^{x+h} f = -\int_{x+h}^x f = -f(c_h)(-h) = f(c_h)\,h$ with $c_h \in [x + h, x]$. In both cases

$$\frac{F(x + h) - F(x)}{h} = f(c_h), \qquad |c_h - x| \leq |h|. \tag{8.3.3}$$

Let $\varepsilon > 0$. Continuity of f at x gives a $\delta > 0$ with $|f(t) - f(x)| < \varepsilon$ whenever $t \in [a, b]$ and $|t - x| < \delta$. For $0 < |h| < \delta$ the point c_h satisfies $|c_h - x| < \delta$ and the difference quotient of (8.3.3) lies within ε of $f(x)$. Hence the limit of the difference quotient as $h \to 0$ is $f(x)$, two-sided for $a < x < b$ and one-sided at a and b. With B a bound for $|f|$ on $[a, b]$ (Theorem 5.4.6), (8.3.2) gives $|F(x + h) - F(x)| \leq B|h|$. Hence F is continuous on $[a, b]$. ∎

Theorem 8.3.5 (Fundamental Theorem of Calculus (Part 2))**.** *If f is continuous on $[a, b]$ and F is any antiderivative of f there (more generally, any F continuous on $[a, b]$ with $F' = f$ on (a, b)), then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{8.3.4}$$

Proof. Let $G(x) = \int_a^x f(t)\,\mathrm{d}t$. By Theorem 8.3.4 G is continuous on $[a, b]$ with $G' = f$ on (a, b). The difference $H = F - G$ is continuous on $[a, b]$ with $H' = f - f = 0$ on (a, b). By Corollary 7.2.2 H equals a constant C on (a, b). Continuity of H at a and b extends the equality $H = C$ to $[a, b]$. Hence $F = G + C$ on $[a, b]$ and

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f(t)\,\mathrm{d}t - 0.$$

∎

By the sign rule Eq. (8.3.4) also holds when $b < a$ since both sides change sign when a and b are exchanged. The difference $F(b) - F(a)$ is written $\big[F(x)\big]_a^b$. A constant added to F cancels in the subtraction. The constant of integration therefore plays no part in a definite integral.

Second proof of Part 2

When F is differentiable on $[a, b]$ with $F' = f$, every partition gives an exact Riemann sum. The Mean Value Theorem (Theorem 7.1.9) on each step supplies a $c_k \in (x_{k-1}, x_k)$ with $F(x_k) - F(x_{k-1}) = f(c_k)\,\Delta x_k$. The sum telescopes:

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} f(c_k)\,\Delta x_k. \tag{8.3.5}$$

The right side is a Riemann sum for every partition. Letting the mesh tend to 0 gives Eq. (8.3.4) for every integrable f that has an antiderivative, continuous or not. The derivative found in Problem 8.1.9 is discontinuous at 0. The argument applies to it on any interval on which it is known to be integrable.

8.3.6 Integral of a differential

Because $\mathrm{d}F = f(x)\,\mathrm{d}x$ by Eq. (6.2.6), Eq. (8.3.4) is a statement about the 1-form $\mathrm{d}F$ (Definition 8.2.2):

$$\int_{[a,b]} \mathrm{d}F = F(b) - F(a). \tag{8.3.6}$$

Integrating the differential of F over $[a, b]$ gives the values of F on the boundary of $[a, b]$, with b counted positively and a negatively. The values of F inside the interval do not enter.

Corollary 8.3.7 (Net Change Theorem)**.** *The net change of a quantity with continuous derivative equals the integral of its rate:* $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$. *This identity is Eq. (8.3.4) read with* $f = F'$.

Corollary 8.3.8 (Integral as antiderivative)**.** *Let f be continuous on an interval I and let $a \in I$. Differentiation and integration are mutually inverse. On the one hand the* area-so-far *function* $x \mapsto \int_a^x f(t)\,\mathrm{d}t$ *is an antiderivative of f by Theorem 8.3.4:*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x). \tag{8.3.7}$$

On the other hand by Theorem 8.3.5 integrating a continuous derivative recovers the function up to its starting value. For F with continuous derivative on I and $x \in I$,

$$\int_a^x F'(t)\,\mathrm{d}t = F(x) - F(a). \tag{8.3.8}$$

In particular every continuous function on an interval has an antiderivative, namely the area-so-far function of Eq. (8.3.7).

Proof. Let f be continuous on an interval I, let $a \in I$ and let $x \in I$. Choose a closed bounded interval $[c, d] \subseteq I$ that contains a and x and has x in its interior unless x is an endpoint of I. By additivity (Theorem 8.2.15), $\int_a^x f = \int_c^x f - \int_c^a f$. Theorem 8.3.4 on $[c, d]$ differentiates the first term and gives Eq. (8.3.7) at x, one-sided at an endpoint of I. The identity Eq. (8.3.8) is Theorem 8.3.5 on the interval between a and x together with the sign rule when $x < a$. ∎

> *If you first differentiate a function and then integrate the result, you get the function back (adjusted by a constant). In a sense, the processes of integration and differentiation are inverses of each other.*
>
> Thomas, Weir, and Hass [2]

8.3.9 Operator form

Chapter 7 treated differentiation as a linear map between function spaces. The corollary above then becomes a pair of identities between maps. Here $C([a,b])$ denotes the continuous functions on $[a,b]$ and $C^1([a,b])$ the functions with a continuous derivative on $[a,b]$, one-sided at the endpoints (Definition 7.3.2).

Theorem 8.3.10 (Fundamental Theorem in operator form)**.** *Let $a<b$. Define $D : C^1([a,b]) \to C([a,b])$ by $Df = f'$, define $J : C([a,b]) \to C^1([a,b])$ by*

$$(Jf)(x) = \int_a^x f(t)\,\mathrm{d}t, \tag{8.3.9}$$

and define $E_a : C^1([a,b]) \to C^1([a,b])$ by $(E_a f)(x) = f(a)$, the constant function with the value of f at a. Then J is linear and

$$DJ = \mathrm{id} \ \text{ on } C([a,b]), \qquad JD = \mathrm{id} - E_a \ \text{ on } C^1([a,b]), \tag{8.3.10}$$

where id *denotes the identity map of the space in question.*

Proof. For $f \in C([a,b])$, Theorem 8.3.4 shows that Jf is differentiable on $[a,b]$ with derivative f, one-sided at the endpoints. Because that derivative is continuous, $Jf \in C^1([a,b])$. Hence J is well defined and $DJf = f$. Linearity of J is Eq. (8.2.5) with upper limit x. For $f \in C^1([a,b])$ and $a < x \le b$, Theorem 8.3.5 applied on $[a,x]$ to the continuous function f' with antiderivative f gives $(JDf)(x) = \int_a^x f'(t)\,\mathrm{d}t = f(x) - f(a)$. At $x = a$ both sides are 0. Hence $JDf = f - E_a f$. ∎

The two identities differ. Differentiating the area-so-far function returns the integrand exactly. Integrating a derivative from a returns the function less its value at a. Differentiation sends every constant to 0 (Section 7.3) and integration from a produces a function with value 0 at a. The map $f \mapsto f(a)$ restores exactly the constant term, the one degree of freedom of Theorem 8.1.3.

The two maps also behave differently with respect to size. Give $C([a,b])$ and $C^1([a,b])$ the sup norm of Definition 3.6.9.

Proposition 8.3.11 (Norms of integration and differentiation)**.** *Let $a<b$. For every $f \in C([a,b])$, $\|Jf\|_\infty \le (b-a)\,\|f\|_\infty$. Hence J is a bounded linear map with $\|J\| = b-a$ (Definition 3.6.19). The map $f \mapsto f'$ from $(C^1([a,b]), \|\cdot\|_\infty)$ to $(C([a,b]), \|\cdot\|_\infty)$ is not bounded.*

Proof. For $a < x \le b$ the estimate Eq. (8.2.6) on $[a,x]$ gives $|(Jf)(x)| \le (x-a)\|f\|_\infty \le (b-a)\|f\|_\infty$. Since $(Jf)(a) = 0$ this gives $\|J\| \le b-a$. For the constant 1, $(J1)(x) = x-a$ has sup norm $b-a = (b-a)\|1\|_\infty$. Therefore $\|J\| = b-a$.

For $n \ge 1$ let $f_n(x) = \big((x-a)/(b-a)\big)^n$. It increases from 0 to 1 on $[a,b]$ and has $\|f_n\|_\infty = 1$. Its derivative $f_n'(x) = n(x-a)^{n-1}/(b-a)^n$ is largest at b, where it equals $n/(b-a)$. A constant K with $\|f'\|_\infty \le K\|f\|_\infty$ for every $f \in C^1([a,b])$ would give $n \le K(b-a)$ for every n, against Theorem 2.4.10. ∎

By Theorem 3.6.21, integration is continuous for the sup norm and differentiation is not. Functions close to 0 in the sup norm can have derivatives far from 0 (Problem 8.3.9).

8.3.12 Variable limits

When the upper limit is itself a function Theorem 8.3.4 combines with the chain rule (Theorem 6.3.4). Let f be continuous on an open interval K, let $a \in K$ and let h be differentiable with values in K. For an antiderivative F of f on K (Corollary 8.3.8), differentiating the identity $\int_a^{h(t)} f(u)\,\mathrm{d}u = F(h(t)) - F(a)$ from Eq. (8.3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{h(t)} f(u)\,\mathrm{d}u = F'(h(t))\,h'(t) = f(h(t))\,h'(t). \tag{8.3.11}$$

This formula is Eq. (8.3.1) read at the moving endpoint $h(t)$ and multiplied by the derivative of the endpoint. For instance $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \sin u\,\mathrm{d}u = \sin(x^2)\cdot 2x$. When both limits move, additivity splits the integral at a fixed point a, and Eq. (8.3.11) applied to each piece gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(t)\,\mathrm{d}t = f(v(x))\,v'(x) - f(u(x))\,u'(x). \tag{8.3.12}$$

Worked Example 8.3.1

Evaluate $\displaystyle\int_1^2 \left(3x^2 - \frac{1}{x^2}\right)\mathrm{d}x$.

An antiderivative of $3x^2 - x^{-2}$ on $[1,2]$ is $x^3 + x^{-1}$ (Proposition 8.1.5). By Eq. (8.3.4),

$$\int_1^2 \left(3x^2 - \frac{1}{x^2}\right)\mathrm{d}x = \left[x^3 + \frac{1}{x}\right]_1^2 = \left(8 + \frac{1}{2}\right) - (1+1) = \frac{13}{2}.$$

The theorem applies because the integrand is continuous on $[1,2]$.

Worked Example 8.3.2

Find $G'(x)$ for $G(x) = \displaystyle\int_x^{x^2} \frac{1}{t}\,\mathrm{d}t$ with $x > 1$. Check the answer by evaluating G.

The integrand $\dfrac{1}{t}$ is continuous on the interval $(0,\infty)$ that contains both limits. By Eq. (8.3.12) with $v(x) = x^2$ and $u(x) = x$,

$$G'(x) = \frac{1}{x^2}\cdot 2x - \frac{1}{x}\cdot 1 = \frac{2}{x} - \frac{1}{x} = \frac{1}{x}.$$

As a check, Eq. (8.3.4) with the antiderivative $\ln t$ gives $G(x) = \ln(x^2) - \ln x = \ln x$. Its derivative is $\dfrac{1}{x}$.

Worked Example 8.3.3

A particle moves along a line with velocity $v(t) = t^2 - 4t + 3$ for $0 \le t \le 3$. Find its net displacement and the total distance it travels.

By Corollary 8.3.7 the displacement is

$$\int_0^3 (t^2 - 4t + 3)\,\mathrm{d}t = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^3 = 9 - 18 + 9 = 0.$$

The velocity factors as $(t-1)(t-3)$. It is positive on $(0,1)$ and negative on $(1,3)$. Distance is the integral of the speed $|v|$. Additivity splits that integral at $t = 1$:

$$\int_0^1 v\,\mathrm{d}t - \int_1^3 v\,\mathrm{d}t = \frac{4}{3} - \left(0 - \frac{4}{3}\right) = \frac{8}{3}.$$

The particle moves forwards $\frac{4}{3}$ and back $\frac{4}{3}$. It travels $\frac{8}{3}$ in total with zero net displacement.

Common misconception

The formula $\int_a^b f = F(b) - F(a)$ needs f continuous on the whole closed interval. Applied blindly it gives $\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_{-1}^1 = -2$ for a positive integrand. The integrand is undefined at 0. Whatever value it is given there, it is unbounded on $[-1,1]$ and is not integrable by Lemma 8.2.6.

Caution

Part 1 differentiates with respect to the upper limit. The derivative of $\int_a^x f(t)\,\mathrm{d}t$ is $f(x)$, with no term from the fixed lower limit a. When the variable sits in the lower limit reverse the limits first: $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\,\mathrm{d}t = -f(x)$.

Study guide

Part 2 evaluates integrals. Find an antiderivative F of the continuous integrand, write $\left[F(x)\right]_a^b$ and subtract. Check continuity on the closed interval before using it. Part 1 differentiates integrals. The derivative of $\int_a^x f(t)\,\mathrm{d}t$ is the integrand at the upper limit. For a limit $h(x)$ multiply by $h'(x)$. For a variable lower limit include a minus sign.
The proofs use three earlier results. The Mean Value Theorem for integrals comes from the Extreme and Intermediate Value Theorems. Part 1 follows from it and continuity of f. Part 2 follows from Part 1 and the fact that functions with zero derivative on an interval are constant. In operator language differentiation after integration is the identity and integration after differentiation loses the value at a.

Problems

8.3.1 Evaluate $\int_0^\pi \sin x\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.3.2 Find $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \cos t\,\mathrm{d}t$. ⋆⋆ Hint · Answer · Solution

8.3.3 Find $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^5 3t\sin t\,\mathrm{d}t$. ⋆⋆ Hint · Answer · Solution

8.3.4 Find $F'(x)$ for $F(x) = \displaystyle\int_x^{3x} t^2\,\mathrm{d}t$ in two ways: by Eq. (8.3.12) and by evaluating the integral first. ⋆⋆ Hint · Answer · Solution

8.3.5 Find a continuous function f on $(0,\infty)$ and a number $a>0$ such that $6+\displaystyle\int_a^x \frac{f(t)}{t^2}\,\mathrm{d}t = 2\sqrt{x}$ for every $x>0$. ⋆⋆ Hint · Answer · Solution

8.3.6 Let f be continuous on $[a,b]$ with $\int_a^x f(t)\,\mathrm{d}t = 0$ for every $x\in[a,b]$. Prove that $f=0$ on $[a,b]$. Deduce that two continuous functions with the same area-so-far function from a are equal. ⋆⋆ Hint · Answer · Solution

8.3.7 Let sgn be the sign function on $[-1,1]$ (Problem 8.1.7) and $F(x)=\int_{-1}^x \operatorname{sgn} t\,\mathrm{d}t$. Show that $F(x)=|x|-1$, that F is continuous on $[-1,1]$ and that F has no derivative at 0. Which hypothesis of Theorem 8.3.4 fails? ⋆⋆ Hint · Answer · Solution

8.3.8 Let f be continuous on $[a,b]$ with $\int_a^b f(x)g(x)\,\mathrm{d}x = 0$ for every $g\in C([a,b])$. Prove that $f=0$ on $[a,b]$. ⋆⋆ Hint · Answer · Solution

8.3.9 For $n\ge 1$ let $g_n(x)=\dfrac{\sin(n^2x)}{n}$ on $[0,1]$. Show that $\|g_n\|_\infty\to 0$ while $\|g_n'\|_\infty = n$. Show that $J(g_n') = g_n$ with J as in Theorem 8.3.10, and explain how the two facts illustrate Proposition 8.3.11. ⋆⋆ Hint · Answer · Solution

8.3.10 Let f be continuous and even on $\mathbb{R}$ and $F(x)=\int_0^x f(t)\,\mathrm{d}t$. Prove that F is odd by differentiating $G(x)=F(x)+F(-x)$. ⋆⋆ Hint · Answer · Solution

8.4 Indefinite Integrals and the Substitution Method

Overview

Objectives: Prove the substitution rule (Theorem 8.4.2) as the chain rule for differentials read backwards. Recognise an integrand of the form $f(g(x))\,g'(x)$ as the pullback of $f(u)\,\mathrm{d}u$, choose the substitution $u = g(x)$, rewrite the integral in u, integrate and return to the original variable.

Prerequisites: The chain rule (Theorem 6.3.4); pullback and the chain rule for differentials (Definition 6.2.11 and Theorem 6.3.5); the basic antiderivatives (Proposition 8.1.5); existence of antiderivatives (Corollary 8.3.8).

Key results: $\displaystyle\int f(g(x))\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$ with $u = g(x)$ and $\mathrm{d}u = g'(x)\,\mathrm{d}x$, the pullback $g^*(\mathrm{d}u) = \mathrm{d}g$ (Definition 6.2.11 and Theorem 6.3.5).

Notation

$u = g(x)$	the inner function chosen as the new variable
$\mathrm{d}u = g'(x)\,\mathrm{d}x$	the pullback of $\mathrm{d}u$, matched against the integrand
$g^*\omega$	pullback of the 1-form ω by g (Eq. (6.2.8))

The table of Proposition 8.1.5 has no entry for $\int 2x\cos(x^2)\,\mathrm{d}x$. The chain rule produces exactly this integrand: $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(x^2) = \cos(x^2)\cdot 2x$. An integrand of the form "outer derivative at an inner function, times the derivative of the inner function" is therefore a derivative whose antiderivative the substitution rule finds. In the language of 1-forms the integrand is a pullback.

8.4.1 Pullback and the substitution rule

For a differentiable g and a 1-form $\omega = f(u)\,\mathrm{d}u$, the pullback $g^*\omega = f(g(x))\,g'(x)\,\mathrm{d}x$ of Eq. (6.2.8) evaluates ω at $g(p)$ on the image displacement $g'(p)\,\mathbf{v}$. The chain rule for differentials Eq. (6.3.15) states $\mathrm{d}(F\circ g) = g^*(\mathrm{d}F)$. Read backwards, it says that an antiderivative of a pulled-back 1-form is an antiderivative of the original 1-form composed with g.

Theorem 8.4.2 (Substitution Rule)**.** *Let g be differentiable on an interval I with values in an interval K and let f be continuous on K. Then*

$$\int f(g(x))\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u, \tag{8.4.1}$$

in the sense that $F(g(x))+C$ is the general antiderivative on I of the left side whenever $F(u)+C$ is the general antiderivative on K of the right side.

Proof. If K contains an endpoint, extend f beyond it by its value there. The extension is continuous on an open interval containing K. By Corollary 8.3.8 it has an antiderivative F there, differentiable at every point of K with $F' = f$. Every other antiderivative of f on K is $F + C$ by Theorem 8.1.3 and leads to the same family below. With $\mathrm{d}F = f(u)\,\mathrm{d}u$ the chain rule for differentials (Theorem 6.3.5) gives

$$\mathrm{d}(F\circ g) = g^*(\mathrm{d}F) = g^*\big(f(u)\,\mathrm{d}u\big) = f(g(x))\,g'(x)\,\mathrm{d}x, \tag{8.4.2}$$

the 1-form on the left of Eq. (8.4.1). At an endpoint of I the one-sided chain rule (Theorem 6.3.4) gives the same identity. Hence $F \circ g$ is an antiderivative of it on I. By Theorem 8.1.3 its antiderivatives are exactly the functions $F(g(x)) + C$. ∎

In practice choose $u = g(x)$ to be the inner function and write the pullback $\mathrm{d}u = g'(x)\,\mathrm{d}x$ (Definition 6.2.11). Check that $g'(x)\,\mathrm{d}x$ is present in the integrand up to a constant factor. Linearity (Proposition 8.1.6) moves such a factor outside the integral. The integral then becomes $\int f(u)\,\mathrm{d}u$. After integrating, replace u by $g(x)$.

Example 8.4.3 (Successful and unsuccessful substitutions)**.** In $\int 2x\cos(x^2)\,\mathrm{d}x$ put $u = x^2$ with $\mathrm{d}u = 2x\,\mathrm{d}x$. The integrand is the pullback of $\cos u\,\mathrm{d}u$ and the integral is $\sin u + C = \sin(x^2) + C$.

In $\int \cos(x^2)\,\mathrm{d}x$ the same choice leaves no factor $2x$ to match $\mathrm{d}u$. On $(0, \infty)$ the identity of 1-forms $\mathrm{d}x = \mathrm{d}u/(2x) = \mathrm{d}u/(2\sqrt{u})$ gives $\int \dfrac{\cos u}{2\sqrt{u}}\,\mathrm{d}u$. The new integral is no simpler than the old one. No antiderivative of $\cos(x^2)$ is expressible with the functions of Proposition 8.1.5 by a theorem of Liouville that this book does not prove.

Worked Example 8.4.1

Evaluate $\displaystyle\int x^2 e^{x^3}\,\mathrm{d}x$.

The inner function is $u = x^3$ with $\mathrm{d}u = 3x^2\,\mathrm{d}x$. The integrand has $x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$. By Theorem 8.4.2

$$\int x^2 e^{x^3}\,\mathrm{d}x = \frac{1}{3}\int e^u\,\mathrm{d}u = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

The derivative of $\frac{1}{3}e^{x^3}$ is $\frac{1}{3}e^{x^3}\cdot 3x^2$ and equals the integrand.

Worked Example 8.4.2

Evaluate $\displaystyle\int \tan x\,\mathrm{d}x$ on an interval on which $\cos x \neq 0$.

Write $\tan x = \dfrac{\sin x}{\cos x}$ and put $u = \cos x$ with $\mathrm{d}u = -\sin x\,\mathrm{d}x$. The values of u lie in $(0, \infty)$ or in $(-\infty, 0)$. On either interval $\dfrac{1}{u}$ is continuous. Then

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{1}{u}\,\mathrm{d}u = -\ln|u| + C = -\ln|\cos x| + C.$$

Since $-\ln|\cos x| = \ln|\sec x|$, the result is also written $\ln|\sec x| + C$.

Worked Example 8.4.3

Evaluate $\displaystyle\int x\sqrt{x+1}\,\mathrm{d}x$ on $(-1, \infty)$.

Put $u = x + 1$ with $\mathrm{d}u = \mathrm{d}x$. The factor x outside the root is expressed through u as $x = u - 1$. Then

$$\int x\sqrt{x+1}\,\mathrm{d}x = \int (u-1)u^{1/2}\,\mathrm{d}u = \int (u^{3/2} - u^{1/2})\,\mathrm{d}u = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C.$$

Returning to x, $\int x\sqrt{x+1}\,\mathrm{d}x = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$.

Caution

A constant factor is easily lost in a substitution when the derivative $g'(x)$ is only proportional to what is present in the integrand. If $u = g(x)$ and the integrand is $f(g(x))$ alone with no visible $g'(x)$, use the identity of 1-forms $\mathrm{d}x = \mathrm{d}u/g'(x)$ (valid where $g'(x) \neq 0$), keep the resulting factor $1/g'(x)$ and express it through u. When g is a bijection with $g' \neq 0$, the 1-form $f(g(x))\,\mathrm{d}x$ is the pullback of $h(u)\,\mathrm{d}u$ with $h(u) = \dfrac{f(u)}{g'(g^{-1}(u))}$. By Eq. (6.2.8) that pullback is $h(g(x))\,g'(x)\,\mathrm{d}x = f(g(x))\,\mathrm{d}x$. Replacing $\mathrm{d}x$ by $\mathrm{d}u$, a different 1-form, is the most common substitution error.

Common misconception

The answer $\int \cos(3x)\,\mathrm{d}x = \sin(3x) + C$ is wrong. Its derivative is $3\cos(3x)$. With $u = 3x$ the 1-form $\mathrm{d}x$ equals $\frac{1}{3}\,\mathrm{d}u$, and the correct result is $\frac{1}{3}\sin(3x) + C$.

Study guide

Substitution is the chain rule read backwards. Look for an inner function $g(x)$ whose derivative $g'(x)$ appears as a factor of the integrand, up to a constant. Write $u = g(x)$ and $\mathrm{d}u = g'(x)\,\mathrm{d}x$, replace every x in the integrand, integrate in u, and substitute back. No x may remain in the u-integral. A leftover factor of x is either expressed through u (Worked Example 8.4.3) or signals a poor choice of u.

The proof of Theorem 8.4.2 is two lines. The chain rule for differentials gives $\mathrm{d}(F \circ g) = g^*(\mathrm{d}F)$ and Theorem 8.1.3 supplies the constant. Check every answer by differentiating it.

Problems

8.4.1 Evaluate $\displaystyle\int (3x+2)\,(3x^2+4x)^4\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.4.2 Evaluate $\displaystyle\int \frac{9r^2}{\sqrt{1-r^3}}\,\mathrm{d}r$. ⋆⋆ Hint · Answer · Solution

8.4.3 Evaluate $\displaystyle\int \sin^5\!\left(\frac{x}{3}\right)\cos\!\left(\frac{x}{3}\right)\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

8.4.4 Let f be continuous on $\mathbb{R}$ with antiderivative F and let $\alpha \neq 0$ and β be constants. Prove that $\displaystyle\int f(\alpha x+\beta)\,\mathrm{d}x = \frac{1}{\alpha}F(\alpha x+\beta) + C$. ⋆⋆ Hint · Answer · Solution

8.4.5 Evaluate $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$ on $(0,\infty)$. ⋆ Hint · Answer · Solution

8.4.6 Evaluate $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x$ on $(1,\infty)$. ⋆ Hint · Answer · Solution

8.4.7 Let g have a continuous derivative on an interval I with $g(x) \neq 0$ for every $x \in I$. Prove that $\displaystyle\int \frac{g'(x)}{g(x)}\,\mathrm{d}x = \ln|g(x)| + C$ on I. ⋆⋆ Hint · Answer · Solution

8.4.8 Find every antiderivative of $\dfrac{1}{x\ln x}$ on the set $(0,1)\cup(1,\infty)$. ⋆⋆ Hint · Answer · Solution

8.4.9 Evaluate $\int 2x\cos(x^2)\,dx$ on $\mathbb{R}$ and explain why Theorem 8.4.2 applies although $g(x) = x^2$ takes every positive value twice. ⋆⋆ Hint · Answer · Solution

8.5 Definite Integral Substitutions and the Area Between Curves

Overview

Objectives: Prove and apply substitution in a definite integral by transforming the limits (Theorem 8.5.2), read as integration of a pullback over the image interval (Eq. (8.5.3)). Shortcut symmetric integrals with even and odd symmetry (Corollary 8.5.4). Find the intersection points of two curves and compute the area between them as the integral of top minus bottom (Definition 8.5.6).

Prerequisites: The substitution method (Section 8.4); the Fundamental Theorem (Theorems 8.3.4 and 8.3.5); the Extreme and Intermediate Value Theorems (Theorems 5.4.2 and 5.4.6); the chain rule (Theorem 6.3.4).

Key results: $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$; $\int_{-a}^{a}(\text{odd}) = 0$ and $\int_{-a}^{a}(\text{even}) = 2\int_0^a$; $A = \int_a^b(\text{top} - \text{bottom})\,\mathrm{d}x$.

Notation

$g(a)$, $g(b)$	the transformed limits after $u = g(x)$
$\int_{-a}^{a}$	symmetric interval (test the integrand for parity)
top − bottom	the integrand for an area between curves

A definite integral computed by substitution can return to x and evaluate there. It is usually shorter to stay in u and evaluate at the values of u that correspond to the original limits. The theorem below justifies the change of limits. When g decreases the image interval is traversed backwards.

8.5.1 Substitution in definite integrals

Theorem 8.5.2 (Substitution in definite integrals)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of g, then*

$$\int_a^b f(g(x))\,g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u. \tag{8.5.1}$$

Proof. By the Extreme and Intermediate Value Theorems (Theorems 5.4.2 and 5.4.6) the range of the continuous g on $[a, b]$ is a closed interval $[m, M]$. Extend f to all of $\mathbb{R}$ by setting $f(u) = f(m)$ for $u < m$ and $f(u) = f(M)$ for $u > M$. The extension is continuous. By Corollary 8.3.8 the function $F(u) = \int_m^u f(t)\,\mathrm{d}t$ satisfies $F' = f$ at every real u. By the chain rule (Theorem 6.3.4) $F \circ g$ has the continuous derivative $f(g(x))\,g'(x)$ on $[a, b]$. Part 2 of the Fundamental Theorem (Theorem 8.3.5) gives

$$\int_a^b f(g(x))\,g'(x)\,\mathrm{d}x = F(g(b)) - F(g(a)). \tag{8.5.2}$$

The same theorem applied to f on the interval between $g(a)$ and $g(b)$, in either orientation, gives $\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = F(g(b)) - F(g(a))$. ∎

Exchanging a and b changes the sign of both sides of Eq. (8.5.1). So the formula also holds when $b < a$.

The left side of Eq. (8.5.1) integrates the pullback $g^*(f(u)\,\mathrm{d}u)$ of Eq. (6.2.8). For every 1-form $\omega = f(u)\,\mathrm{d}u$ with f continuous the theorem therefore reads

$$\int_a^b g^*\omega = \int_{g(a)}^{g(b)} \omega, \tag{8.5.3}$$

where $\int_c^d$ with $c > d$ denotes the integral over $[d, c]$ traversed from c to d. Integrating a pullback over $[a, b]$ equals integrating the 1-form over the image interval oriented from $g(a)$ to $g(b)$. An increasing g keeps the orientation. A decreasing g reverses it and the sign rule accounts for the reversal. When g is not monotone the parts of the image traversed twice in opposite directions cancel.

Example 8.5.3 (Orientation reversed by a decreasing substitution)**.** In $\int_0^1 2x(1-x^2)^3\,\mathrm{d}x$ the substitution $u = g(x) = 1 - x^2$ decreases on $[0, 1]$ and has $\mathrm{d}u = -2x\,\mathrm{d}x$. The image interval runs from $g(0) = 1$ to $g(1) = 0$:

$$\int_0^1 2x(1-x^2)^3\,\mathrm{d}x = -\int_1^0 u^3\,\mathrm{d}u = \int_0^1 u^3\,\mathrm{d}u = \frac{1}{4}.$$

The two minus signs cancel: one comes from $\mathrm{d}u$ and one from the reversed orientation. Forgetting only one of them gives the wrong value $-\frac{1}{4}$.

Corollary 8.5.4 (Even and odd functions on symmetric intervals)**.** *If f is continuous on $[-a, a]$, then $\int_{-a}^a f = 2\int_0^a f$ when f is even and $\int_{-a}^a f = 0$ when f is odd.*

Proof. Apply Theorem 8.5.2 with $g(t) = -t$ and $g'(t) = -1$ on $[0, a]$:

$$\int_0^a f(-t)\,(-1)\,\mathrm{d}t = \int_{g(0)}^{g(a)} f(x)\,\mathrm{d}x = \int_0^{-a} f(x)\,\mathrm{d}x.$$

Multiplying by -1 and using the sign rule gives $\int_0^a f(-t)\,\mathrm{d}t = \int_{-a}^0 f(x)\,\mathrm{d}x$. For even f this integral is $\int_0^a f$. For odd f it is $-\int_0^a f$. Adding $\int_0^a f$ by additivity (Theorem 8.2.15) gives $\int_{-a}^a f = 2\int_0^a f$ in the even case and 0 in the odd case. ∎

8.5.5 Area between curves

When $f \geq g \geq 0$ on $[a, b]$ the region between the graphs is the region under f with the region under g removed. Its area is $\int_a^b f - \int_a^b g = \int_a^b (f - g)$ by linearity. Raising both graphs by a constant leaves $f - g$ and the region's shape unchanged. The same integral is therefore the area whenever $f \geq g$, whatever the signs of f and g.

Definition 8.5.6 (Area between curves)**.** If f, g are continuous with $f(x) \geq g(x)$ on $[a, b]$, the area of the region between $y = f(x)$ and $y = g(x)$ is (Fig. 8.5.1)

$$A = \int_a^b (f(x) - g(x))\,\mathrm{d}x. \tag{8.5.4}$$

Example 8.5.7 (Area with and without crossing)**.** On $[0, 1]$ the line $y = x$ lies above $y = x^3$ and the area between them is $\int_0^1 (x - x^3)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. On $[-1, 1]$ the curves cross at 0. There $\int_{-1}^1 (x - x^3)\,\mathrm{d}x = 0$ by Corollary 8.5.4, while the area is $\frac{1}{2}$ (Problem 8.5.4). The hypothesis $f \geq g$ of Definition 8.5.6 fails on $[-1, 0]$ because $x^3 \geq x$ there.

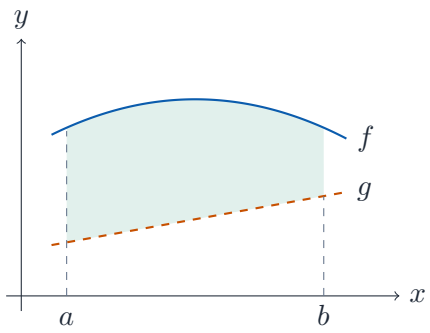


Figure 8.5.1: Area between curves $\int_a^b (f-g)\,dx$ as the integral of top minus bottom.

Worked Example 8.5.1

Evaluate $\displaystyle\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$.

Put $u = \cos x$ with $du = -\sin x\,dx$. The function cos decreases on $[0, \pi/3]$ and the image interval runs from $\cos 0 = 1$ to $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$. By Eq. (8.5.3),

$$\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx = -\int_1^{1/2} u^{-2}\,du = -\Big[-u^{-1}\Big]_1^{1/2} = \big[u^{-1}\big]_1^{1/2} = 2 - 1 = 1.$$

Worked Example 8.5.2

Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} (x^3\cos x + \cos x)\,dx$.

The function $x^3\cos x$ is odd because $(-x)^3\cos(-x) = -x^3\cos x$. The function $\cos x$ is even. By linearity and Corollary 8.5.4,

$$\int_{-\pi/2}^{\pi/2} (x^3\cos x + \cos x)\,dx = 0 + 2\int_0^{\pi/2} \cos x\,dx = 2\big[\sin x\big]_0^{\pi/2} = 2.$$

No antiderivative of $x^3\cos x$ was needed.

Worked Example 8.5.3

Find the area of the region between $y = \sin x$ and $y = \cos x$ for $0 \le x \le \pi/2$.

The curves cross where $\sin x = \cos x$, at $x = \pi/4$. The cosine is on top on $[0, \pi/4]$ and the sine on $[\pi/4, \pi/2]$. Applying Definition 8.5.6 on each piece and adding,

$$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2.$$

The first integral is $\big[\sin x + \cos x\big]_0^{\pi/4} = \sqrt{2} - 1$. The second is $\big[-\cos x - \sin x\big]_{\pi/4}^{\pi/2} = -1 + \sqrt{2}$.

Common misconception

Integrating $f-g$ over an interval on which the curves cross does not give the area. Positive and negative parts cancel (Example 8.5.7). Find the crossing points, split the interval there and integrate top minus bottom on each piece.

Caution

After substituting $u=g(x)$ in a definite integral the limits must be values of u. Writing $\int_0^1$ in u when 0 and 1 were the original x-limits mixes the two variables. Either change the limits to $g(0)$ and $g(1)$ or return to x before evaluating.

Study guide

For a definite integral by substitution choose $u=g(x)$ as in Section 8.4, compute $\mathrm{d}u$, and replace the limits a and b by $g(a)$ and $g(b)$ in that order. Do not reorder the new limits when $g(a)>g(b)$. The sign rule handles the orientation. Before integrating over a symmetric interval $[-a,a]$, test the integrand for parity and use Corollary 8.5.4.
For an area sketch the curves, find their intersections, decide which curve is on top on each piece and integrate top minus bottom. The answer cannot be negative.

Problems

8.5.1 Evaluate $\displaystyle\int_0^1 x\,(x^2+1)^3\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.5.2 Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x$. ⋆ Hint · Answer · Solution

8.5.3 Find the area of the region enclosed by $y=4-x^2$ and $y=x+2$. ⋆⋆ Hint · Answer · Solution

8.5.4 Find the area of the region enclosed by $y=x^3$ and $y=x$. ⋆⋆ Hint · Answer · Solution

8.5.5 Prove that $\displaystyle\int_0^{\pi/2} \sin^n x\,\mathrm{d}x=\int_0^{\pi/2}\cos^n x\,\mathrm{d}x$ for every positive integer n. ⋆⋆ Hint · Answer · Solution

8.5.6 Let f be continuous on $\mathbb{R}$ and periodic with period $p>0$, that is, $f(x+p)=f(x)$ for every x. Prove that $\displaystyle\int_a^{a+p} f(x)\,\mathrm{d}x=\int_0^p f(x)\,\mathrm{d}x$ for every real a. ⋆⋆ Hint · Answer · Solution

8.5.7 Let f be continuous on $[0,1]$. Prove that

$$\int_0^{\pi} x\,f(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$

Deduce the value of $\displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$. ⋆⋆⋆ Hint · Answer · Solution

8.5.8 Evaluate $\int_{-1}^{2} 2x\, e^{x^2}\, \mathrm{d}x$ by Theorem 8.5.2 with $u = g(x) = x^2$. The function g is not monotone on $[-1, 2]$. Explain with Eq. (8.5.3) why the value is correct all the same.

⋆⋆ Hint · Answer · Solution

8.5.9 A student substitutes $u = 1/x$ in $\int_{-1}^{1} \dfrac{\mathrm{d}x}{1+x^2}$. Since $\mathrm{d}u = -\mathrm{d}x/x^2$ and the limits become $u = -1$ and $u = 1$, the student finds $\int_{-1}^{1} \dfrac{-\mathrm{d}u}{1+u^2} = -\dfrac{\pi}{2}$. The value is negative although the integrand is positive. Find the correct value and the error.

⋆⋆ Hint · Answer · Solution

8.6 Arc Length

Overview

Objectives: Derive the arc-length integral $L = \int_a^b \sqrt{1+(f')^2}\,\mathrm{d}x$ (Definition 8.6.2 and Eq. (8.6.1)) as the limit of the lengths of inscribed polygons. Read it as the integral of the arc-length density $\mathrm{d}s = \sqrt{1+(f')^2}\,|\mathrm{d}x|$ (Eq. (8.6.3)), equal to the 1-form $\sqrt{1+(f')^2}\,\mathrm{d}x$ on $[a,b]$ traversed from left to right. Recognise the engineered integrands in which $1+(f')^2$ is a perfect square and evaluate the resulting integral.

Prerequisites: The Mean Value Theorem (Theorem 7.1.9); Riemann sums and integrability (Definition 8.2.4 and Theorem 8.2.13); the Fundamental Theorem (Section 8.3); substitution (Section 8.5).

Key results: $L = \displaystyle\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,\mathrm{d}x$. Before integrating, check whether $1+(f')^2$ is a perfect square. Arc length on the unit circle completes radian measure and gives $\sin\theta < \theta < \tan\theta$ (Theorem 8.6.6).

Notation

L	arc length of $y = f(x)$ over $[a,b]$
$\mathrm{d}s$	arc-length density $\sqrt{1+(f')^2}\,\lvert\mathrm{d}x\rvert$ (Eq. (8.6.3))
$1+(f')^2$	the radicand (write it as a perfect square when possible)

The length of a straight segment from (x_1,y_1) to (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ by Pythagoras. A curved graph has no such formula. Joining points of the graph by segments gives a polygon of computable length. Finer partitions give polygons that follow the curve more closely. The length of the curve is the limit of these polygon lengths. The integral of this section computes that limit.

8.6.1 Arc-length integral

Definition 8.6.2 (Arc length)**.** If f' is continuous on $[a,b]$, the length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,\mathrm{d}x. \tag{8.6.1}$$

The integrand is continuous and the integral exists by Theorem 8.2.13. The derivation below shows that L is the limit of the lengths of inscribed polygons as the mesh tends to 0.

Derivation. Let P be a partition of $[a,b]$ and join the points $(x_k, f(x_k))$ by segments (Fig. 8.6.1). With $\Delta y_k = f(x_k) - f(x_{k-1})$ the segment over $[x_{k-1}, x_k]$ has length

$$\sqrt{\Delta x_k^2 + \Delta y_k^2} = \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k.$$

The Mean Value Theorem (Theorem 7.1.9) gives a point $c_k \in (x_{k-1}, x_k)$ with $\Delta y_k/\Delta x_k = f'(c_k)$. The length of the polygon is therefore

$$\Lambda(P) = \sum_{k=1}^{n} \sqrt{1+f'(c_k)^2}\,\Delta x_k, \tag{8.6.2}$$

a Riemann sum Eq. (8.2.2) of the continuous function $\sqrt{1+(f')^2}$ with sample points c_k. By Definition 8.2.4, for every $\varepsilon>0$ there is a $\delta>0$ with $|\Lambda(P)-L|<\varepsilon$ whenever $\|P\|<\delta$. ∎

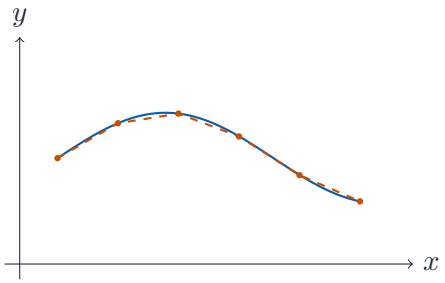


Figure 8.6.1: Arc length as the limit of polygonal approximations: $L=\int_a^b \sqrt{1+(f')^2}\,\mathrm{d}x$.

Example 8.6.3 (Segment and cusp)**.** For the line $f(x)=mx+q$ the integrand is the constant $\sqrt{1+m^2}$ and $L=\sqrt{1+m^2}\,(b-a)$. This value is the distance $\sqrt{(b-a)^2+(mb-ma)^2}$ between the endpoints.

For $f(x)=(\sqrt[3]{x})^2$ on $[-1,1]$ the derivative $\dfrac{2}{3\sqrt[3]{x}}$ is unbounded near 0 and undefined at 0. The graph has a cusp at the origin. Definition 8.6.2 does not apply on $[-1,1]$. The length is obtained from limits of lengths over $[-1,-\varepsilon]$ and $[\varepsilon,1]$ by the improper integrals of Section 9.8.

8.6.4 Arc-length density

The length of a curve is the integral of a density. It is the one integrand in the book outside the class of 1-forms. For the graph $\gamma(x)=(x,f(x))$, a displacement $\mathbf{v}$ at p maps to the displacement $(\mathbf{v},f'(p)\,\mathbf{v})$ in the plane with Euclidean length

$$(\mathrm{d}s)_p(\mathbf{v})=\sqrt{\mathbf{v}^2+\big(f'(p)\,\mathbf{v}\big)^2}=\sqrt{1+f'(p)^2}\;|\mathbf{v}|. \tag{8.6.3}$$

The absolute value makes $\mathrm{d}s$ a *density*. It is positive on every non-zero displacement and keeps its sign when the displacement reverses. A 1-form changes sign under the same reversal (Definition 8.2.2). The customary $\mathrm{d}s=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$ states Eq. (8.6.3). Here $\mathrm{d}x^2+\mathrm{d}y^2$ is the squared length of a displacement in the plane. On an interval traversed from left to right every displacement is positive. There $\mathrm{d}s$ and the 1-form $\sqrt{1+f'(x)^2}\,\mathrm{d}x$ give the same sums and the same integral Eq. (8.6.1). Traversed from right to left the 1-form would give $-L$ and the density still gives L.

Worked Example 8.6.1

Find the length of $y=x^{3/2}$ for $0\le x\le 4$.
Here $f'(x)=\frac{3}{2}x^{1/2}$ and $1+(f')^2=1+\frac{9}{4}x$. Substitute $u=1+\frac{9}{4}x$ with $\mathrm{d}u=\frac{9}{4}\,\mathrm{d}x$. The limits become $u=1$ and $u=10$, and

$$L=\int_0^4\sqrt{1+\tfrac{9}{4}x}\,\mathrm{d}x=\frac{4}{9}\int_1^{10}u^{1/2}\,\mathrm{d}u=\frac{4}{9}\cdot\frac{2}{3}\Big[u^{3/2}\Big]_1^{10}=\frac{8}{27}(10\sqrt{10}-1)\approx 9.073.$$

The chord from $(0,0)$ to $(4,8)$ has length $\sqrt{80}\approx 8.944$. The curve is slightly longer as Problem 8.6.4 requires.

Worked Example 8.6.2

Find the length of $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ for $1 \le x \le 2$.

Here $f'(x) = \frac{x^3}{2} - \frac{1}{2x^3}$. Squaring gives $(f')^2 = \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$. Adding 1 changes the middle sign:

$$1 + (f')^2 = \frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6} = \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2.$$

The square root is $\frac{x^3}{2} + \frac{1}{2x^3}$, positive on $[1, 2]$. Then

$$L = \int_1^2 \left(\frac{x^3}{2} + \frac{x^{-3}}{2}\right) \mathrm{d}x = \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2 = \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) = \frac{33}{16}.$$

Worked Example 8.6.3

Find the length of the arc of the unit circle $y = \sqrt{1 - x^2}$ for $0 \le x \le \frac{1}{2}$.

Here $f'(x) = -\dfrac{x}{\sqrt{1 - x^2}}$ and $1 + (f')^2 = 1 + \dfrac{x^2}{1 - x^2} = \dfrac{1}{1 - x^2}$, continuous on $[0, \frac{1}{2}]$. Then

$$L = \int_0^{1/2} \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \left[\arcsin x\right]_0^{1/2} = \frac{\pi}{6}.$$

The arc runs from $(0, 1)$ to $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and subtends the angle $\frac{\pi}{6}$ at the centre. Its length equals the angle by the definition of radian measure (Definition 4.5.2).

Common misconception

The radicand is $1 + (f')^2$, built from the derivative. Writing $\sqrt{1 + f^2}$ measures nothing. Also $\sqrt{1 + (f')^2} \neq 1 + f'$. The square root of a sum differs from the sum of the square roots, and even that sum is $1 + |f'|$ because $\sqrt{(f')^2} = |f'|$.

Caution

The formula needs f' continuous on the closed interval. At a vertical tangent or a cusp, such as $(\sqrt[3]{x})^2$ at 0 in Example 8.6.3, the integrand is unbounded and the integral is improper.

8.6.5 Unit circle and radian measure

Definition 4.5.2 measures an angle by the length of an arc of the unit circle and promises that Definition 8.6.2 completes the definition. The arc from $(1, 0)$ to the point $(\sqrt{1 - s^2}, s)$ with $0 \le s < 1$ is the graph of $x = \sqrt{1 - y^2}$ over $0 \le y \le s$. Its derivative $-y/\sqrt{1 - y^2}$ is continuous on $[0, s]$. With the roles of x and y exchanged, Definition 8.6.2 gives the length

$$1 + \left(\frac{-y}{\sqrt{1 - y^2}}\right)^2 = \frac{1}{1 - y^2}, \qquad \Theta(s) = \int_0^s \frac{\mathrm{d}y}{\sqrt{1 - y^2}}, \tag{8.6.4}$$

which is Eq. (4.5.1). The integrand is an algebraic function of y. The theorem below uses only integrability and the properties of the integral (Theorems 8.2.13 and 8.2.15), the Fundamental Theorem, the derivative of a square root, completeness and the Intermediate Value Theorem. No trigonometric function enters their proofs.

Theorem 8.6.6 (Arc length on the unit circle)**.** *Let* $\Theta(s) = \displaystyle\int_0^s \frac{\mathrm{d}y}{\sqrt{1-y^2}}$ *for* $0 \le s < 1$.

1. Θ *is continuous and strictly increasing on* $[0,1)$*, and* $\Theta(s) < 2$ *for every* s.
2. *The supremum of* Θ *on* $[0,1)$ *is* $\pi/2$*, the length of the quarter circle (Proposition A.1.12). Each* $\theta \in [0, \pi/2)$ *equals* $\Theta(s)$ *for exactly one* $s \in [0,1)$.
3. *For* $0 < s < 1$,
$$s < \Theta(s) < \frac{s}{\sqrt{1-s^2}}. \tag{8.6.5}$$
With $\theta = \Theta(s)$ (8.6.5) *is* $\sin\theta < \theta < \tan\theta$ *for* $0 < \theta < \pi/2$.

Proof. Write $g(y) = 1/\sqrt{1-y^2}$. Every $\Theta(s)$ exists because g is continuous on $[0,1)$ (Theorem 8.2.13). The function g is increasing on $[0,1)$ with $g(0) = 1$.

(1) For $0 \le t < s < 1$, additivity and the order property of Theorem 8.2.15 give $s - t \le \Theta(s) - \Theta(t) = \int_t^s g\,\mathrm{d}y \le g(s)(s-t)$. The left inequality shows that Θ is strictly increasing. The right one gives $|\Theta(s) - \Theta(t)| \le g(b)|s-t|$ for s, t in any $[0,b]$ with $b < 1$. Hence Θ is continuous. Since $1 - y^2 = (1-y)(1+y) \ge 1-y$, the integrand satisfies $g(y) \le (1-y)^{-1/2}$. The Fundamental Theorem (Theorem 8.3.5) with the antiderivative $-2\sqrt{1-y}$ then gives

$$\Theta(s) \le \int_0^s \frac{\mathrm{d}y}{\sqrt{1-y}} = 2 - 2\sqrt{1-s} < 2.$$

(2) The supremum exists by Assumption 2.4.5 and is $\pi/2$ by Proposition A.1.12. Uniqueness follows from strict monotonicity. Given $\theta < \pi/2$, some $s_1 \in [0,1)$ has $\Theta(s_1) > \theta$ because θ is not an upper bound. The Intermediate Value Theorem (Theorem 5.4.2) applied to the continuous function Θ on $[0, s_1]$ with $\Theta(0) = 0 \le \theta < \Theta(s_1)$ gives s with $\Theta(s) = \theta$.

(3) On $[0, s/2]$ the integrand satisfies $1 \le g \le g(s/2)$ and on $[s/2, s]$ it satisfies $g(s/2) \le g \le g(s)$. Here $1 < g(s/2) < g(s)$. Splitting the integral at $s/2$,

$$s < \frac{s}{2} + \frac{s}{2}\,g\Big(\frac{s}{2}\Big) \le \Theta(s) \le \frac{s}{2}\,g\Big(\frac{s}{2}\Big) + \frac{s}{2}\,g(s) < s\,g(s) = \frac{s}{\sqrt{1-s^2}}. \tag{8.6.6}$$

By Definition 4.5.2 the point at arc length $\theta = \Theta(s)$ from $(1,0)$ is $(\sqrt{1-s^2}, s)$. Hence $\sin\theta = s$, $\cos\theta = \sqrt{1-s^2} > 0$ and $\tan\theta = s/\sqrt{1-s^2}$. By (2) every $\theta \in (0, \pi/2)$ arises in this way. ∎

An increasing function bounded above has its supremum as its limit (Proposition 5.2.23). The number $\pi/2$ of Theorem 8.6.6 is therefore the improper integral $\int_0^1 \mathrm{d}y/\sqrt{1-y^2}$ of Section 9.8. Proposition A.1.12 proves that this integral is the length of the quarter circle and that the upper semicircle has length π, as Definition 4.5.2 requires. Part (3) is the inequality Eq. (5.2.4) from which Chapter 5 derives $\lim_{x\to 0}(\sin x)/x = 1$. Nothing in this chain uses a derivative of a trigonometric function. The derivatives of sin and arcsin rest on this theorem through Theorem 5.2.18, and Worked Example 8.6.3 uses them only to recompute an arc whose length the definition already fixes.

Line integrals

The remark on forms on manifolds (Page 197) describes 1-forms on a manifold and the generalised Stokes theorem Eq. (6.2.11). On a manifold M the integral of a 1-form ω along a curve $\gamma : [a,b] \to M$ is $\int_{[a,b]} \gamma^*\omega$, defined by Eq. (8.2.1).

Study guide

To set up an arc length compute f', form $1+(f')^2$, and write $L=\int_a^b \sqrt{1+(f')^2}\,\mathrm{d}x$ with $a<b$. Then simplify the radicand before integrating. Exam integrands are engineered so that $1+(f')^2$ is a perfect square, often because $(f')^2$ has the form $A^2-\frac{1}{2}+B^2$ with $AB=\frac{1}{4}$. Adding 1 then gives $(A+B)^2$. When the radicand is linear in x, substitute. Otherwise expect a standard integral such as $\int\sqrt{1+u^2}\,\mathrm{d}u$ (Section 9.4).
The integrand is a density, positive on every displacement. On an interval traversed from left to right it agrees with the 1-form $\sqrt{1+(f')^2}\,\mathrm{d}x$. A length is always positive and is at least the length of the chord.

Problems

8.6.1 Find the length of $y=\frac{2}{3}(x^2+1)^{3/2}$ for $0\le x\le 1$. ⋆ Hint · Answer · Solution

8.6.2 Find the length of $y=\dfrac{x^3}{6}+\dfrac{1}{2x}$ for $1\le x\le 2$. ⋆⋆ Hint · Answer · Solution

8.6.3 Find the length of $y=\dfrac{x^2}{8}-\ln x$ for $1\le x\le 2$. ⋆⋆ Hint · Answer · Solution

8.6.4 Let f' be continuous on $[a,b]$. Prove that the length L of the graph is at least the length $\sqrt{(b-a)^2+\big(f(b)-f(a)\big)^2}$ of the chord joining its endpoints. ⋆⋆ Hint · Answer · Solution

8.6.5 Let f' be continuous on $[a,b]$ and let h and k be constants. Prove that the graph of $g(x)=f(x-h)+k$ over $[a+h,b+h]$ has the same length as the graph of f over $[a,b]$. ⋆⋆ Hint · Answer · Solution

8.6.6 Let f' be continuous on $[a,b]$, let $\lambda>0$ and let $g(x)=\lambda f(x/\lambda)$ on $[\lambda a,\lambda b]$. Prove that the graph of g is λ times as long as the graph of f. ⋆⋆ Hint · Answer · Solution

8.6.7 Find the lengths of the graphs of $f(x)=x$ and of $2f$ over $[0,1]$. Is the map $f\mapsto L(f)$ from $C^1([0,1])$ to $\mathbb{R}$ linear? ⋆ Hint · Answer · Solution

8.6.8 For $n\ge 1$ let $f_n(x)=\dfrac{\sin(n^2x)}{n}$ on $[0,\pi]$. Show that $\|f_n\|_\infty\le\frac{1}{n}$ and that the graph of f_n has length at least $2n$. Conclude that $f\mapsto L(f)$ is not continuous with respect to the sup norm (Definition 3.6.9). ⋆⋆ Hint · Answer · Solution

Chapter 9

Techniques of Integration

Chapter 8 reduced the evaluation of the integral of a continuous function to the search for an antiderivative and proved that one always exists (Corollary 8.3.8). Existence does not produce a formula. This chapter proves the theorems that do. Each is a statement about 1-forms with hypotheses: rewriting the integrand as an equal function, linearity and pullback by a substitution or by a bijection (Section 9.1), the product rule for differentials integrated (Section 9.2), reduction formulae for trigonometric integrands (Section 9.3), pullback by trigonometric bijections (Section 9.4) and the partial fraction decomposition of a rational function (Section 9.5). Section 9.6 explains what a table entry or a computer algebra system asserts and how to check it. Section 9.7 approximates integrals that no formula reaches. The one-step error lemmas are proved in Appendix A and the bounds in Section 9.7. Section 9.8 extends the integral to unbounded intervals and unbounded integrands. The chapter uses the algebra of Section 4.3, the trigonometric identities of Section 4.5 and the substitution rule of Section 8.4 throughout.

Notation local to one technique stays with that technique: u and $\mathrm{d}v$ and LIATE in Section 9.2, the exponents m, n in Section 9.3, the substitution angle θ and its reference triangle in Section 9.4, the undetermined coefficients in Section 9.5, and the variable bound inside a limit in Section 9.8.

9.1 Strategy for a General Integrand

Overview

Objectives: State the operations on a 1-form that preserve its antiderivatives: replacement by an equal function on the interval, linearity and pullback. Prove the substitution theorem for a bijection (Theorem 9.1.4) and the arctangent and arcsine templates (Proposition 9.1.7). Choose a method by classifying the integrand, and check every answer by differentiation on a stated interval.

Prerequisites: Antiderivatives on an interval (Theorem 8.1.3 and Corollary 8.3.8); the substitution rule (Theorem 8.4.2); bijections and inverse functions (Definitions 1.5.15 and 1.5.24); the derivatives of arctan and arcsin (Theorem 6.4.10); completing the square (Section 4.3).

Key results: If $\varphi : J \to I$ is a differentiable bijection of intervals and $\mathrm{d}G = \varphi^*(f(x)\,\mathrm{d}x)$, then $\mathrm{d}(G \circ \varphi^{-1}) = f(x)\,\mathrm{d}x$. $\int \mathrm{d}u/(a^2+u^2) = \frac{1}{a}\arctan\frac{u}{a} + C$ and $\int \mathrm{d}u/\sqrt{a^2-u^2} = \arcsin\frac{u}{a} + C$ for $a > 0$.

Notation

I, J	intervals of the old variable x and of the new variable t
$\varphi : J \to I$	a substitution $x = \varphi(t)$ that is a bijection (Theorem 9.1.4)
$\varphi^*\omega$	pullback of the 1-form ω by φ (Definition 6.2.11)
a	the positive scale in the templates of Proposition 9.1.7

The table of Proposition 8.1.5 and the substitution rule integrate only integrands that already have a recognisable form. A method of integration transforms the 1-form $f(x)\,\mathrm{d}x$ into one whose antiderivative is known, and it is valid only when the transformation preserves the set of antiderivatives on the interval in question. Four operations have this property. Every method in this chapter combines them.

9.1.1 Legitimate operations on an integrand

Let f be continuous on an interval I.

1. *Replacement by an equal function.* If $g(x) = f(x)$ for every $x \in I$, then f and g have the same antiderivatives on I. Algebraic simplification, a trigonometric identity and a partial fraction decomposition are replacements of this kind. Each is valid only as an identity at every point of I.

2. *Linearity.* By Proposition 8.1.6 the antiderivatives of a sum are the sums of antiderivatives of the terms. In particular a numerator sum over a single denominator splits term by term:

$$\frac{f_1(x) + f_2(x) + \cdots + f_m(x)}{g(x)} = \frac{f_1(x)}{g(x)} + \frac{f_2(x)}{g(x)} + \cdots + \frac{f_m(x)}{g(x)}. \tag{9.1.1}$$

3. *Pullback.* By Theorem 8.4.2 an integrand of the form $h(g(x))\,g'(x)$ is the pullback $g^*(h(u)\,\mathrm{d}u)$ and has the antiderivatives $H(g(x)) + C$ with $H' = h$.

4. *Pullback by a bijection.* A substitution $x = \varphi(t)$ in the other direction replaces $f(x)\,\mathrm{d}x$ by $\varphi^*(f(x)\,\mathrm{d}x)$ on the t-interval. An antiderivative found there returns to x through φ^{-1} (Theorem 9.1.4 below).

The product rule integrated (Theorem 9.2.1) is a fifth operation and has a section of its own.

Example 9.1.2 (Replacement and its domain)**.** On each of the intervals $(-\infty, 4)$ and $(4, \infty)$, $\dfrac{x^2 - 16}{x - 4} = \dfrac{(x-4)(x+4)}{x-4} = x + 4$. The antiderivatives of the left side on $\mathbb{R} \setminus \{4\}$ are therefore $\frac{1}{2}x^2 + 4x + C_1$ on $(-\infty, 4)$ and $\frac{1}{2}x^2 + 4x + C_2$ on $(4, \infty)$, with two independent constants (Theorem 7.3.8). The polynomial $x + 4$ is continuous on all of $\mathbb{R}$. The quotient is undefined at 4. A cancellation is an identity on the set where both sides are defined and on no larger set.

Common misconception

The identity Eq. (9.1.1) distributes a numerator over a common denominator. Its mirror image for a denominator is false: $\dfrac{1}{g_1(x) + g_2(x)} \neq \dfrac{1}{g_1(x)} + \dfrac{1}{g_2(x)}$ in general. With $g_1 = x^2$ and $g_2 = 1$ at $x = 1$ the left side is $\frac{1}{2}$ and the right side is 2. The integrand $\dfrac{9}{x^2 + 1}$ has the antiderivative $9 \arctan x$. Splitting it as $\dfrac{9}{x^2} + 9$ changes the function.

9.1.3 Substitution by a bijection

The integrand of $\int \mathrm{d}x/\sqrt{1-x^2}$ contains no factor $g'(x)$ for an inner function g. The substitution $x = \sin t$ writes the old variable as a function of a new one. The pullback $\varphi^*(f(x)\,\mathrm{d}x) = f(\varphi(t))\varphi'(t)\,\mathrm{d}t$ can be simpler than $f(x)\,\mathrm{d}x$. An antiderivative in t must then be expressed in x. That step needs φ to be a bijection.

Theorem 9.1.4 (Substitution by a bijection)**.** *Let I and J be intervals, let $\varphi : J \to I$ be a differentiable bijection (Definition 1.5.15) and let f be continuous on I. If G is an antiderivative on J of $f(\varphi(t))\varphi'(t)$, that is, $\mathrm{d}G = \varphi^*(f(x)\,\mathrm{d}x)$, then $G \circ \varphi^{-1}$ is an antiderivative of f on I:*

$$\int f(x)\,\mathrm{d}x = G(\varphi^{-1}(x)) + C \qquad (x \in I). \tag{9.1.2}$$

Proof. By Corollary 8.3.8 the continuous function f has an antiderivative F on the interval I. By Theorem 8.4.2, $F \circ \varphi$ is an antiderivative of $f(\varphi(t))\varphi'(t)$ on J. So is G. Because J is an interval, Theorem 8.1.3 gives a constant C_0 with $G = F \circ \varphi + C_0$ on J. The inverse $\varphi^{-1} : I \to J$ exists because φ is a bijection (Definition 1.5.24). For $x \in I$,

$$G(\varphi^{-1}(x)) = F(\varphi(\varphi^{-1}(x))) + C_0 = F(x) + C_0.$$

Hence $G \circ \varphi^{-1}$ differs from F by a constant and is an antiderivative of f on I. ∎

The proof uses injectivity of φ to define φ^{-1} and surjectivity to define it on all of I. It uses no derivative of φ^{-1}. Many substitutions are bijections only after their domain is restricted. The restriction is part of the substitution.

Example 9.1.5 (Choice of the domain of a substitution)**.** The sine is a bijection from $J = (-\frac{\pi}{2}, \frac{\pi}{2})$ onto $I = (-1, 1)$ (Theorem 4.5.13) with inverse arcsin. On J the cosine is positive and $\sqrt{1-\sin^2 t} = \cos t$. The pullback of $\mathrm{d}x/\sqrt{1-x^2}$ is therefore $\cos t\,\mathrm{d}t/\cos t = \mathrm{d}t$ with antiderivative $G(t) = t$, and Theorem 9.1.4 gives $\int \mathrm{d}x/\sqrt{1-x^2} = \arcsin x + C$ on $(-1, 1)$. On $\mathbb{R}$ the sine is not injective. On the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ the pullback is undefined at the two endpoints where $\cos t = 0$. Neither choice satisfies the hypotheses. A second bijection onto $(-1, 1)$ gives an antiderivative that differs from $\arcsin x$ by a constant (Problem 9.1.5).

9.1.6 Arctangent and arcsine templates

Two antiderivatives occur so often after a substitution that this section states them with a scale $a > 0$.

Proposition 9.1.7 (Arctangent and arcsine templates)**.** *For $a > 0$,*

$$\int \frac{\mathrm{d}u}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C \qquad (u \in \mathbb{R}), \tag{9.1.3}$$

$$\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C \qquad (-a < u < a). \tag{9.1.4}$$

Proof. By the chain rule and Theorem 6.4.10,

$$\frac{\mathrm{d}}{\mathrm{d}u} \frac{1}{a} \arctan \frac{u}{a} = \frac{1}{a} \cdot \frac{1}{1 + u^2/a^2} \cdot \frac{1}{a} = \frac{1}{a^2 + u^2}$$

for every real u. For $-a < u < a$ the number u/a lies in $(-1, 1)$ and

$$\frac{\mathrm{d}}{\mathrm{d}u} \arcsin \frac{u}{a} = \frac{1}{\sqrt{1 - u^2/a^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2 - u^2}},$$

using $\sqrt{a^2 - u^2} = a\sqrt{1 - u^2/a^2}$ for $a > 0$. Each formula follows by Theorem 8.1.3 on the stated interval. ∎

A quadratic $x^2 + bx + c$ in a denominator is brought to one of these shapes by completing the square (Eq. (4.3.3)):

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4}. \tag{9.1.5}$$

When $b^2 - 4c < 0$ the constant is positive and equals a^2 with $a = \frac{1}{2}\sqrt{4c - b^2}$. The translation $u = x + \frac{b}{2}$ then leads to Eq. (9.1.3).

Caution

Completing the square identifies a^2 only when the leading coefficient is 1. In $\int \mathrm{d}x/(2x^2 + 4x + 10)$ factor out the 2 first: $2x^2 + 4x + 10 = 2((x + 1)^2 + 4)$.

Example 9.1.8 (Completing the square for an arctangent)**.** Since $x^2 - 6x + 13 = (x - 3)^2 + 2^2$, the substitution $u = x - 3$ and Eq. (9.1.3) with $a = 2$ give

$$\int \frac{\mathrm{d}x}{x^2 - 6x + 13} = \frac{1}{2} \arctan \frac{x - 3}{2} + C \qquad (x \in \mathbb{R}). \tag{9.1.6}$$

Example 9.1.9 (Completing the square for an arcsine)**.** Since $8 - 2x - x^2 = 9 - (x + 1)^2$, the radicand is positive exactly for $-4 < x < 2$. On that interval $u = x + 1$ and Eq. (9.1.4) with $a = 3$ give

$$\int \frac{\mathrm{d}x}{\sqrt{8 - 2x - x^2}} = \arcsin \frac{x + 1}{3} + C. \tag{9.1.7}$$

9.1.10 Choosing a method

No finite list of rules integrates every elementary function. The antiderivative $\int_0^x e^{t^2}\,\mathrm{d}t$ of e^{x^2} exists by Corollary 8.3.8. No function built from polynomials, exponentials, logarithms, trigonometric functions and their inverses by finitely many arithmetic operations and compositions equals it (a theorem of Liouville whose proof lies beyond this book). A strategy therefore orders the methods by what each can guarantee. Figure 9.1.1 sets out the order.

Substitution and parts keep the shape of the integrand recognisable and finish quickly when they apply. Neither is an algorithm. No theorem guarantees that a given g or a given factorisation leads anywhere. The last three classes are complete once each denominator is factored into real linear factors and irreducible quadratics. A rational function with a factored denominator has an antiderivative built from rational functions, logarithms and arctangents (Theorem 9.5.7). The factorisation exists for every real polynomial by the fundamental theorem of algebra. This book uses that theorem without proof. The Weierstrass substitution (Theorem 9.3.15) turns every rational function of $\sin x$ and $\cos x$ into a rational function of t. Trigonometric substitution turns a rational function of x and the square root of a quadratic into a rational function of $\sin\theta$ and $\cos\theta$ (Proposition 9.4.4). An integrand outside every class still has the area-so-far antiderivative of Corollary 8.3.8. Its definite integrals are computed numerically (Section 9.7).

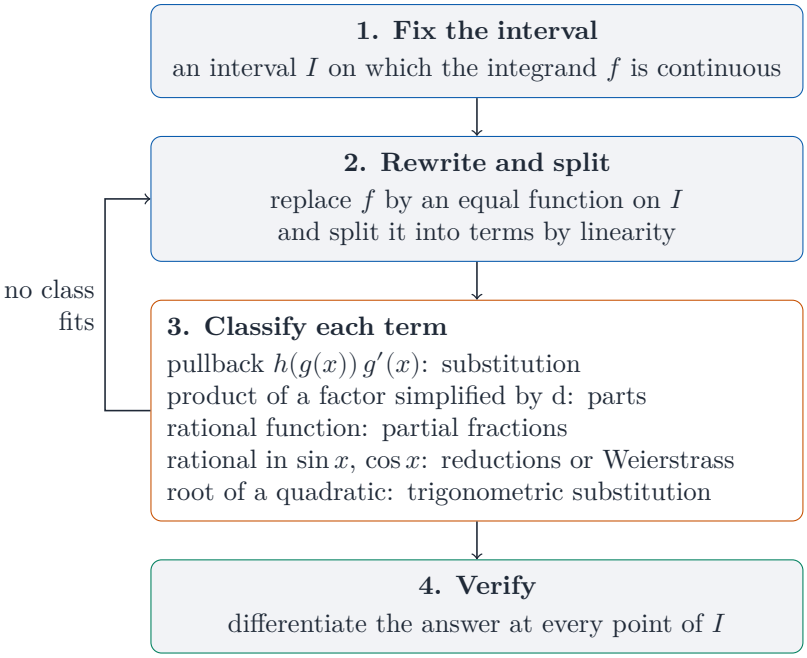


Figure 9.1.1: Strategy for an integral $\int f(x)\,\mathrm{d}x$. The first two steps and the last are compulsory. Of the classes in step 3 the last three always succeed once each denominator is factored (Theorems 9.3.15 and 9.5.7 and Proposition 9.4.4).

Worked Example 9.1.1

Evaluate $\displaystyle\int \frac{\sec\theta}{\sec\theta-\cos\theta}\,\mathrm{d}\theta$.

The integrand is defined where $\cos\theta\neq 0$ and $\sec\theta\neq\cos\theta$, that is, where $\cos^2\theta\neq 1$ as well. Both conditions together say $\sin\theta\neq 0$ and $\cos\theta\neq 0$. Work therefore on an interval I between consecutive multiples of $\frac{\pi}{2}$. On I the Pythagorean identity of Theorem 4.5.7 gives

$$\sec\theta-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta},\qquad \frac{\sec\theta}{\sec\theta-\cos\theta}=\frac{1/\cos\theta}{\sin^2\theta/\cos\theta}=\frac{1}{\sin^2\theta}.$$

Since $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta=-\dfrac{1}{\sin^2\theta}$ by the quotient rule, the answer on I is $-\cot\theta+C$. The function $1/\sin^2\theta$ is defined at the multiples of $\frac{\pi}{2}$ where $\cos\theta=0$. The original integrand is not. The replacement is an identity on I only.

Worked Example 9.1.2

Evaluate $\displaystyle\int \frac{2x+3}{x^2+2x+5}\,\mathrm{d}x$.

The denominator has discriminant $4-20<0$ and no real zero. Hence $I=\mathbb{R}$. The derivative of the denominator is $2x+2$. Splitting the numerator as $(2x+2)+1$ and using Eq. (9.1.1),

$$\int \frac{2x+3}{x^2+2x+5}\,\mathrm{d}x=\int \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x+\int \frac{\mathrm{d}x}{(x+1)^2+4}.$$

The first term is a pullback by $g(x) = x^2 + 2x + 5 > 0$ and integrates to $\ln(x^2 + 2x + 5)$ (Problem 8.4.7). The second is Eq. (9.1.3) with $u = x + 1$ and $a = 2$. Hence

$$\int \frac{2x+3}{x^2+2x+5}\,\mathrm{d}x = \ln(x^2+2x+5) + \frac{1}{2}\arctan\frac{x+1}{2} + C.$$

Worked Example 9.1.3

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x}\,(1+x)}$ on $(0,\infty)$.

No factor of the integrand is the derivative of an inner function. The root suggests $x = \varphi(t) = t^2$. On $J = (0,\infty)$ the map φ is a differentiable bijection onto $I = (0,\infty)$ with inverse $\sqrt{x}$. Since $\sqrt{t^2} = t$ for $t > 0$, the pullback is

$$\varphi^*\Big(\frac{\mathrm{d}x}{\sqrt{x}\,(1+x)}\Big) = \frac{2t\,\mathrm{d}t}{t\,(1+t^2)} = \frac{2\,\mathrm{d}t}{1+t^2},$$

with antiderivative $G(t) = 2\arctan t$. By Theorem 9.1.4,

$$\int \frac{\mathrm{d}x}{\sqrt{x}\,(1+x)} = 2\arctan\sqrt{x} + C \qquad (x > 0).$$

Differentiating confirms the answer: $\dfrac{\mathrm{d}}{\mathrm{d}x} 2\arctan\sqrt{x} = \dfrac{2}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$.

Study guide

Before integrating, name the interval. An antiderivative is a statement about one interval. Every rewriting of the integrand must be an identity on that interval. Rewrite and split the integrand and then classify each term in the order of Fig. 9.1.1. A substitution $u = g(x)$ needs no injectivity. A substitution $x = \varphi(t)$ must be a bijection between the two intervals. Its domain is part of its statement. Finish by differentiating the answer.

Completing the square reduces every quadratic $x^2 + bx + c$ to $(x + \frac{b}{2})^2$ plus a constant. When the constant is positive the arctangent template applies. When it is zero or negative the quadratic factors over $\mathbb{R}$ and partial fractions apply.

Problems

9.1.1 Evaluate $\displaystyle\int \frac{x\sqrt{x} + 2\sqrt{x}}{x^2}\,\mathrm{d}x$ on $(0,\infty)$. ⋆ Hint · Answer · Solution

9.1.2 Evaluate $\displaystyle\int \frac{\mathrm{d}x}{2x^2+4x+10}$. ⋆ Hint · Answer · Solution

9.1.3 Evaluate $\displaystyle\int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x$ on $(0,\infty)$ with the substitution $x = t^2$ of Worked Example 9.1.3. ⋆⋆ Hint · Answer · Solution

9.1.4 Let $b^2 - 4c < 0$ and $d = \sqrt{4c - b^2}$. Prove that

$$\int \frac{\mathrm{d}x}{x^2+bx+c} = \frac{2}{d}\arctan\frac{2x+b}{d} + C \qquad (x \in \mathbb{R}).$$

⋆⋆ Hint · Answer · Solution

9.1.5 Show that $\varphi(t) = \sin t$ is a bijection from $(\frac{\pi}{2}, \frac{3\pi}{2})$ onto $(-1, 1)$ with inverse $\varphi^{-1}(x) = \pi - \arcsin x$. Use Theorem 9.1.4 with this φ to find $\int \mathrm{d}x/\sqrt{1-x^2}$ on $(-1, 1)$ and compare the result with Example 9.1.5. ⋆⋆ Hint · Answer · Solution

9.1.6 Find every antiderivative of $\dfrac{x^2 - 16}{x - 4}$ on $\mathbb{R} \setminus \{4\}$. Is $\frac{1}{2}x^2 + 4x$ an antiderivative of this function on $\mathbb{R}$? ⋆ Hint · Answer · Solution

9.1.7 Explain why e^{x^2} has an antiderivative on $\mathbb{R}$, give one, and find its derivative at 0 and its value at 0. Which class of Fig. 9.1.1 does the integrand belong to?
⋆⋆ Hint · Answer · Solution

9.1.8 Let f be continuous and odd on $(-a, a)$. Prove that every antiderivative of f on $(-a, a)$ is even by pulling $f(x)\,\mathrm{d}x$ back with the bijection $\varphi(t) = -t$.
⋆⋆ Hint · Answer · Solution

9.2 Integration by Parts

Overview

Objectives: Prove integration by parts as the product rule for differentials integrated, with its hypotheses (Theorem 9.2.1). Choose the two factors so that the new integral is simpler and recognise when the LIATE heuristic fails. Prove reduction formulae and their closed forms by induction (Theorems 9.2.6 and 9.2.7). Prove the tabular method (Theorem 9.2.9) and resolve a cyclic integral by solving for it.
Prerequisites: The product rule for differentials (Eq. (6.3.6)); the Fundamental Theorem (Theorem 8.3.5 and Eq. (8.3.6)); existence of antiderivatives (Corollary 8.3.8); induction (Theorem 1.4.2); the spaces $C^k(I)$ (Definition 7.3.2).
Key results: For $u, v \in C^1(I)$, $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$ and $\int_a^b u\,\mathrm{d}v = [uv]_a^b - \int_a^b v\,\mathrm{d}u$. For a polynomial u of degree n and iterated antiderivatives V_k of V_0, $\int uV_0\,\mathrm{d}x = \sum_{k=0}^{n}(-1)^k u^{(k)}V_{k+1} + C$.

Notation

$u,\ \mathrm{d}v$	a function and a 1-form whose product is the integrand
$\mathrm{d}u,\ v$	$\mathrm{d}u = u'(x)\,\mathrm{d}x$ (Eq. (6.2.6)) and v an antiderivative of $\mathrm{d}v$
$C^1(I)$	functions with a continuous derivative on the interval I (Definition 7.3.2)
V_k	iterated antiderivatives: $V_k' = V_{k-1}$ (Theorem 9.2.9)
LIATE	a heuristic order of kinds for the choice of u

The table of Proposition 8.1.5 has no entry for $x\cos x$. No substitution reduces it to one. The product rule produces such integrands: $\dfrac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = \sin x + x\cos x$. Substitution is the chain rule for differentials integrated (Theorem 8.4.2). Integration by parts is the product rule for differentials Eq. (6.3.6) integrated in the same way.

Theorem 9.2.1 (Integration by parts)**.** *Let u and v have continuous derivatives on an interval I.*

1. *If G is an antiderivative of $v\,u'$ on I, then $uv - G$ is an antiderivative of $u\,v'$ on I. In the notation of Definition 8.1.4,*
$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u. \tag{9.2.1}$$
2. *For $a, b \in I$,*
$$\int_a^b u\,\mathrm{d}v = [uv]_a^b - \int_a^b v\,\mathrm{d}u. \tag{9.2.2}$$

Proof. By the product rule for differentials Eq. (6.3.6), $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$ on I. Both terms are continuous multiples of $\mathrm{d}x$ because u, v, u' and v' are continuous.
(i) Indefinite form. If $\mathrm{d}G = v\,\mathrm{d}u$, then $\mathrm{d}(uv - G) = u\,\mathrm{d}v + v\,\mathrm{d}u - v\,\mathrm{d}u = u\,\mathrm{d}v$. The identity of families Eq. (9.2.1) follows by Theorem 8.1.3.
(ii) Definite form. The two 1-forms $u\,\mathrm{d}v$ and $v\,\mathrm{d}u$ have continuous coefficients on the closed interval between a and b and are integrable there (Theorem 8.2.13). Integrating $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$ and applying Eq. (8.3.6) to the left side gives $[uv]_a^b = \int_a^b u\,\mathrm{d}v + \int_a^b v\,\mathrm{d}u$. ∎

Only one antiderivative v of $\mathrm{d}v$ is needed. Replacing v by $v+k$ for a constant k adds ku to uv and $k\,\mathrm{d}u$ to $v\,\mathrm{d}u$. The second addition contributes ku to $\int v\,\mathrm{d}u$. The two contributions cancel in Eq. (9.2.1).

Example 9.2.2 (Hypotheses of the definite form)**.** The logarithm is undefined at 0. Hence Eq. (9.2.2) does not apply to $\int_0^1 \ln x\,\mathrm{d}x$ with $u=\ln x$. On $[\varepsilon,1]$ with $0<\varepsilon<1$ the hypotheses are satisfied by $u=\ln x$ and $v=x$:

$$\int_\varepsilon^1 \ln x\,\mathrm{d}x = \big[x\ln x\big]_\varepsilon^1 - \int_\varepsilon^1 \mathrm{d}x = -\varepsilon\ln\varepsilon - 1 + \varepsilon.$$

As $\varepsilon\to 0^+$ this tends to -1 because $\varepsilon\ln\varepsilon = -(\ln x)/x \to 0$ with $x=1/\varepsilon\to\infty$ (Theorem 5.2.26). The value -1 is the improper integral of Section 9.8. Integration by parts reaches it only through integrals over intervals on which the hypotheses hold.

Worked Example 9.2.1

Evaluate $\displaystyle\int xe^{2x}\,\mathrm{d}x$.

Differentiation simplifies x to 1. The factor e^{2x} has the antiderivative $\frac{1}{2}e^{2x}$. On $I=\mathbb{R}$ take

$$u=x,\qquad \mathrm{d}v=e^{2x}\,\mathrm{d}x,\qquad \mathrm{d}u=\mathrm{d}x,\qquad v=\tfrac{1}{2}e^{2x}.$$

Then

$$\int xe^{2x}\,\mathrm{d}x \overset{(9.2.1)}{=\!=} \tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}\,\mathrm{d}x = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C.$$

The derivative of the answer is $\frac{1}{2}e^{2x} + xe^{2x} - \frac{1}{2}e^{2x} = xe^{2x}$.

Worked Example 9.2.2

Evaluate $\displaystyle\int \arcsin x\,\mathrm{d}x$ on $(-1,1)$.

The derivative of arcsin is algebraic while arcsin itself has no entry in the table. With no second factor visible, take $\mathrm{d}v=\mathrm{d}x$:

$$u=\arcsin x,\qquad \mathrm{d}v=\mathrm{d}x,\qquad \mathrm{d}u=\frac{\mathrm{d}x}{\sqrt{1-x^2}},\qquad v=x.$$

Both u and v have continuous derivatives on $(-1,1)$, and

$$\int \arcsin x\,\mathrm{d}x \overset{(9.2.1)}{=\!=} x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = x\arcsin x + \sqrt{1-x^2} + C.$$

The last integral is a pullback by $w=1-x^2$ with $\mathrm{d}w=-2x\,\mathrm{d}x$: $\int x(1-x^2)^{-1/2}\,\mathrm{d}x = -\frac{1}{2}\int w^{-1/2}\,\mathrm{d}w = -\sqrt{1-x^2}+C$.

9.2.3 Choice of the factors

Theorem 9.2.1 replaces $\int u\,\mathrm{d}v$ by $\int v\,\mathrm{d}u$. It helps when two conditions hold together: an antiderivative v of $\mathrm{d}v$ is known, and $v\,u'$ is simpler than $u\,v'$. A polynomial factor loses one degree under differentiation. A logarithm or an inverse trigonometric function becomes algebraic. An

exponential, a sine or a cosine keeps its kind under both differentiation and integration. The mnemonic LIATE (logarithm, inverse trigonometric, algebraic, trigonometric, exponential) lists the kinds in the order in which they are usually best chosen as u. It is a heuristic. The two conditions above decide the choice.

Worked Example 9.2.3

Evaluate $\int x^3 e^{x^2}\,\mathrm{d}x$.

LIATE suggests $u = x^3$ and $\mathrm{d}v = e^{x^2}\,\mathrm{d}x$. The first condition then fails: no elementary v has $\mathrm{d}v = e^{x^2}\,\mathrm{d}x$ by the theorem of Liouville quoted in Section 9.1. The factor $x\,\mathrm{d}x$ is the pullback of $\frac{1}{2}\mathrm{d}w$ by $w = x^2$. Hence $x\,e^{x^2}$ has the antiderivative $\frac{1}{2}e^{x^2}$. Take

$$u = x^2, \qquad \mathrm{d}v = x\,e^{x^2}\,\mathrm{d}x, \qquad \mathrm{d}u = 2x\,\mathrm{d}x, \qquad v = \tfrac{1}{2}e^{x^2}.$$

Then

$$\int x^3 e^{x^2}\,\mathrm{d}x \overset{(9.2.1)}{=\!=} \tfrac{1}{2}x^2 e^{x^2} - \int x\,e^{x^2}\,\mathrm{d}x = \tfrac{1}{2}(x^2-1)e^{x^2} + C.$$

The split puts part of the algebraic factor into $\mathrm{d}v$, against the mnemonic.

Common misconception

LIATE does not decide the choice of u. It fails in Worked Example 9.2.3. For $\int x\sqrt{x+1}\,\mathrm{d}x$ the substitution $u = x+1$ of Worked Example 8.4.3 is shorter than any choice of parts. Check the two conditions: a computable v and a new integral simpler than the old one.

9.2.4 Cyclic integrals

When neither factor simplifies under differentiation, two applications of Theorem 9.2.1 can return the original integral with a coefficient different from 1. The equation can then be solved for it.

Worked Example 9.2.4

Evaluate $\int e^x \sin x\,\mathrm{d}x$.

On $\mathbb{R}$ take $u = \sin x$ and $v = e^x$:

$$\int e^x \sin x\,\mathrm{d}x \overset{(9.2.1)}{=\!=} e^x \sin x - \int e^x \cos x\,\mathrm{d}x.$$

With $u = \cos x$ and $v = e^x$ again,

$$\int e^x \cos x\,\mathrm{d}x \overset{(9.2.1)}{=\!=} e^x \cos x + \int e^x \sin x\,\mathrm{d}x.$$

Both applications take the trigonometric factor as u. Choosing $u = e^x$ in the second would undo the first and return an empty identity. Let I be one antiderivative of $e^x \sin x$. By Theorem 9.2.1(i) applied twice, $e^x \sin x - e^x \cos x - I$ is an antiderivative of $e^x \sin x$. By Theorem 8.1.3 it equals $I + C_0$ for a constant C_0. Hence $2I = e^x(\sin x - \cos x) - C_0$. Restoring the constant,

$$\int e^x \sin x\,\mathrm{d}x = \frac{e^x}{2}(\sin x - \cos x) + C.$$

9.2.5 Reduction formulae

A *reduction formula* expresses an integral that depends on an integer n through the same integral with a smaller n. Integration by parts proves one. Induction turns it into a closed form.

Theorem 9.2.6 (Exponential-power reduction formula)**.** *For an integer $n \geq 1$ and a constant $a \neq 0$, on $\mathbb{R}$,*

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x. \tag{9.2.3}$$

Proof. Apply Theorem 9.2.1 with $u = x^n$ and $\mathrm{d}v = e^{ax}\,\mathrm{d}x$, for which $\mathrm{d}u = nx^{n-1}\,\mathrm{d}x$ and $v = e^{ax}/a$. Both have continuous derivatives on $\mathbb{R}$. With these choices Eq. (9.2.1) is Eq. (9.2.3). ∎

Theorem 9.2.7 (Closed form for $\int x^n e^{ax}\,\mathrm{d}x$)**.** *For every integer $n \geq 0$ and every constant $a \neq 0$,*

$$\int x^n e^{ax}\,\mathrm{d}x = e^{ax}\sum_{k=0}^{n}(-1)^k \frac{n!}{(n-k)!}\,\frac{x^{n-k}}{a^{k+1}} + C. \tag{9.2.4}$$

Proof. By induction on n (Theorem 1.4.2). For $n = 0$ the right side is $e^{ax}/a + C$, the antiderivative of e^{ax} (Proposition 8.1.5). Let $n \geq 1$ and assume Eq. (9.2.4) for $n-1$. By Eq. (9.2.3),

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a}\,e^{ax}\sum_{k=0}^{n-1}(-1)^k \frac{(n-1)!}{(n-1-k)!}\,\frac{x^{n-1-k}}{a^{k+1}} + C.$$

With $j = k+1$ the term of index k in the sum, multiplied by $-n/a$, is

$$(-1)^j\,\frac{n\,(n-1)!}{(n-j)!}\,\frac{x^{n-j}}{a^{j+1}} = (-1)^j\frac{n!}{(n-j)!}\,\frac{x^{n-j}}{a^{j+1}}, \qquad j = 1, \dots, n.$$

The first term $x^n e^{ax}/a$ is the term $j = 0$ of Eq. (9.2.4). Hence Eq. (9.2.4) holds for n. ∎

9.2.8 Tabular integration

Repeated integration by parts on a product $u\,V_0$ differentiates u and integrates V_0 at every step, with alternating signs. The bookkeeping can be arranged in two columns. The theorem below states what the columns compute and why the process stops.

Theorem 9.2.9 (Tabular integration)**.** *Let $n \geq 0$, let $u \in C^{n+1}(I)$ on an interval I, let V_0 be continuous on I, and let $V_1, \dots, V_{n+1}$ satisfy $V_k' = V_{k-1}$ on I for $k = 1, \dots, n+1$. Then*

$$\int u\,V_0\,\mathrm{d}x = \sum_{k=0}^{n}(-1)^k u^{(k)} V_{k+1} + (-1)^{n+1}\int u^{(n+1)}V_{n+1}\,\mathrm{d}x. \tag{9.2.5}$$

If u is a polynomial of degree at most n, then $\int u\,V_0\,\mathrm{d}x = \sum_{k=0}^{n}(-1)^k u^{(k)}V_{k+1} + C$.

Proof. The functions V_k exist by Corollary 8.3.8. For $n = 0$, Eq. (9.2.5) is Eq. (9.2.1) with $\mathrm{d}v = V_0\,\mathrm{d}x$ and $v = V_1$. Let $n \geq 1$ and assume Eq. (9.2.5) for $n-1$ and every $u \in C^n(I)$. For $u \in C^{n+1}(I)$ the last integral of the case $n-1$ is $(-1)^n\int u^{(n)}V_n\,\mathrm{d}x$. Since $u^{(n)}$ has a continuous derivative, Theorem 9.2.1 with $\mathrm{d}v = V_n\,\mathrm{d}x$ and $v = V_{n+1}$ gives

$$(-1)^n\int u^{(n)}V_n\,\mathrm{d}x = (-1)^n u^{(n)}V_{n+1} + (-1)^{n+1}\int u^{(n+1)}V_{n+1}\,\mathrm{d}x. \tag{9.2.6}$$

Substituting (9.2.6) into the case $n-1$ gives Eq. (9.2.5). When u is a polynomial of degree at most n, its derivative $u^{(n+1)}$ is 0 and the last integral in Eq. (9.2.5) is a constant. ∎

The polynomial case is the usual tabular method: list $u, u', u'', \ldots$ down to 0 beside $V_1, V_2, \ldots$, multiply along the diagonals and alternate the signs. For $\int e^x \sin x \, dx$ no derivative of $\sin x$ vanishes and the remainder in Eq. (9.2.5) never disappears. The method then gives the cyclic equation of Worked Example 9.2.4 (Problem 9.2.10).

Worked Example 9.2.5

Evaluate $\int x^3 e^{2x} \, dx$ by Theorem 9.2.9 and compare with Theorem 9.2.7.

Take the cubic $u = x^3$ and $V_0 = e^{2x}$ with $V_k = e^{2x}/2^k$. The columns are

k	0	1	2	3	4
$u^{(k)}$	x^3	$3x^2$	$6x$	6	0
V_{k+1}	$\frac{1}{2}e^{2x}$	$\frac{1}{4}e^{2x}$	$\frac{1}{8}e^{2x}$	$\frac{1}{16}e^{2x}$	

and Eq. (9.2.5) with $n = 3$ gives

$$\int x^3 e^{2x} \, dx = \Big(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{6x}{8} - \frac{6}{16}\Big)e^{2x} + C = \Big(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\Big)e^{2x} + C.$$

Equation (9.2.4) with $n = 3$ and $a = 2$ has the coefficients $\frac{3!}{3!\,2} = \frac{1}{2}$, $-\frac{3!}{2!\,4} = -\frac{3}{4}$, $\frac{3!}{1!\,8} = \frac{3}{4}$ and $-\frac{3!}{0!\,16} = -\frac{3}{8}$, the same polynomial.

Caution

In the definite form Eq. (9.2.2) the term uv is evaluated at both limits. Two errors are common: writing uv at the upper limit alone and dropping the minus sign in front of $\int v \, du$. A definite computation leaves no antiderivative to differentiate. Derive the indefinite form Eq. (9.2.1) first, check it by differentiation and then evaluate it at the limits.

Study guide

Integration by parts is the product rule for differentials integrated. It needs u and v with continuous derivatives on the interval. The definite form needs them on the closed interval of integration. Before applying it, write u, dv, du and v on one line and check that v is known and that $v\,du$ is simpler than $u\,dv$. LIATE suggests a choice and can be wrong.
For a polynomial times a function whose iterated antiderivatives are known, use the tabular method. It is Theorem 9.2.1 applied $n+1$ times and stops because the $(n+1)$th derivative of the polynomial is 0. A reduction formula proved by parts becomes a closed form by induction. When the original integral reappears after a second application, solve for it.

Problems

9.2.1 Evaluate $\int x^2 e^x \, dx$ by Theorem 9.2.9. ⋆ Hint · Answer · Solution

9.2.2 Evaluate $\int e^x \cos x \, dx$. ⋆⋆ Hint · Answer · Solution

9.2.3 Evaluate $\int \arctan x \, dx$. ⋆ Hint · Answer · Solution

9.2.4 Evaluate $\int_0^1 x e^{-x} \, dx$. ⋆ Hint · Answer · Solution

9.2.5 Let $f \in C^{n+1}(I)$ on an open interval I and let $a, x \in I$. Prove by induction on n that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t.$$

Deduce that the remainder is at most $\dfrac{M|x-a|^{n+1}}{(n+1)!}$ when $|f^{(n+1)}| \le M$ between a and x. ⋆⋆⋆ Hint · Answer · Solution

9.2.6 Prove that for every integer $n \ge 1$, on $(0,\infty)$,

$$\int (\ln x)^n\,\mathrm{d}x = x(\ln x)^n - n\int (\ln x)^{n-1}\,\mathrm{d}x,$$

and use the formula to evaluate $\int (\ln x)^3\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.2.7 Integration by parts with $u = \dfrac{1}{x}$ and $\mathrm{d}v = \mathrm{d}x$ on $(0,\infty)$ gives

$$\int \frac{\mathrm{d}x}{x} = \frac{1}{x}\cdot x - \int x\cdot\Big(-\frac{1}{x^2}\Big)\mathrm{d}x = 1 + \int \frac{\mathrm{d}x}{x}.$$

Subtracting $\int \mathrm{d}x/x$ from both sides seems to give $0 = 1$. Find the error.
⋆⋆ Hint · Answer · Solution

9.2.8 Evaluate $\displaystyle\int x^5 e^{x^3}\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.2.9 Let $f \in C^1([a,b])$ and $\lambda > 0$. Prove that

$$\Big|\int_a^b f(x)\sin(\lambda x)\,\mathrm{d}x\Big| \le \frac{|f(a)| + |f(b)| + (b-a)\|f'\|_\infty}{\lambda},$$

and deduce that the integral tends to 0 as $\lambda \to \infty$. ⋆⋆ Hint · Answer · Solution

9.2.10 Apply Theorem 9.2.9 with $n = 1$, $u = \sin x$ and $V_0 = e^x$. Show that the formula contains the original integral again and solve for it. Why does the tabular method never terminate for this integrand? ⋆⋆ Hint · Answer · Solution

9.3 Trigonometric Integrals

Overview

Objectives: Prove that an odd power of $\sin x$ or $\cos x$ turns $\sin^m x \cos^n x \, dx$ into the pullback of a polynomial 1-form (Theorem 9.3.4) and that a product of even powers is a trigonometric polynomial (Theorem 9.3.7). Prove the reduction formulae for $\int \sin^n x \, dx$ and $\int \sec^n x \, dx$ (Theorems 9.3.9 and 9.3.13) and the closed forms of the integrals $\int_0^{\pi/2} \sin^n x \, dx$ (Proposition 9.3.10). State the Weierstrass substitution as the pullback by a bijection onto $(-\pi, \pi)$ and use it on a rational function of $\sin x$ and $\cos x$ (Theorem 9.3.15).
Prerequisites: The Pythagorean identity, the angle-sum formulas and the half-angle formulas (Theorems 4.5.7 and 4.5.9 and Eq. (4.5.5)); substitution and substitution by a bijection (Theorems 8.4.2 and 9.1.4); integration by parts (Theorem 9.2.1); induction (Theorem 1.4.2).
Key results: $\sin^2 x + \cos^2 x = 1$ and $\tan^2 x + 1 = \sec^2 x$; the half-angle formulas Eq. (9.3.2); the reduction formulae Eqs. (9.3.7) and (9.3.12); the substitution $x = 2 \arctan t$ with Eq. (9.3.13).

Notation

m, n	non-negative integer exponents in $\sin^m x \cos^n x$ or $\tan^m x \sec^n x$
p, q	frequencies in a product $\sin px \cos qx$
W_n	the integral $\int_0^{\pi/2} \sin^n x \, dx$ (Proposition 9.3.10)
t	the Weierstrass variable, $x = 2 \arctan t$ and $t = \tan(x/2)$
$R(s, c)$	a rational function of two variables

A product of powers of $\sin x$ and $\cos x$ is continuous on $\mathbb{R}$ and has an antiderivative there (Corollary 8.3.8). The table of Proposition 8.1.5 contains only the first powers. This section proves that every such product has an antiderivative expressible in elementary terms. The same holds for every rational function of $\sin x$ and $\cos x$ once the denominators that arise are factored (Theorems 9.3.15 and 9.5.7).

9.3.1 Identities

The Pythagorean identity of Theorem 4.5.7 is restated here for reference. For every real x,

$$\sin^2 x + \cos^2 x = 1. \tag{9.3.1}$$

Dividing by $\cos^2 x$ where $\cos x \neq 0$ gives $\tan^2 x + 1 = \sec^2 x$. Solving the double-angle formulas Eq. (4.5.4) for the squares gives the half-angle formulas Eq. (4.5.5), restated as

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}. \tag{9.3.2}$$

Proposition 9.3.2 (Product-to-sum identities)**.** *For all real p, q and x,*

$$\begin{aligned} \sin px \cos qx &= \tfrac{1}{2}[\sin(p-q)x + \sin(p+q)x], \\ \sin px \sin qx &= \tfrac{1}{2}[\cos(p-q)x - \cos(p+q)x], \\ \cos px \cos qx &= \tfrac{1}{2}[\cos(p-q)x + \cos(p+q)x]. \end{aligned} \tag{9.3.3}$$

Proof. Put $\alpha = px$ and $\beta = qx$ in the four formulas of Theorem 4.5.9. Adding the formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ gives $2\sin\alpha\cos\beta$. Subtracting the formula for $\cos(\alpha+\beta)$ from that for $\cos(\alpha-\beta)$ gives $2\sin\alpha\sin\beta$. Adding the same two formulas gives $2\cos\alpha\cos\beta$. ∎

9.3.3 Odd powers

Theorem 9.3.4 (Sine-cosine products with an odd power)**.** *Let $m, n \geq 0$ be integers.*

1. *If $m = 2k+1$ is odd, then on $\mathbb{R}$*

$$\sin^m x \cos^n x = (1-\cos^2 x)^k \cos^n x \sin x, \tag{9.3.4}$$

and $\sin^m x\cos^n x\,dx$ is the pullback by $g(x) = \cos x$ of the polynomial 1-form $-(1-u^2)^k u^n\,du$. If P is a polynomial with $P'(u) = (1-u^2)^k u^n$, then $\int \sin^m x\cos^n x\,dx = -P(\cos x) + C$.

2. *If $n = 2k+1$ is odd, then on $\mathbb{R}$*

$$\sin^m x \cos^n x = \sin^m x\,(1-\sin^2 x)^k \cos x, \tag{9.3.5}$$

and if $Q'(u) = u^m(1-u^2)^k$ for a polynomial Q, then $\int \sin^m x\cos^n x\,dx = Q(\sin x) + C$.

Proof. **(i)** Odd power of sine. By Eq. (9.3.1), $\sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x$. This is Eq. (9.3.4). With $h(u) = -(1-u^2)^k u^n$ and $g = \cos$, Eq. (6.2.8) and Theorem 6.4.8 give

$$g^*(h(u)\,du) = -(1-\cos^2 x)^k \cos^n x \cdot (-\sin x)\,dx = \sin^m x \cos^n x\,dx.$$

The polynomial $-P$ is an antiderivative of h on $\mathbb{R}$, and Theorem 8.4.2 gives the antiderivatives $-P(\cos x) + C$.
(ii) Odd power of cosine. The argument is the same with sin and cos exchanged and $g = \sin$, for which $g^*(du) = \cos x\,dx$. ∎

When m and n are both odd, either part applies. The two answers differ by a constant (Problem 9.6.2).

Worked Example 9.3.1

Evaluate $\displaystyle\int \sin^3 x \cos^4 x\,dx$.

The exponent $m = 3$ is odd. By Theorem 9.3.4(i) with $k = 1$ and $n = 4$, the integrand is the pullback by $u = \cos x$ of $-(1-u^2)u^4\,du = (u^6 - u^4)\,du$. Hence

$$\int \sin^3 x \cos^4 x\,dx = \int (u^6 - u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.$$

Rewriting the even factor $\cos^4 x$ in terms of $\sin x$ would leave $\sin^3 x$ times a polynomial in $\sin x$ with no factor $\cos x$ to form $d(\sin x)$.

Worked Example 9.3.2

Evaluate $\displaystyle\int \sin^2 x \cos^5 x\,dx$.

The exponent $n = 5$ is odd. By Theorem 9.3.4(ii) with $k = 2$ and $m = 2$, the integrand is the pullback by $u = \sin x$ of $u^2(1-u^2)^2\,du$. Hence

$$\int \sin^2 x \cos^5 x\,dx = \int (u^2 - 2u^4 + u^6)\,du = \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C.$$

9.3.5 Even powers

When m and n are both even, no factor $\sin x$ or $\cos x$ is left over to form a pullback. The half-angle formulas trade each square for a first power of $\cos 2x$. The following theorem shows that the process ends.

Lemma 9.3.6 (Powers of the cosine)**.** *For every integer $k \geq 0$ there are constants $a_0, \dots, a_k$ with $\cos^k\theta = \sum_{j=0}^{k} a_j \cos(j\theta)$ for every real θ.*

Proof. By induction on k. For $k = 0$ and $k = 1$ take $\cos^0\theta = \cos 0\theta$ and $\cos^1\theta = \cos\theta$. If $\cos^k\theta = \sum_{j=0}^{k} a_j\cos(j\theta)$, then by Eq. (9.3.3) with $p = 1$ and $q = j$,

$$\cos^{k+1}\theta = \sum_{j=0}^{k} a_j \cos\theta\cos(j\theta) = \sum_{j=0}^{k} \frac{a_j}{2}\big[\cos((1-j)\theta) + \cos((1+j)\theta)\big].$$

Since the cosine is even, $\cos((1-j)\theta) = \cos((j-1)\theta)$. Every frequency on the right lies between 0 and $k+1$. ∎

Theorem 9.3.7 (Sine-cosine products with even powers)**.** *Let m and n be even and $N = (m+n)/2$. There are constants $c_0, \dots, c_N$ with $\sin^m x\cos^n x = \sum_{j=0}^{N} c_j\cos(2jx)$ on $\mathbb{R}$, and*

$$\int \sin^m x\cos^n x\,dx = c_0 x + \sum_{j=1}^{N} \frac{c_j}{2j}\sin(2jx) + C. \tag{9.3.6}$$

Proof. By Eq. (9.3.2), $\sin^m x\cos^n x$ equals the polynomial $\left(\frac{1-\cos 2x}{2}\right)^{m/2}\left(\frac{1+\cos 2x}{2}\right)^{n/2}$ of degree N in $\cos 2x$. By Lemma 9.3.6 with $\theta = 2x$, each power $\cos^k 2x$ with $k \leq N$ is a combination of $\cos(2jx)$ with $j \leq k$. Collecting terms gives the constants c_j. The antiderivative of $\cos(2jx)$ is $\sin(2jx)/(2j)$ for $j \geq 1$ (Problem 8.4.4). Linearity gives the formula. ∎

The proof is also the computation: apply Eq. (9.3.2) and expand, and apply it again to every square of a cosine that appears.

Worked Example 9.3.3

Evaluate $\displaystyle\int \sin^4 x\,dx$.

Both exponents $m = 4$ and $n = 0$ are even. By Eq. (9.3.2) twice,

$$\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4} = \frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}.$$

Theorem 9.3.7 with $c_0 = \frac{3}{8}$, $c_1 = -\frac{1}{2}$ and $c_2 = \frac{1}{8}$ gives

$$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

9.3.8 Reduction formulae and the integrals of Wallis

For a single power the two theorems above apply. A reduction formula lowers the exponent by 2 at each step and leads to closed forms for definite integrals.

Theorem 9.3.9 (Sine reduction formula)**.** *For an integer $n \geq 2$, on $\mathbb{R}$,*

$$\int \sin^n x \, \mathrm{d}x = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, \mathrm{d}x. \tag{9.3.7}$$

Proof. Integration by parts with $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x \, \mathrm{d}x$ suggests the formula. It is proved by differentiation. Let G be an antiderivative of $\sin^{n-2} x$ and $F = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} G$. By the product rule and Eq. (9.3.1),

$$\begin{aligned} F'(x) &= -\frac{1}{n}\big((n-1)\sin^{n-2} x \cos^2 x - \sin^n x\big) + \frac{n-1}{n} \sin^{n-2} x \\ &= -\frac{n-1}{n} \sin^{n-2} x \,(1 - \sin^2 x) + \frac{1}{n} \sin^n x + \frac{n-1}{n} \sin^{n-2} x = \sin^n x. \end{aligned}$$

Every antiderivative of $\sin^{n-2} x$ arises as $G + C$. The corresponding F runs over all antiderivatives of $\sin^n x$ (Theorem 8.1.3). ∎

Proposition 9.3.10 (Wallis integrals)**.** *Let $W_n = \int_0^{\pi/2} \sin^n x \, \mathrm{d}x$. Then $W_0 = \frac{\pi}{2}$, $W_1 = 1$ and $W_n = \frac{n-1}{n} W_{n-2}$ for $n \geq 2$. For every integer $m \geq 0$*

$$W_{2m} = \frac{\pi}{2} \prod_{j=1}^{m} \frac{2j-1}{2j}, \qquad W_{2m+1} = \prod_{j=1}^{m} \frac{2j}{2j+1}. \tag{9.3.8}$$

Proof. The values $W_0 = \frac{\pi}{2}$ and $W_1 = \big[-\cos x\big]_0^{\pi/2} = 1$ follow from Theorem 8.3.5. For $n \geq 2$ the definite form of Eq. (9.3.7) is $W_n = \big[-\frac{1}{n} \sin^{n-1} x \cos x\big]_0^{\pi/2} + \frac{n-1}{n} W_{n-2}$, and the bracket vanishes because $\sin 0 = 0$ and $\cos \frac{\pi}{2} = 0$. The products in Eq. (9.3.8) are empty for $m = 0$ and give W_0 and W_1. If Eq. (9.3.8) holds for $m - 1$, the recursion multiplies W_{2m-2} by $\frac{2m-1}{2m}$ and W_{2m-1} by $\frac{2m}{2m+1}$, and Eq. (9.3.8) holds for m (Theorem 1.4.2). ∎

9.3.11 Tangent and secant

The identity $\tan^2 x + 1 = \sec^2 x$ replaces Eq. (9.3.1). The differentials $\mathrm{d}(\tan x) = \sec^2 x \, \mathrm{d}x$ and $\mathrm{d}(\sec x) = \sec x \tan x \, \mathrm{d}x$ replace $\mathrm{d}(\sin x)$ and $\mathrm{d}(\cos x)$. On an interval where $\cos x \neq 0$, an even power $n = 2k \geq 2$ of the secant gives

$$\tan^m x \sec^{2k} x = \tan^m x \,(1 + \tan^2 x)^{k-1} \sec^2 x, \tag{9.3.9}$$

the pullback by $u = \tan x$ of $u^m (1 + u^2)^{k-1} \, \mathrm{d}u$. An odd power $m = 2k + 1$ of the tangent gives

$$\tan^{2k+1} x \sec^n x = (\sec^2 x - 1)^k \sec^{n-1} x \cdot \sec x \tan x, \tag{9.3.10}$$

the pullback by $u = \sec x$ of $(u^2 - 1)^k u^{n-1} \, \mathrm{d}u$. For $n \geq 1$ this is a polynomial 1-form. For $n = 0$ the factor u^{-1} makes it a rational 1-form. With $k = 0$ it gives $\int \tan x \, \mathrm{d}x = \ln|\sec x| + C$ (Problem 8.4.7). The proofs are those of Theorem 9.3.4 with these differentials. An odd power of $\sec x$ alone fits neither case and needs the next two results. An even power m of the tangent with an odd power n of the secant fits neither case either. The identity $\tan^2 x = \sec^2 x - 1$ turns

$\tan^m x \sec^n x$ into a sum of odd powers of $\sec x$. Proposition 9.3.12 and Eq. (9.3.12) integrate each of them.

Proposition 9.3.12 (Integral of the secant)**.** *On every interval on which* $\cos x \neq 0$*,*

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C. \tag{9.3.11}$$

Proof. Write $g = \sec x + \tan x = (1 + \sin x)/\cos x$. If $1 + \sin x = 0$, then $\sin x = -1$ and $\cos x = 0$ by Eq. (9.3.1). Hence $g \neq 0$ wherever $\cos x \neq 0$. Its derivative is $g' = \sec x \tan x + \sec^2 x = \sec x\,(\tan x + \sec x) = \sec x \cdot g$. By Problem 8.4.7 the antiderivatives of $g'/g = \sec x$ are $\ln|g| + C$. ∎

Theorem 9.3.13 (Secant reduction formula)**.** *For an integer* $n \geq 2$*, on any interval on which* $\cos x \neq 0$*,*

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx. \tag{9.3.12}$$

Proof. Integration by parts with $u = \sec^{n-2} x$ and $dv = \sec^2 x \, dx$ suggests the formula. Let $G' = \sec^{n-2} x$ and $F = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} G$. Using $(\sec^{n-2} x)' = (n-2) \sec^{n-2} x \tan x$ and $\tan^2 x = \sec^2 x - 1$,

$$F' = \frac{1}{n-1}\big((n-2) \sec^{n-2} x \,(\sec^2 x - 1) + \sec^n x\big) + \frac{n-2}{n-1} \sec^{n-2} x = \sec^n x.$$

As in Theorem 9.3.9, F runs over all antiderivatives of $\sec^n x$ as G runs over those of $\sec^{n-2} x$. ∎

Worked Example 9.3.4

Evaluate $\int \sec^3 x \, dx$ on an interval where $\cos x \neq 0$.

By Eq. (9.3.12) with $n = 3$ and Proposition 9.3.12,

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

On $(-\frac{\pi}{2}, \frac{\pi}{2})$ the quantity $\sec x + \tan x = (1 + \sin x)/\cos x$ is positive and the bars may be dropped. On an interval where $\cos x < 0$ it is negative. Only the barred form is defined there.

9.3.14 Weierstrass substitution

A rational function of $\sin x$ and $\cos x$, such as $1/(2 + \cos x)$, is covered by none of the cases above. The substitution $t = \tan(x/2)$ turns it into a rational function of t. Read in the direction $x = 2 \arctan t$, it is a pullback by a bijection to which Theorem 9.1.4 applies.

Theorem 9.3.15 (Weierstrass substitution)**.** *The map* $\varphi(t) = 2 \arctan t$ *is a differentiable bijection from* $\mathbb{R}$ *onto* $(-\pi, \pi)$ *with inverse* $x \mapsto \tan(x/2)$*, and*

$$\sin \varphi(t) = \frac{2t}{1+t^2}, \qquad \cos \varphi(t) = \frac{1-t^2}{1+t^2}, \qquad \varphi^*(dx) = \frac{2\,dt}{1+t^2}. \tag{9.3.13}$$

Let $R(s, c)$ *be a rational function and let* $R(\sin x, \cos x)$ *be continuous on an interval* $I \subseteq (-\pi, \pi)$*. The pullback of* $R(\sin x, \cos x)\,dx$ *is* $r(t)\,dt$ *with the rational function* $r(t) = R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}) \frac{2}{1+t^2}$*.*

If G is an antiderivative of r on the interval $J = \varphi^{-1}(I)$, then $G(\tan(x/2))$ is an antiderivative of $R(\sin x, \cos x)$ on I.

Proof. The arctangent is a bijection from $\mathbb{R}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2})$ with inverse tan (Definition 4.5.14). Hence φ is a bijection onto $(-\pi, \pi)$ with inverse $\tan(x/2)$. By Theorem 6.4.10 $\varphi'(t) = 2/(1+t^2)$. This is the third identity. Let $\theta = \arctan t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\cos\theta > 0$ and $\sec^2\theta = 1 + \tan^2\theta = 1 + t^2$. Hence $\cos^2\theta = 1/(1+t^2)$ and $\sin\theta = t\cos\theta$. By the double-angle formulas Eq. (4.5.4),

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2t}{1+t^2}, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1-t^2}{1+t^2},$$

which are the first two identities since $\varphi(t) = 2\theta$. The pullback of $R(\sin x, \cos x)\,dx$ is $r(t)\,dt$ by Eq. (6.2.8). The inverse $x \mapsto \tan(x/2)$ is continuous and increasing on $(-\pi, \pi)$, and the image J of the interval I is an interval (Corollary 5.4.4). The restriction of φ is a differentiable bijection from J onto I. Theorem 9.1.4 gives the last statement. ∎

The theorem is a statement about one interval inside $(-\pi, \pi)$. At $x = \pm\pi$ the variable $t = \tan(x/2)$ is undefined. On the next interval $(\pi, 3\pi)$ the substitution is $x = 2\pi + 2\arctan t$. An antiderivative built from the two pieces need not be continuous at π (Problem 9.3.9).

Caution

Convert a product of powers to sines and cosines of multiples of x only after deciding the case. In $\int \sin^5 x\cos^2 x\,dx$ the odd power makes Theorem 9.3.4 the short route. Replacing $\cos^2 x$ by $\frac{1}{2}(1+\cos 2x)$ first produces terms $\sin^5 x\cos 2x$ that need further work.

Study guide

Identify the exponents before writing anything. An odd power of $\sin x$ or of $\cos x$ makes the integrand the pullback of a polynomial 1-form by $\cos x$ or by $\sin x$ (Theorem 9.3.4). Two even powers make it a combination of $\cos(2jx)$, reached by the half-angle formulas (Theorem 9.3.7). For $\tan^m x\sec^n x$ an even n gives a pullback by $\tan x$ and an odd m a pullback by $\sec x$. An odd power of $\sec x$ alone needs Eq. (9.3.12) and Proposition 9.3.12. An even m with an odd n becomes a sum of odd powers of $\sec x$ by $\tan^2 x = \sec^2 x - 1$. Reduction formulae are found by parts and proved by differentiation. Their definite forms give closed values such as Eq. (9.3.8). A rational function of $\sin x$ and $\cos x$ becomes a rational function of t under $x = 2\arctan t$, valid on an interval inside $(-\pi, \pi)$.

Problems

9.3.1 Evaluate $\displaystyle\int \sin 4x\cos 2x\,dx$. ⋆ Hint · Answer · Solution

9.3.2 Evaluate $\displaystyle\int \tan^3 x\sec^3 x\,dx$ on $(-\tfrac{\pi}{2}, \tfrac{\pi}{2})$. ⋆ Hint · Answer · Solution

9.3.3 Evaluate $\displaystyle\int \sec^5 x\,dx$ on an interval where $\cos x \neq 0$. ⋆⋆ Hint · Answer · Solution

9.3.4 Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$ on $(-\pi, \pi)$. ⋆⋆ Hint · Answer · Solution

9.3.5 Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx$. ⋆ Hint · Answer · Solution

9.3.6 With W_n as in Proposition 9.3.10, show that $W_{2m+1} \le W_{2m} \le W_{2m-1}$ for $m \ge 1$. Deduce Wallis's product

$$\lim_{m\to\infty} \prod_{j=1}^{m} \frac{(2j)^2}{(2j-1)(2j+1)} = \frac{\pi}{2}.$$

⋆⋆⋆ Hint · Answer · Solution

9.3.7 Let m and n be positive integers. Prove that

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0, & m \neq n, \\ \pi, & m = n, \end{cases}$$

and $\int_{-\pi}^{\pi} \sin mx \cos nx \, \mathrm{d}x = 0$. ⋆⋆ Hint · Answer · Solution

9.3.8 Prove that for an integer $n \ge 2$, on any interval on which $\cos x \neq 0$,

$$\int \tan^n x \, \mathrm{d}x = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, \mathrm{d}x,$$

and use it to evaluate $\int \tan^4 x \, \mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.3.9 The function $F(x) = \frac{2}{\sqrt{3}} \arctan(\frac{1}{\sqrt{3}} \tan \frac{x}{2})$ of Problem 9.3.4 satisfies $F(2\pi) - F(0) = 0$. Explain why $\int_0^{2\pi} \mathrm{d}x/(2 + \cos x)$ is not 0 and compute it.

⋆⋆⋆ Hint · Answer · Solution

9.3.10 Evaluate $\displaystyle\int \frac{\mathrm{d}x}{1 + \sin^2 x}$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by the bijective substitution $x = \arctan t$ from $\mathbb{R}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2})$. Why does this substitution give a simpler rational function than Theorem 9.3.15? ⋆⋆ Hint · Answer · Solution

9.4 Trigonometric Substitutions

Overview

Objectives: State each trigonometric substitution as a bijection between open intervals with an explicit inverse (Proposition 9.4.1) and integrate by pulling back along it (Theorem 9.1.4). Simplify the radical by its sign on the stated interval and return to x with the compositions of Lemma 9.4.2. Complete the square on a quadratic under a root (Eq. (9.4.4)). Use the closed interval in a definite integral, for which no bijection is needed.
Prerequisites: Substitution by a bijection (Theorem 9.1.4) and in definite integrals (Theorem 8.5.2); the inverse trigonometric functions (Definition 4.5.14 and Theorem 4.5.13); the Pythagorean identity (Eq. (9.3.1)); trigonometric integrals (Section 9.3).
Key results: The substitutions of Table 9.4.1 with their intervals and inverses; $\cos(\arcsin s) = \sqrt{1-s^2}$, $\sec(\arctan s) = \sqrt{1+s^2}$ and $\tan(\arccos(1/s)) = \pm\sqrt{s^2-1}$ with the sign of s.

Notation

a	the positive constant fixed by the shape of the radical
θ	the new variable, with the interval of Table 9.4.1
φ	the substitution $x = \varphi(\theta)$, a bijection
u	the shifted variable $x + \beta/(2\alpha)$ after completing the square

No inner function of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$ has its derivative as a factor of a typical integrand such as $1/(x^2\sqrt{25-x^2})$. The substitution therefore takes $x = \varphi(\theta)$ with a trigonometric φ chosen so that a Pythagorean identity turns the radicand into a square. By Theorem 9.1.4 the method is valid when φ is a differentiable bijection between intervals. That requirement fixes the interval of θ. The interval in turn fixes the sign of the square root.

Table 9.4.1: Trigonometric substitutions as bijections, with their inverses and the radical written in θ

Radical	Substitution	Bijection	Inverse	Radical
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$(-\frac{\pi}{2}, \frac{\pi}{2}) \to (-a, a)$	$\theta = \arcsin(x/a)$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$	$\theta = \arctan(x/a)$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$(0, \frac{\pi}{2}) \to (a, \infty)$	$\theta = \arccos(a/x)$	$a\tan\theta$
		$(\frac{\pi}{2}, \pi) \to (-\infty, -a)$	$\theta = \arccos(a/x)$	$-a\tan\theta$

Proposition 9.4.1 (Trigonometric substitutions)**.** *Let $a > 0$. Each map in Table 9.4.1 is a differentiable bijection between the stated open intervals with the stated inverse. On its interval the radical equals the expression in the last column:*

$$a^2 - a^2\sin^2\theta = a^2\cos^2\theta, \qquad \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta, \tag{9.4.1}$$

$$a^2 + a^2\tan^2\theta = a^2\sec^2\theta, \qquad \sqrt{a^2 + a^2\tan^2\theta} = a\sec\theta, \tag{9.4.2}$$

$$a^2\sec^2\theta - a^2 = a^2\tan^2\theta, \qquad \sqrt{a^2\sec^2\theta - a^2} = a|\tan\theta|. \tag{9.4.3}$$

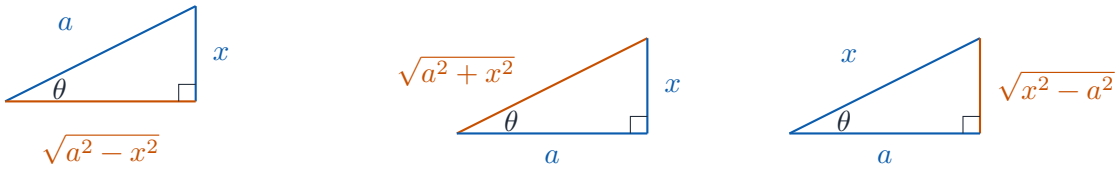
Proof. The left identities follow from Eq. (9.3.1) and from it divided by $\cos^2\theta$. For every real y, $\sqrt{y^2} = |y|$.

The sine is a strictly increasing bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ onto $[-1, 1]$ with inverse arcsin (Theorem 4.5.13 and Definition 4.5.14). It maps the endpoints to the endpoints and therefore maps $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $(-1, 1)$. Multiplying by a gives the first row with inverse $\arcsin(x/a)$. On this interval $\cos\theta > 0$ and $\sqrt{a^2\cos^2\theta} = a\cos\theta$.

The tangent is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$ with inverse arctan (Definition 4.5.14). Since $\cos\theta > 0$ there, $\sec\theta > 0$ and $\sqrt{a^2\sec^2\theta} = a\sec\theta$.

The cosine is a strictly decreasing bijection from $[0, \pi]$ onto $[-1, 1]$ with inverse arccos. It maps $(0, \frac{\pi}{2})$ onto $(0, 1)$ and $(\frac{\pi}{2}, \pi)$ onto $(-1, 0)$. The reciprocal $c \mapsto 1/c$ is a bijection from $(0, 1)$ onto $(1, \infty)$ and from $(-1, 0)$ onto $(-\infty, -1)$. Hence $\theta \mapsto a\sec\theta = a/\cos\theta$ is a bijection from $(0, \frac{\pi}{2})$ onto (a, ∞) and from $(\frac{\pi}{2}, \pi)$ onto $(-\infty, -a)$. The equation $a\sec\theta = x$ is $\cos\theta = a/x$, with the solution $\theta = \arccos(a/x)$ in $[0, \pi]$. The tangent is positive on $(0, \frac{\pi}{2})$ and negative on $(\frac{\pi}{2}, \pi)$. These signs give the two entries in the last column. All three maps are differentiable on their intervals by Theorem 6.4.8 and the quotient rule. ∎

Figure 9.4.1 shows the reference triangles. For θ strictly between 0 and $\frac{\pi}{2}$ each triangle has an acute angle θ with two sides matching the substitution. Its third side comes from the Pythagorean theorem. The triangles give the compositions of the next lemma for positive arguments. The lemma proves them on the whole interval, with the sign that a triangle cannot show.



(a) $x = a\sin\theta$ (b) $x = a\tan\theta$ (c) $x = a\sec\theta$

Figure 9.4.1: Reference triangles for the substitutions of Table 9.4.1 with $0 < \theta < \frac{\pi}{2}$ and the radical on the highlighted side

Lemma 9.4.2 (Compositions with inverse functions)**.** *For $-1 < s < 1$, $\cos(\arcsin s) = \sqrt{1 - s^2}$ and $\tan(\arcsin s) = s/\sqrt{1 - s^2}$. For every real s, $\sec(\arctan s) = \sqrt{1 + s^2}$ and $\sin(\arctan s) = s/\sqrt{1 + s^2}$. For $|s| > 1$, $\tan(\arccos(1/s)) = \sqrt{s^2 - 1}$ when $s > 1$ and $-\sqrt{s^2 - 1}$ when $s < -1$.*

Proof. Each statement is the last column of Table 9.4.1 with $a = 1$ and θ the inverse function of s. For instance with $\theta = \arcsin s$, Eq. (9.4.1) gives $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - s^2}$, and $\tan\theta = \sin\theta/\cos\theta$. With $\theta = \arccos(1/s)$ the identity $\sec\theta = s$ and Eq. (9.4.3) give $|\tan\theta| = \sqrt{s^2 - 1}$. By Proposition 9.4.1 the sign of $\tan\theta$ is that of s. ∎

A definite integral needs no bijection. Theorem 8.5.2 asks only for a continuous derivative of the substitution on the closed interval. The endpoints that Table 9.4.1 excludes are then allowed.

Example 9.4.3 (Area of a disc)**.** The disc of radius $r > 0$ has area $2\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx$. Apply Theorem 8.5.2 with $g(\theta) = r\sin\theta$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. There $g'(\theta) = r\cos\theta$ is continuous and $\cos\theta \geq 0$. Then $\sqrt{r^2 - g(\theta)^2} = r\cos\theta$ and, by Eq. (9.3.2),

$$2\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx = 2r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = r^2\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_{-\pi/2}^{\pi/2} = \pi r^2.$$

A radicand $\alpha x^2+\beta x+\gamma$ with $\alpha \neq 0$ fits a row of Table 9.4.1 after completing the square:

$$\alpha x^2+\beta x+\gamma = \alpha\left(x+\frac{\beta}{2\alpha}\right)^2+\left(\gamma-\frac{\beta^2}{4\alpha}\right). \tag{9.4.4}$$

With $u = x+\beta/(2\alpha)$ the radicand is a multiple of u^2 plus a constant. For instance $x^2+2x+5 = (x+1)^2+4$ is the tangent case with $u = x+1$ and $a = 2$. The row is determined by the signs of the leading coefficient and of the constant.

Proposition 9.4.4 (Rationalisation of the square root of a quadratic)**.** *Let $R(x,y)$ be a rational function of two variables and $Q(x) = \alpha x^2+\beta x+\gamma$ with $\alpha \neq 0$. Let I be an open interval on which $Q > 0$ and $f(x) = R(x, \sqrt{Q(x)})$ is continuous. Then either f equals a rational function of x on I, or there are a differentiable bijection φ from an open interval $J \subseteq (-\pi,\pi)$ onto I and a rational function $R_1(s,c)$ with*

$$\varphi^*\big(f(x)\,\mathrm{d}x\big) = R_1(\sin\theta,\cos\theta)\,\mathrm{d}\theta \qquad (\theta \in J). \tag{9.4.5}$$

Proof. **1.** Translation. Put $\delta = \gamma-\beta^2/(4\alpha)$ and $u = x+\beta/(2\alpha)$. By Eq. (9.4.4), $Q = \alpha u^2+\delta$. The translation maps I onto an open interval I' of u, and f is a rational function of u and $\sqrt{\alpha u^2+\delta}$.
2. Shape of the radicand. If $\alpha < 0$ and $\delta \leq 0$, then $Q \leq 0$ everywhere. Since $Q > 0$ on I, one of four cases occurs.
(i) $\alpha > 0$ and $\delta = 0$. Then $\sqrt{Q} = \sqrt{\alpha}\,|u|$ and $u \neq 0$ on I'. An interval that misses 0 lies on one side of it. Hence $|u| = u$ on all of I' or $|u| = -u$ on all of I', and f equals a rational function of x on I.
(ii) $\alpha > 0$ and $\delta > 0$. Then $\sqrt{Q} = \sqrt{\alpha}\sqrt{a^2+u^2}$ with $a = \sqrt{\delta/\alpha}$. Take the tangent row of Table 9.4.1.
(iii) $\alpha > 0$ and $\delta < 0$. Then $\sqrt{Q} = \sqrt{\alpha}\sqrt{u^2-a^2}$ with $a = \sqrt{-\delta/\alpha}$ and $|u| > a$ on I'. An interval with points on both sides of $[-a,a]$ contains $[-a,a]$. Hence $I' \subseteq (a,\infty)$ or $I' \subseteq (-\infty,-a)$. Take the secant row for that half-line.
(iv) $\alpha < 0$ and $\delta > 0$. Then $\sqrt{Q} = \sqrt{-\alpha}\sqrt{a^2-u^2}$ with $a = \sqrt{-\delta/\alpha}$ and $I' \subseteq (-a,a)$. Take the sine row.
3. Pullback. In the last three cases let ψ be the bijection of the chosen row from its interval J_0 onto the u-interval (a,∞), $(-\infty,-a)$, $(-a,a)$ or $\mathbb{R}$ that contains I'. Every interval J_0 of Table 9.4.1 lies in $(-\pi,\pi)$. The inverse of ψ in Table 9.4.1 is continuous. By Corollary 5.4.4 it maps I' onto an open interval $J \subseteq J_0$, and ψ is a differentiable bijection from J onto I'. Put $\varphi(\theta) = \psi(\theta)-\beta/(2\alpha)$. By Proposition 9.4.1 the radical $\sqrt{Q}$ is a constant times $a\cos\theta$, $a\sec\theta$ or $\pm a\tan\theta$ on J. The differential $\varphi^*(\mathrm{d}x)$ is $a\cos\theta\,\mathrm{d}\theta$, $a\sec^2\theta\,\mathrm{d}\theta$ or $a\sec\theta\tan\theta\,\mathrm{d}\theta$ by Theorem 6.4.8. Since $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$, the functions $\psi(\theta)$, $\sqrt{Q(\varphi(\theta))}$ and $\varphi'(\theta)$ are rational functions of $\sin\theta$ and $\cos\theta$. So is $f(\varphi(\theta))\varphi'(\theta)$, and Eq. (6.2.8) gives Eq. (9.4.5). ∎

With Theorem 9.1.4 an antiderivative of $R_1(\sin\theta,\cos\theta)$ on J returns to x through φ^{-1}. The Weierstrass substitution (Theorem 9.3.15) applies on J because $J \subseteq (-\pi,\pi)$, and it leaves a rational function of t for Section 9.5.

Worked Example 9.4.1

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{25-x^2}}$.

The integrand is continuous on $(-5,0)$ and on $(0,5)$. Work on $(0,5)$. By Proposition 9.4.1 the map $x = 5\sin\theta$ is a bijection from $(0,\frac{\pi}{2})$ onto $(0,5)$ with inverse $\arcsin(x/5)$, and $\sqrt{25-x^2} = 5\cos\theta$. The pullback is

$$\frac{5\cos\theta\,\mathrm{d}\theta}{25\sin^2\theta\cdot 5\cos\theta} = \frac{1}{25}\frac{\mathrm{d}\theta}{\sin^2\theta},$$

with antiderivative $-\frac{1}{25}\cot\theta$. By Lemma 9.4.2, $\cot(\arcsin s) = \sqrt{1-s^2}/s$. Theorem 9.1.4 gives

$$\int \frac{\mathrm{d}x}{x^2\sqrt{25-x^2}} = -\frac{\sqrt{25-x^2}}{25x} + C.$$

The same computation on $(-\frac{\pi}{2},0)$ gives the same formula on $(-5,0)$ with an independent constant.

Worked Example 9.4.2

Evaluate $\displaystyle\int \frac{x^3\,\mathrm{d}x}{\sqrt{x^2+4}}$.

By Proposition 9.4.1 the map $x = 2\tan\theta$ is a bijection from $(-\frac{\pi}{2},\frac{\pi}{2})$ onto $\mathbb{R}$, and $\sqrt{x^2+4} = 2\sec\theta$. The pullback is

$$\frac{8\tan^3\theta\cdot 2\sec^2\theta\,\mathrm{d}\theta}{2\sec\theta} = 8\tan^3\theta\sec\theta\,\mathrm{d}\theta = 8(\sec^2\theta-1)\sec\theta\tan\theta\,\mathrm{d}\theta.$$

By Eq. (9.3.10) this is the pullback by $w = \sec\theta$ of $8(w^2-1)\,\mathrm{d}w$, with antiderivative $\frac{8}{3}\sec^3\theta - 8\sec\theta$. By Lemma 9.4.2, $\sec(\arctan\frac{x}{2}) = \frac{1}{2}\sqrt{x^2+4}$. Hence

$$\int \frac{x^3\,\mathrm{d}x}{\sqrt{x^2+4}} = \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C.$$

Worked Example 9.4.3

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-9}}$ for $x>3$.

By Proposition 9.4.1 the map $x = 3\sec\theta$ is a bijection from $(0,\frac{\pi}{2})$ onto $(3,\infty)$ with $\sqrt{x^2-9} = 3\tan\theta$. The pullback is $3\sec\theta\tan\theta\,\mathrm{d}\theta/(3\tan\theta) = \sec\theta\,\mathrm{d}\theta$ with antiderivative $\ln(\sec\theta+\tan\theta)$ (Proposition 9.3.12). Both terms are positive on $(0,\frac{\pi}{2})$. With $\sec\theta = x/3$ and $\tan\theta = \sqrt{x^2-9}/3$ from Lemma 9.4.2,

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-9}} = \ln\frac{x+\sqrt{x^2-9}}{3} + C = \ln(x+\sqrt{x^2-9}) + C \qquad (x>3),$$

with $-\ln 3$ absorbed into the constant. The second interval $(-\infty,-a)$ of this radical shape is treated with $a=5$ in Problem 9.4.4.

Common misconception

The identity $\sqrt{a^2\cos^2\theta} = a\cos\theta$ is false whenever $\cos\theta < 0$. The simplification of the radical is a consequence of the interval in Table 9.4.1. A substitution stated without its interval gives an answer whose sign is unknown (Problem 9.4.6).

Study guide

Identify the radical, complete the square if the radicand has a linear term and choose the row of Table 9.4.1. State the interval of θ. It makes the substitution the bijection that Theorem 9.1.4 requires and fixes the sign of the radical. Pull back, integrate in θ with the methods of Section 9.3 and return to x with Lemma 9.4.2 or the reference triangle. For $x = a\sec\theta$ the triangle covers only $x > a$. On $x < -a$ the tangent is negative.
A definite integral needs no bijection. With Theorem 8.5.2 the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is allowed for $x = a\sin\theta$. No return to x is needed.

Problems

9.4.1 Evaluate $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$ for $x > 2$. ⋆⋆ Hint · Answer · Solution

9.4.2 Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+13}}$. ⋆⋆ Hint · Answer · Solution

9.4.3 Evaluate $\displaystyle\int x^2\sqrt{4-x^2}\,dx$ on $(-2,2)$. ⋆⋆ Hint · Answer · Solution

9.4.4 Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$ for $x < -5$. ⋆⋆ Hint · Answer · Solution

9.4.5 Prove that $\dfrac{d}{dx}\arccos\dfrac{1}{x} = \dfrac{1}{|x|\sqrt{x^2-1}}$ for $|x| > 1$. Deduce the antiderivatives of $\dfrac{1}{x\sqrt{x^2-1}}$ on $(1,\infty)$ and on $(-\infty,-1)$, and obtain the first one again with $x = \sec\theta$. ⋆⋆ Hint · Answer · Solution

9.4.6 The map $x = \sin\theta$ also sends $[\frac{\pi}{2}, \pi]$ onto $[0,1]$, from 1 down to 0. Use Theorem 8.5.2 with θ running from π to $\frac{\pi}{2}$ to evaluate $\int_0^1 \sqrt{1-x^2}\,dx$. What value results if the radical is replaced by $\cos\theta$ on this interval? ⋆⋆ Hint · Answer · Solution

9.4.7 Let $a > 0$. Prove that on $\mathbb{R}$

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C.$$

⋆⋆ Hint · Answer · Solution

9.4.8 Set up and evaluate the length of $y = x^2$ for $0 \le x \le 1$. ⋆⋆ Hint · Answer · Solution

9.4.9 Let $a, b > 0$. Show that the ellipse $x^2/a^2 + y^2/b^2 \le 1$ has area $2b\int_{-a}^{a}\sqrt{1-x^2/a^2}\,dx = \pi ab$. The substitution $x = a\sin\theta$ on the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is allowed here although Table 9.4.1 excludes the endpoints. Explain why. ⋆⋆ Hint · Answer · Solution

9.4.10 Evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x$ using area. ⋆ Hint · Answer · Solution

9.4.11 Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$. ⋆ Hint · Answer · Solution

9.5 Integration of Rational Functions by Partial Fractions

Overview

Objectives: State the partial fraction decomposition with its uniqueness (Theorem 9.5.2, proved in Appendix A.2). Derive from it the methods that find the coefficients: equating coefficients and substituting values in the cleared identity, and the cover-up formula (Proposition 9.5.4). Prove that every rational function has an antiderivative built from rational functions, logarithms and arctangents once its denominator is factored (Theorem 9.5.7), with the reduction formula Eq. (9.5.6) for powers of an irreducible quadratic.
Prerequisites: Polynomial division and the identity theorem for polynomials (Theorem 4.2.4 and Corollary 4.2.8); partial fractions with two linear factors (Proposition 4.2.20); completing the square and the arctangent template (Eq. (9.1.5) and Proposition 9.1.7); substitution (Theorem 8.4.2).
Key results: A proper fraction is a unique sum of terms $A/(x-r)^k$ and $(Bx+C)/(x^2+bx+c)^k$. The top coefficient of a factor $(x-r)^m$ is $p(r)/q_1(r)$ with $q=(x-r)^m q_1$. $\int \mathrm{d}x/(x-r) = \ln|x-r| + C$ on each side of r.

Notation

$p,\ q$	numerator and denominator of $f = p/q$
$r,\ m$	a real root of q and its multiplicity
$b,\ c,\ n$	an irreducible factor x^2+bx+c $(b^2<4c)$ and its multiplicity
$A_k,\ B_k,\ C_k$	the coefficients of the decomposition
$u,\ k$	the shifted variable $x+\frac{b}{2}$ and $k=\sqrt{c-b^2/4}$

A rational function $f = p/q$ is continuous on every interval that contains no real root of q. It has an antiderivative there by Corollary 8.3.8. This section shows that the antiderivative is elementary once q is factored and finds it. The method rests on one algebraic theorem. Its uniqueness statement justifies every way of computing the coefficients.

9.5.1 Decomposition theorem

The fraction $f = p/q$ is *proper* when $\deg p < \deg q$. If it is not proper, polynomial division (Theorem 4.2.4) gives

$$p(x) = q(x)\,s(x) + t(x), \qquad t = 0 \text{ or } \deg t < \deg q, \tag{9.5.1}$$

and $f = s + t/q$ is a polynomial plus a proper fraction.

Theorem 9.5.2 (Partial fraction decomposition)**.** *Let $f = p/q$ be proper and let $q(x) = a\prod_i (x-r_i)^{m_i}\prod_j (x^2+b_jx+c_j)^{n_j}$ with $a\neq 0$, distinct real r_i, distinct pairs (b_j,c_j) and $b_j^2-4c_j<0$. At every x with $q(x)\neq 0$, $f(x)$ is the sum over the factors of q of the following terms. A linear factor $(x-r)^m$ contributes*

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}, \tag{9.5.2}$$

and an irreducible quadratic factor $(x^2+bx+c)^n$ *contributes*

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}. \tag{9.5.3}$$

The real coefficients A_k, B_k *and* C_k *are unique.*

Appendix A.2 proves the theorem from the division algorithm, in the way that Theorem 1.4.14 is proved for integers. The factorisation of q is an assumption of the theorem. Its existence for every real polynomial follows from the fundamental theorem of algebra. This book does not prove that theorem. For a distinct linear factor ($m=1$) the block Eq. (9.5.2) is a single term

$$\frac{A}{x-r}. \tag{9.5.4}$$

9.5.3 Finding the coefficients

Multiplying the decomposition by $q(x)$ gives an identity between polynomials, the *cleared identity*. It holds on the infinite set of x with $q(x)\neq 0$ and therefore at every real x (Corollary 4.2.8). Two consequences give the two standard methods.

1. *Equating coefficients.* The coefficients of each power of x on the two sides are equal. This is a square linear system in the unknown coefficients. By the uniqueness in Theorem 9.5.2 it has exactly one solution.
2. *Substituting values.* The cleared identity may be evaluated at any real x. The roots of q are allowed, although f itself is undefined there.

Substituting a root kills every term that contains the corresponding factor. The next proposition states the result.

Proposition 9.5.4 (Cover-up formula)**.** *Let* $f=p/q$ *be proper and* $q(x)=(x-r)^m q_1(x)$ *with* $q_1(r)\neq 0$. *The coefficient* A_m *of* $(x-r)^{-m}$ *in Eq. (9.5.2) is*

$$A_m=\lim_{x\to r}(x-r)^m f(x)=\frac{p(r)}{q_1(r)}. \tag{9.5.5}$$

Proof. Multiply the decomposition by $(x-r)^m$. For x near r with $x\neq r$,

$$\frac{p(x)}{q_1(x)}=(x-r)^m f(x)=\sum_{k=1}^{m}A_k(x-r)^{m-k}+(x-r)^m R(x),$$

where R is the sum of the terms of the other factors. Each of them is a rational function whose denominator does not vanish at r. Both sides are continuous at r (Theorem 5.3.6). Letting $x\to r$ leaves only the term $k=m$ on the right. ∎

Cover-up finds every coefficient at once only when all factors are distinct and linear. The remaining coefficients come from equating coefficients or from Problem 9.5.7.

Worked Example 9.5.1

Evaluate $\displaystyle\int\frac{7x-3}{(x-2)(x+1)}\,\mathrm{d}x$.

The fraction is proper with two distinct linear factors: $\dfrac{7x-3}{(x-2)(x+1)}=\dfrac{A}{x-2}+\dfrac{B}{x+1}$, with cleared identity $7x-3=A(x+1)+B(x-2)$.

By cover-up (Proposition 9.5.4), $A = \dfrac{7\cdot 2-3}{2+1} = \dfrac{11}{3}$ and $B = \dfrac{7(-1)-3}{-1-2} = \dfrac{10}{3}$. Equating coefficients the identity reads $7x-3 = (A+B)x + (A-2B)$. Hence $A+B=7$ and $A-2B=-3$, with the same solution $B=\frac{10}{3}$ and $A=\frac{11}{3}$. Uniqueness guarantees that the two methods agree. By Problem 8.4.7 each term $A/(x-r)$ has the antiderivative $A\ln|x-r|$ on each side of r. Hence

$$\int \frac{7x-3}{(x-2)(x+1)}\,\mathrm{d}x = \frac{11}{3}\ln|x-2| + \frac{10}{3}\ln|x+1| + C$$

on each of the intervals $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$.

Worked Example 9.5.2

Evaluate $\displaystyle\int \frac{2x-1}{(x+1)^3}\,\mathrm{d}x$.

By Theorem 9.5.2, $\dfrac{2x-1}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$ with cleared identity $2x-1 = A(x+1)^2 + B(x+1) + C$. Cover-up gives $C = 2(-1)-1 = -3$. The coefficient of x^2 gives $A=0$. The coefficient of x then gives $2A+B=2$ and $B=2$. Hence

$$\int \frac{2x-1}{(x+1)^3}\,\mathrm{d}x = \int\Big(\frac{2}{(x+1)^2} - \frac{3}{(x+1)^3}\Big)\mathrm{d}x = -\frac{2}{x+1} + \frac{3}{2(x+1)^2} + C.$$

9.5.5 Integration of the terms

Proposition 9.5.6 (Reduction for an irreducible quadratic)**.** *Let $k>0$ and $J_n = \int \mathrm{d}u/(u^2+k^2)^n$. Then $J_1 = \dfrac{1}{k}\arctan\dfrac{u}{k} + C$. For $n \geq 2$*

$$J_n = \frac{u}{2k^2(n-1)(u^2+k^2)^{n-1}} + \frac{2n-3}{2k^2(n-1)}\,J_{n-1}. \tag{9.5.6}$$

Proof. The case $n=1$ is Eq. (9.1.3). For $n\geq 2$ let G be an antiderivative of $(u^2+k^2)^{-(n-1)}$ and F the right side of Eq. (9.5.6) with G for J_{n-1}. Write $Q = u^2+k^2$ and $\kappa = 2k^2(n-1)$. By the quotient rule,

$$F' = \frac{1}{\kappa}\Big(Q^{-(n-1)} - 2(n-1)u^2Q^{-n}\Big) + \frac{2n-3}{\kappa}Q^{-(n-1)} = \frac{2(n-1)}{\kappa}(Q-u^2)Q^{-n} = Q^{-n},$$

because $Q-u^2 = k^2$. As G runs over the antiderivatives of $Q^{-(n-1)}$, F runs over those of Q^{-n} (Theorem 8.1.3). ∎

Theorem 9.5.7 (Integration of rational functions)**.** *Let $f=p/q$ be a rational function with q factored as in Theorem 9.5.2, and let I be an interval containing no real root of q. On I, f has an antiderivative that is a sum of a rational function, constant multiples of $\ln|x-r|$ for real roots r of q, of $\ln(x^2+bx+c)$ for irreducible factors of q, and of arctangents $\arctan\big((2x+b)/\sqrt{4c-b^2}\big)$.*

The factorisation in the hypothesis exists for every real polynomial q by the fundamental theorem of algebra. This book uses that theorem without proof. For a given q the hypothesis is met once the real roots and the irreducible quadratic factors are found.

Proof. By Eq. (9.5.1) and Theorem 9.5.2, f is a polynomial plus a sum of terms of the forms Eq. (9.5.2) and Eq. (9.5.3). By linearity each term may be integrated separately. A polynomial has a polynomial antiderivative. For a linear factor,

$$\int \frac{\mathrm{d}x}{x-r} = \ln|x-r| + C, \qquad \int \frac{\mathrm{d}x}{(x-r)^j} = -\frac{1}{(j-1)(x-r)^{j-1}} + C \quad (j \geq 2), \tag{9.5.7}$$

on each interval not containing r, by Problem 8.4.7 and the power rule. For a quadratic factor put $u = x + \frac{b}{2}$ and $k = \frac{1}{2}\sqrt{4c - b^2}$. Then $x^2 + bx + c = u^2 + k^2$ by Eq. (9.1.5). With $D = C - \frac{Bb}{2}$ the numerator is $Bx + C = Bu + D$ and

$$\frac{Bx + C}{(x^2 + bx + c)^n} = B\,\frac{u}{(u^2 + k^2)^n} + \frac{D}{(u^2 + k^2)^n}.$$

The first term is a pullback by $w = u^2 + k^2 > 0$ with $\mathrm{d}w = 2u\,\mathrm{d}u$ and has the antiderivative $\frac{B}{2}\ln(u^2 + k^2)$ for $n = 1$ and $-\frac{B}{2(n-1)}(u^2 + k^2)^{-(n-1)}$ for $n \geq 2$. For the second, Eq. (9.5.6) and induction on n show that J_n is a rational function of u plus a constant multiple of $\arctan(u/k)$. Since $u/k = (2x + b)/\sqrt{4c - b^2}$, every antiderivative has the stated form. ∎

The bars in Eq. (9.5.7) matter. On an interval to the left of r the function $\ln(x - r)$ is undefined. There $\ln|x - r| = \ln(r - x)$. A quadratic factor needs no bars because $x^2 + bx + c > 0$ when $b^2 < 4c$.

Worked Example 9.5.3

Evaluate $\displaystyle\int \frac{3x + 5}{x^2 + 6x + 13}\,\mathrm{d}x$.

The discriminant $36 - 52$ is negative and the fraction is a single term of Eq. (9.5.3). With $u = x + 3$, $k = 2$ and $3x + 5 = 3u - 4$,

$$\int \frac{3x + 5}{x^2 + 6x + 13}\,\mathrm{d}x = 3\int \frac{u\,\mathrm{d}u}{u^2 + 4} - 4\int \frac{\mathrm{d}u}{u^2 + 4} = \frac{3}{2}\ln(x^2 + 6x + 13) - 2\arctan\frac{x + 3}{2} + C.$$

Worked Example 9.5.4

Evaluate $\displaystyle\int \frac{x^3 + 1}{x^2 - x}\,\mathrm{d}x$.

The fraction is improper. Division gives $x^3 + 1 = (x^2 - x)(x + 1) + (x + 1)$ and hence $\dfrac{x^3 + 1}{x^2 - x} = x + 1 + \dfrac{x + 1}{x(x - 1)}$. Cover-up on the proper part gives the coefficients $\dfrac{0 + 1}{0 - 1} = -1$ at 0 and $\dfrac{1 + 1}{1} = 2$ at 1. Hence, on each interval avoiding 0 and 1,

$$\int \frac{x^3 + 1}{x^2 - x}\,\mathrm{d}x = \frac{x^2}{2} + x - \ln|x| + 2\ln|x - 1| + C.$$

Common misconception

The decomposition applies only to a proper fraction. For $\dfrac{x^2+3x-1}{x^2-1}$ the ansatz $\dfrac{A}{x-1}+\dfrac{B}{x+1}$ gives the cleared identity $x^2+3x-1=A(x+1)+B(x-1)$. Its right side has no term in x^2. The system has no solution. Divide first (Problem 9.5.1).

Study guide

Divide if the fraction is improper. Factor the denominator into linear factors and irreducible quadratics after checking each discriminant. Write the decomposition of Theorem 9.5.2. Clear denominators. The cleared identity holds for every real x. Substitute the roots to get the top coefficients (Proposition 9.5.4) and equate coefficients for the rest. Uniqueness guarantees one answer however it is found.

Integrate a linear term to a logarithm with bars or to a negative power. For a quadratic term complete the square and split the numerator into a multiple of u and a constant. The first part integrates to a logarithm or a power, the second to an arctangent through Eq. (9.5.6). State the intervals: one constant per interval between consecutive real roots.

Problems

9.5.1 Evaluate $\displaystyle\int \frac{x^2+3x-1}{x^2-1}\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.5.2 Evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+2)^2}\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.5.3 Evaluate $\displaystyle\int \frac{\mathrm{d}x}{x^2-9}$ on $(-3,3)$ and explain why the absolute values in Eq. (9.5.7) cannot be dropped there. ⋆⋆ Hint · Answer · Solution

9.5.4 Evaluate $\displaystyle\int \frac{3x+1}{x^2-x-20}\,\mathrm{d}x$. What goes wrong if the denominator is treated as an irreducible quadratic? ⋆⋆ Hint · Answer · Solution

9.5.5 Evaluate $\displaystyle\int \frac{2x^3-2x^2+4x}{(x-1)^2(x^2+1)}\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.5.6 Evaluate $\displaystyle\int_0^1 \frac{4x-1}{(x+1)(x+2)}\,\mathrm{d}x$. ⋆⋆ Hint · Answer · Solution

9.5.7 Let $f=p/q$ be proper and $q(x)=(x-r)^m q_1(x)$ with $q_1(r)\neq 0$, and let $g(x)=p(x)/q_1(x)$. Prove that the coefficients of Eq. (9.5.2) are

$$A_{m-j}=\frac{g^{(j)}(r)}{j!}\qquad (j=0,1,\dots,m-1).$$

Use the formula to redo Worked Example 9.5.2. ⋆⋆⋆ Hint · Answer · Solution

9.5.8 Let $r_1,\dots,r_N$ be distinct positive numbers, $q(x)=\prod_{i=1}^{N}(x^2-r_i^2)$ and p a polynomial with $\deg p<2N$ and $p(-x)=p(x)$. Write $\dfrac{p}{q}=\sum_i\left(\dfrac{A_i}{x-r_i}+\dfrac{B_i}{x+r_i}\right)$. Prove that $B_i=-A_i$ for every i. ⋆⋆ Hint · Answer · Solution

9.5.9 Let $q(x) = \prod_{i=1}^{n}(x - r_i)$ with $n \geq 2$ distinct real roots. Prove that

$$\frac{1}{q(x)} = \sum_{i=1}^{n} \frac{1}{q'(r_i)\,(x - r_i)} \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{1}{q'(r_i)} = 0.$$

Check the second identity for $q(x) = (x-1)(x-2)(x-4)$.

⋆⋆ Hint · Answer · Solution

9.5.10 Evaluate $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}$. ⋆ Hint · Answer · Solution

9.6 Integral Tables and Computer Algebra Systems

Overview

Objectives: Read a table entry as a theorem with hypotheses. State the content of an antiderivative printed by a computer algebra system: one function with no constant and no interval. Recognise the branch problems of logarithms and arctangents and repair an antiderivative that jumps where the integrand is continuous. Verify an answer by differentiation on a stated interval.
Prerequisites: Antiderivatives on an interval and on a union of intervals (Theorems 7.3.8 and 8.1.3); the Fundamental Theorem (Theorem 8.3.5); the reduction formulae (Theorems 9.2.6 and 9.2.7); the Weierstrass substitution (Theorem 9.3.15).
Key results: Two antiderivatives of f on a finite union U of disjoint open intervals differ by a locally constant function (Proposition 9.6.4). An antiderivative found by any means is verified by differentiating it at every point of the interval. A definite integral needs an antiderivative continuous on the whole interval of integration.

Notation

U	an open set, here a union of finitely many disjoint open intervals
$F_1,\ F_2$	two antiderivatives of the same integrand
log	the logarithm as printed by a computer algebra system

A table of integrals lists antiderivatives derived once with general constants. A computer algebra system (CAS) derives them on request by algorithms that include the decomposition and reduction methods of this chapter. Both are reliable within their conventions. Neither states the interval on which its formula is an antiderivative. The reader supplies that interval.

9.6.1 Table entries

An entry such as Theorem 9.2.6, or

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \tag{9.6.1}$$

is a theorem with hypotheses. Here they are $a > 0$ and $-a < x < a$ (Proposition 9.1.7). With $a < 0$ the right side is still defined for $|x| < |a|$. Its derivative is then $-1/\sqrt{a^2-x^2}$ (Problem 9.6.4). A table is used correctly when the integral is matched to an entry and the hypotheses of the entry are checked. A reduction formula is applied once per step until the exponent is low enough to finish (Problem 9.6.1).

9.6.2 Content of an antiderivative from a CAS

A CAS returns one function F with $F' = f$ at the points where it regards both as defined. It adds no constant and names no interval. It may work with complex values. For $\int \mathrm{d}x/x$ a typical system returns $\log x$. Over the real numbers this is an antiderivative on $(0,\infty)$ only. With the complex logarithm, $\log x = \ln|x| + i\pi$ for $x < 0$. This differs from the real antiderivative $\ln|x|$ by a constant. On a domain made of several intervals the general antiderivative has one constant per interval.

Definition 9.6.3 (Locally constant function)**.** A function g on an open set $U \subseteq \mathbb{R}$ is *locally constant* if every point of U has a neighbourhood in U on which g is constant.

The function $\operatorname{sgn} x$ on $\mathbb{R} \setminus \{0\}$ is locally constant and not constant. On an interval every locally constant function is constant. Its derivative is 0 and Corollary 7.2.2 applies to it.

Proposition 9.6.4 (Antiderivatives on an open set)**.** *Let U be a union of finitely many disjoint open intervals and let F_1 and F_2 be differentiable on U with $F_1' = F_2'$. Then $F_1 - F_2$ is locally constant on U, constant on each of the intervals. Conversely adding a locally constant function to an antiderivative gives an antiderivative.*

Proof. The difference has derivative 0 on U. By Theorem 7.3.8 it is constant on each interval. A function constant on each interval is constant near every point of U. Conversely a locally constant function has derivative 0 at every point of U. ∎

Two printed antiderivatives of one integrand are therefore consistent exactly when their difference is locally constant on the common domain.

Worked Example 9.6.1

Evaluate $\displaystyle\int \tan x \sec^2 x \, \mathrm{d}x$ by two substitutions and reconcile the answers.

With $u = \tan x$ the integrand is the pullback of $u \, \mathrm{d}u$ and an antiderivative is $\frac{1}{2}\tan^2 x$. Writing the integrand as $\sec x \cdot \sec x \tan x$ and putting $w = \sec x$ gives $\frac{1}{2}\sec^2 x$. On every interval where $\cos x \neq 0$,

$$\frac{\sec^2 x}{2} - \frac{\tan^2 x}{2} = \frac{1}{2}(\sec^2 x - \tan^2 x) = \frac{1}{2}$$

by Eq. (9.3.1). The difference is constant on each interval, as Proposition 9.6.4 requires. A table that prints one form and a CAS that prints the other agree.

9.6.5 Branches and repairs

A CAS answer can be an antiderivative on each of several intervals while jumping at the points between them. On an interval where the integrand is continuous such a function is not an antiderivative. Theorem 8.3.5 applied to it gives wrong values.

Worked Example 9.6.2

A CAS returns $F_0(x) = \frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right)$ for $\int \mathrm{d}x/(2 + \cos x)$. Find an antiderivative on all of $\mathbb{R}$ and evaluate $\int_0^{2\pi} \mathrm{d}x/(2 + \cos x)$.

By Problem 9.3.4, F_0 is an antiderivative on $(-\pi, \pi)$ and, by periodicity, on every interval $((2k-1)\pi, (2k+1)\pi)$. It is undefined at the odd multiples of π and jumps by $-2\pi/\sqrt{3}$ there (Problem 9.3.9). The integrand is continuous on $\mathbb{R}$. Let $c = 2 - \sqrt{3}$ and

$$F(x) = \frac{x}{\sqrt{3}} - \frac{2}{\sqrt{3}} \arctan \frac{c \sin x}{1 + c \cos x}.$$

Since $0 < c < 1$, the denominator satisfies $1 + c\cos x \geq 1 - c > 0$ and F is differentiable on $\mathbb{R}$. With $h = c \sin x/(1 + c\cos x)$, the quotient rule gives $h' = c(\cos x + c)/(1 + c\cos x)^2$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan h = \frac{h'}{1 + h^2} = \frac{c(\cos x + c)}{1 + 2c\cos x + c^2}.$$

Since $c^2 = 7 - 4\sqrt{3}$, we have $1 + c^2 = 4c$ and the denominator equals $2c(2 + \cos x)$. Hence

$$F'(x) = \frac{1}{\sqrt{3}} - \frac{\cos x + c}{\sqrt{3}\,(2 + \cos x)} = \frac{2 - c}{\sqrt{3}\,(2 + \cos x)} = \frac{1}{2 + \cos x},$$

because $2 - c = \sqrt{3}$. Thus F is an antiderivative on $\mathbb{R}$, and Theorem 8.3.5 gives

$$\int_0^{2\pi} \frac{\mathrm{d}x}{2 + \cos x} = F(2\pi) - F(0) = \frac{2\pi}{\sqrt{3}}.$$

The formula $F_0(2\pi) - F_0(0) = 0$ is wrong because F_0 is not an antiderivative on $[0, 2\pi]$.

9.6.6 Verifying an answer

An antiderivative is verified on an interval I in three steps.

1. *Domain.* The proposed F must be defined and differentiable at every point of I. Logarithms of quantities that change sign, square roots of negative numbers and functions such as $\tan(x/2)$ with poles in I are the usual failures.
2. *Derivative.* The identity $F'(x) = f(x)$ must hold at every point of I. A symbolic simplification of $F' - f$ to 0 proves it. So does a computation by hand as in Worked Example 9.6.2. A simplifier that fails to reach 0 proves nothing either way.
3. *Numbers.* Evaluating $F' - f$ at sample points detects errors. So does a comparison of $F(b) - F(a)$ with a numerical approximation of the integral (Section 9.7). A numerical check can refute an answer and cannot prove one. Finitely many values do not determine a function.

A definite integral needs one antiderivative continuous on the whole closed interval of integration (Theorem 8.3.5). An antiderivative on each piece of a partition may be used piece by piece, with additivity.

Common misconception

An answer that differs in form from the one in the book is not wrong for that reason. Its difference from the printed answer must be locally constant (Proposition 9.6.4). Conversely an answer whose derivative is correct at every point where it is defined can still fail as an antiderivative on an interval if it is undefined at a point of that interval.

Study guide

Treat a table entry as a theorem and check its hypotheses and the signs of its constants. Treat a CAS answer as a function whose interval you must supply. Find where it is defined, differentiate it there and compare with the integrand. Look for jumps on intervals where the integrand is continuous. Two correct answers differ by a constant on each interval. For a definite integral use an antiderivative continuous on the whole interval, or split the interval.

Problems

9.6.1 Evaluate $\int x^2 e^{-x}\,\mathrm{d}x$ with Theorem 9.2.6 as a table entry.

⋆ Hint · Answer · Solution

9.6.2 Evaluate $\int \sin x \cos x \, dx$ by the substitutions $u = \sin x$ and $u = \cos x$ and reconcile the two answers. ⋆ Hint · Answer · Solution

9.6.3 A CAS returns $\frac{1}{6}\log(x-3) - \frac{1}{6}\log(x+3)$ for $\int dx/(x^2-9)$. On which intervals is this an antiderivative when log is the real logarithm? Read log as the complex logarithm of this section and find the function it defines on each of $(-\infty,-3)$, $(-3,3)$ and $(3,\infty)$. Give the general real antiderivative on $\mathbb{R}\setminus\{-3,3\}$. ⋆⋆ Hint · Answer · Solution

9.6.4 Let $a<0$ and $|x|<|a|$. Show that the function $\arcsin(x/a)$ is defined there with derivative $-1/\sqrt{a^2-x^2}$. Which form of the arcsine entry is correct for every $a \neq 0$? ⋆⋆ Hint · Answer · Solution

9.6.5 With F as in Worked Example 9.6.2, prove that $F(x+2\pi) = F(x) + \dfrac{2\pi}{\sqrt{3}}$ for every real x and evaluate $\displaystyle\int_0^{4\pi} \frac{dx}{2+\cos x}$. ⋆⋆ Hint · Answer · Solution

9.6.6 Show that $F_1(x) = \arctan x$ and $F_2(x) = \arctan\dfrac{1+x}{1-x}$ both have derivative $\dfrac{1}{1+x^2}$ for $x \neq 1$. Find $F_2 - F_1$ on $(-\infty,1)$ and on $(1,\infty)$. ⋆⋆ Hint · Answer · Solution

9.6.7 Let f be continuous on an open interval I and $c \in I$. Let F be continuous on I and differentiable on $I\setminus\{c\}$ with $F' = f$ there. Prove that F is differentiable at c with $F'(c) = f(c)$. Why does this not rescue F_0 of Worked Example 9.6.2 on $(0,2\pi)$? ⋆⋆ Hint · Answer · Solution

9.6.8 Evaluate $\displaystyle\int_0^1 x^3 e^x \, dx$ with Theorem 9.2.7 as a table entry. ⋆ Hint · Answer · Solution

9.7 Numerical Integration

Overview

Objectives: Define the Trapezoidal and Simpson's Rules (Definitions 9.7.2 and 9.7.6) and read them as linear functionals on $(C([a,b]),\|\cdot\|_\infty)$. State their error formulas precisely (Theorems 9.7.3 and 9.7.7), with the one-step estimates proved in Appendices A.3 and A.4. Define the order of a rule (Definition 9.7.9), prove that the orders 2 and 4 are attained (Proposition 9.7.10) and invert an error bound to choose n.

Prerequisites: Riemann sums and integrability (Definition 8.2.4 and Theorem 8.2.13); the sup-norm estimate (Eq. (8.2.6)); the Extreme and Intermediate Value Theorems (Theorems 5.4.2 and 5.4.6); the Fundamental Theorem (Theorem 8.3.5).

Key results: $T_n - \int_a^b f = \dfrac{(b-a)h^2}{12} f''(\xi)$ and $S_n - \int_a^b f = \dfrac{(b-a)h^4}{180} f^{(4)}(\xi)$ with $h = (b-a)/n$; $|E_T| \le M(b-a)^3/(12n^2)$ and $|E_S| \le M(b-a)^5/(180n^4)$; $n^2 E_T \to \dfrac{(b-a)^2}{12}(f'(b)-f'(a))$.

Notation

$n,\ h$	the number of subintervals and their common width $(b-a)/n$
$x_k,\ y_k$	the kth partition point $a+kh$ and the value $f(x_k)$
$T_n,\ S_n$	the Trapezoidal and Simpson's Rule estimates
$E_T,\ E_S$	the signed errors $T_n - \int_a^b f$ and $S_n - \int_a^b f$
M	an upper bound for $\lvert f''\rvert$ or $\lvert f^{(4)}\rvert$ on $[a,b]$

The integral $\int_0^1 e^{x^2}\,\mathrm{d}x$ exists by Theorem 8.2.13, and no method of this chapter gives a formula for an antiderivative of its integrand (Problem 9.1.7). Its value is computed by replacing f on each piece of a partition with a polynomial that agrees with f at a few points and integrating the polynomials exactly. A rule is useful when its error is bounded in terms of f and n by a theorem. This section proves such bounds and shows that they cannot be improved in order.

9.7.1 Trapezoidal Rule

Partition $[a,b]$ into n subintervals of width $h=(b-a)/n$ with points $x_k = a+kh$ and values $y_k = f(x_k)$. Replacing f on $[x_{k-1},x_k]$ by the linear function through its endpoint values replaces the integral over that subinterval by the trapezoid area $\frac{h}{2}(y_{k-1}+y_k)$ (Fig. 9.7.1). Each interior value occurs in two adjacent trapezoids.

Definition 9.7.2 (Trapezoidal Rule)**.** For f continuous on $[a,b]$ and an integer $n \ge 1$, with h, x_k and y_k as above, the *Trapezoidal Rule* is

$$T_n(f) = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) = \sum_{k=1}^{n} \frac{h}{2}(y_{k-1}+y_k). \tag{9.7.1}$$

The map $f \mapsto T_n(f)$ is linear. Its weights $\frac{h}{2}, h, \dots, h, \frac{h}{2}$ are positive with sum $nh = b-a$. Hence $|T_n(f)| \le (b-a)\|f\|_\infty$ with equality for $f=1$. Like the integral itself (Theorem 8.2.17), T_n is a bounded linear functional of norm $b-a$ on $(C([a,b]),\|\cdot\|_\infty)$. The error functional

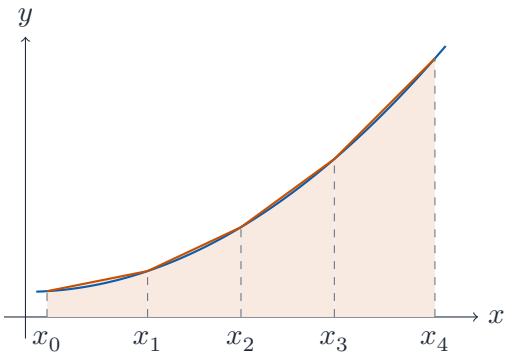


Figure 9.7.1: Trapezoidal Rule with $n = 4$. Each strip under $y = f(x)$ is replaced by the trapezoid under the chord joining consecutive points of the graph. For this convex function every chord lies above its arc and every trapezoid overestimates its strip.

$E_T = T_n - \int_a^b$ therefore satisfies $|E_T(f)| \le 2(b-a)\|f\|_\infty$. Bounds that tend to 0 as $n \to \infty$ need derivatives of f.

Theorem 9.7.3 (Trapezoidal Rule error)**.** *Let f'' be continuous on $[a,b]$ and $h = (b-a)/n$. There is a $\xi \in [a,b]$ with*

$$E_T = T_n(f) - \int_a^b f(x)\,\mathrm{d}x = \frac{(b-a)h^2}{12} f''(\xi). \tag{9.7.2}$$

In particular if $|f''| \le M$ on $[a,b]$ then

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \tag{9.7.3}$$

Proof. By Lemma A.3.1 on each $[x_{k-1}, x_k]$ there is $c_k \in (x_{k-1}, x_k)$ with $\int_{x_{k-1}}^{x_k} f = \frac{h}{2}(y_{k-1} + y_k) - \frac{h^3}{12} f''(c_k)$. Adding over k by additivity (Theorem 8.2.15),

$$E_T = \frac{h^3}{12} \sum_{k=1}^{n} f''(c_k) = \frac{(b-a)h^2}{12} \cdot \frac{1}{n} \sum_{k=1}^{n} f''(c_k). \tag{9.7.4}$$

The average of the values $f''(c_k)$ in (9.7.4) lies between their minimum $f''(c_i)$ and their maximum $f''(c_j)$. By the Intermediate Value Theorem (Theorem 5.4.2) applied to f'' between c_i and c_j, it equals $f''(\xi)$ for some ξ there. This proves Eq. (9.7.2). With $|f''(\xi)| \le M$ and $h = (b-a)/n$ it gives Eq. (9.7.3). A bound M exists by Theorem 5.4.6. ∎

Appendix A.3 proves Lemma A.3.1 by applying Rolle's theorem twice to the difference between the trapezoid and the integral over one subinterval.

9.7.4 Simpson's Rule

Simpson's Rule replaces f on each pair of adjacent subintervals by the quadratic through three consecutive points of the graph. It needs an even number n of subintervals.

Lemma 9.7.5 (Area under a parabola)**.** *Let $y = Ax^2 + Bx + C$ pass through $(-h, y_0)$, $(0, y_1)$ and (h, y_2) with $h > 0$. Then*

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,\mathrm{d}x = \frac{h}{3}(y_0 + 4y_1 + y_2). \tag{9.7.5}$$

Proof. The odd term Bx integrates to 0 (Corollary 8.5.4) and $\int_{-h}^{h}(Ax^2+C)\,\mathrm{d}x = \frac{2Ah^3}{3}+2Ch = \frac{h}{3}(2Ah^2+6C)$. The three points give $y_1 = C$ and $y_0+y_2 = 2Ah^2+2C$. Hence $2Ah^2+6C = y_0+y_2-2y_1+6y_1 = y_0+4y_1+y_2$. ∎

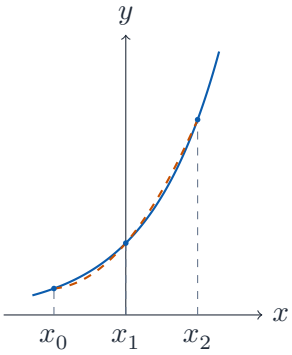


Figure 9.7.2: Simpson's Rule on one panel. The parabola (dashed) through three points of the graph of f (solid) agrees with f at those points. Lemma 9.7.5 gives the area under it.

By the translation $u = x-x_{2i-1}$ and Eq. (8.5.3), the lemma gives the integral of the parabola through (x_{2i-2},y_{2i-2}), (x_{2i-1},y_{2i-1}) and (x_{2i},y_{2i}) as $\frac{h}{3}(y_{2i-2}+4y_{2i-1}+y_{2i})$ (Fig. 9.7.2). Adding over the $n/2$ panels, an interior point of even index ends one panel and begins the next. It receives the weight 2. A point of odd index is the midpoint of one panel and receives the weight 4.

Definition 9.7.6 (Simpson's Rule)**.** For f continuous on $[a,b]$ and an even integer $n\geq 2$, with h, x_k and y_k as in Definition 9.7.2, *Simpson's Rule* is

$$S_n(f) = \frac{h}{3}(y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n). \tag{9.7.6}$$

The weights of S_n are positive with sum $\frac{h}{3}(2+4\cdot\frac{n}{2}+2(\frac{n}{2}-1)) = nh = b-a$, and S_n is again a linear functional of norm $b-a$.

Theorem 9.7.7 (Simpson's Rule error)**.** *Let $f^{(4)}$ be continuous on $[a,b]$, n even and $h=(b-a)/n$. There is a $\xi\in[a,b]$ with*

$$E_S = S_n(f)-\int_a^b f(x)\,\mathrm{d}x = \frac{(b-a)h^4}{180}\,f^{(4)}(\xi). \tag{9.7.7}$$

In particular if $\left|f^{(4)}\right|\leq M$ on $[a,b]$ then

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \tag{9.7.8}$$

Proof. By Lemma A.4.1 on each panel $[x_{2i-2},x_{2i}]$ there is a c_i in the panel with $\int_{x_{2i-2}}^{x_{2i}} f = \frac{h}{3}(y_{2i-2}+4y_{2i-1}+y_{2i}) - \frac{h^5}{90}f^{(4)}(c_i)$. Adding over the $n/2$ panels,

$$E_S = \frac{h^5}{90}\sum_{i=1}^{n/2} f^{(4)}(c_i) = \frac{(b-a)h^4}{180}\cdot\frac{2}{n}\sum_{i=1}^{n/2} f^{(4)}(c_i),$$

using $\frac{h^5}{90}\cdot\frac{n}{2}=\frac{(b-a)h^4}{180}$. As in Theorem 9.7.3, the average of the $n/2$ values equals $f^{(4)}(\xi)$ for some ξ by the Intermediate Value Theorem. The bound follows with $h=(b-a)/n$. ∎

Appendix A.4 proves Lemma A.4.1 with three applications of Rolle's theorem and one of the Mean Value Theorem.

9.7.8 Order of convergence

Definition 9.7.9 (Order of a rule)**.** A sequence of rules Q_n has *order* $p>0$ on a class of functions on $[a,b]$ if for each f in the class there is a constant K_f with $\left|Q_n(f)-\int_a^b f\right|\le K_f\,n^{-p}$ for every admissible n.

By Theorems 9.7.3 and 9.7.7, T_n has order 2 on $C^2([a,b])$ and S_n has order 4 on $C^4([a,b])$. Doubling n divides the error bound by 4 and by 16. The next proposition shows that the orders cannot be raised. The errors are asymptotically equal to a constant times n^{-2} and n^{-4}.

Proposition 9.7.10 (Error asymptotics of the Trapezoidal and Simpson's Rules)**.** *If f'' is continuous on $[a,b]$, then*

$$\lim_{n\to\infty} n^2 E_T=\frac{(b-a)^2}{12}\bigl(f'(b)-f'(a)\bigr). \tag{9.7.9}$$

If $f^{(4)}$ is continuous on $[a,b]$, then along even n

$$\lim_{n\to\infty} n^4 E_S=\frac{(b-a)^4}{180}\bigl(f'''(b)-f'''(a)\bigr). \tag{9.7.10}$$

Proof. From the proof of Theorem 9.7.3 and $nh=b-a$,

$$n^2E_T=n^2\frac{h^3}{12}\sum_{k=1}^{n}f''(c_k)=\frac{(b-a)^2}{12}\sum_{k=1}^{n}f''(c_k)\,h.$$

The last sum is a Riemann sum of the continuous function f'' for a partition of mesh h with sample points c_k. As $n\to\infty$ it tends to $\int_a^b f''=f'(b)-f'(a)$ (Theorems 8.2.13 and 8.3.5). For Simpson's Rule, $n^4E_S=\dfrac{(b-a)^4}{180}\sum_{i=1}^{n/2}f^{(4)}(c_i)\,(2h)$ is a Riemann sum of $f^{(4)}$ over the panels of width $2h$ and tends to $f'''(b)-f'''(a)$. ∎

When $f'(b)\neq f'(a)$ the error of T_n is therefore bounded below by a positive constant times n^{-2} for large n. No order $p>2$ is then possible. The same holds for S_n and the order 4 when $f'''(b)\neq f'''(a)$. The limits also predict the error. For $\int_0^1 e^x\,dx$ the constant is $(e-1)/12\approx 0.1432$, and $0.1432/4^2\approx 0.00895$ is close to the error of T_4 in Problem 9.7.3.

Worked Example 9.7.1

Estimate $\displaystyle\int_0^2 x^3\,dx$ with T_4 and compare the error with Theorem 9.7.3.

Here $h=\frac12$ and $y_k=x_k^3$ takes the values $0,\frac18,1,\frac{27}{8},8$. By Eq. (9.7.1),

$$T_4=\frac14\Bigl(0+\frac14+2+\frac{27}{4}+8\Bigr)=\frac{17}{4}.$$

The integral is 4 and $E_T = \frac{1}{4}$. Since $f''(x) = 6x$ is at most 12 on $[0, 2]$, Eq. (9.7.3) gives $|E_T| \le \frac{12 \cdot 8}{12 \cdot 16} = \frac{1}{2}$. By Eq. (9.7.2), $\frac{1}{4} = \frac{2 \cdot (1/4)}{12} 6\xi$ and hence $\xi = 1$. The error is positive because $f'' > 0$ on $(0, 2]$, in agreement with Fig. 9.7.1.

Worked Example 9.7.2

Estimate $\int_0^2 x^4 \, dx$ with S_4 and compare the error with Theorem 9.7.7.

With $h = \frac{1}{2}$ the values $y_k = x_k^4$ are $0, \frac{1}{16}, 1, \frac{81}{16}, 16$. By Eq. (9.7.6),

$$S_4 = \frac{1}{6}\Big(0 + \frac{1}{4} + 2 + \frac{81}{4} + 16\Big) = \frac{77}{12}.$$

The integral is $\frac{32}{5}$ and $E_S = \frac{1}{60}$. The fourth derivative is the constant 24. With $M = 24$, Eq. (9.7.8) gives exactly $\frac{24 \cdot 32}{180 \cdot 256} = \frac{1}{60}$. The bound is attained because Eq. (9.7.7) holds with $f^{(4)}(\xi) = 24$ for every ξ.

Worked Example 9.7.3

Find an even n for which Simpson's Rule approximates $\int_0^\pi \sin x \, dx$ with error less than 10^{-4}.

Here $f^{(4)} = \sin$ and $M = 1$. By Eq. (9.7.8) it suffices that $\pi^5/(180n^4) < 10^{-4}$, that is, $n^4 > \pi^5 \cdot 10^4/180 \approx 17001.09$ and $n > 11.42$. The smallest even n is 12. The bound is sufficient and need not be necessary. A direct computation gives $S_{10} \approx 2.00011$ with error 1.10×10^{-4} and $S_{12} \approx 2.00005$ with error 5.26×10^{-5}. Here $n = 12$ is also the least even n that works.

Caution

An error bound needs the derivative hypothesis on the whole closed interval. For $\int_0^1 \sqrt{x} \, dx$ the second derivative is unbounded near 0. Theorem 9.7.3 then says nothing. The observed errors do not decrease like n^{-2} (Problem 9.7.7).

Study guide

List h, the points x_k and the values y_k before applying a rule. The weights are $1, 2, \dots, 2, 1$ times $\frac{h}{2}$ for the Trapezoidal Rule and $1, 4, 2, 4, \dots, 4, 1$ times $\frac{h}{3}$ for Simpson's Rule, with n even. For a bound, find M for $|f''|$ or $|f^{(4)}|$ on the whole interval and use Eq. (9.7.3) or Eq. (9.7.8). To choose n, solve the bound for n and round up, to an even integer for Simpson's Rule.

The exact error formulas say more than the bounds. Their sign is the sign of f'' or $f^{(4)}$. By Proposition 9.7.10 the orders 2 and 4 are attained. The one-step estimates from which the formulas follow are proved in Appendices A.3 and A.4.

Problems

9.7.1 Bound the error of T_6 for $\int_0^{\pi/2} \sin(2x) \, dx$, compute T_6 and confirm that the true error lies within the bound. ⋆⋆ Hint · Answer · Solution

9.7.2 Compute S_2 for $\displaystyle\int_1^3 (2x^3 - x)\,\mathrm{d}x$ and explain why it equals the integral. For which polynomials is Simpson's Rule exact? ⋆ Hint · Answer · Solution

9.7.3 Estimate $\displaystyle\int_0^1 e^x\,\mathrm{d}x$ with T_4 and S_4, compare the errors and compare them with the predictions of Proposition 9.7.10. ⋆⋆ Hint · Answer · Solution

9.7.4 Bound the error of S_4 for $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x}$, compute S_4 and confirm that the true error lies within the bound. ⋆⋆ Hint · Answer · Solution

9.7.5 Let f'' be continuous on $[\mu - h, \mu + h]$. Prove that there is a c in the interval with

$$\int_{\mu-h}^{\mu+h} f(x)\,\mathrm{d}x = 2h\,f(\mu) + \frac{h^3}{3}\,f''(c).$$

Deduce that the Midpoint Rule $M_n = \Delta \sum_{k=1}^{n} f(a+(k-\tfrac{1}{2})\Delta)$, $\Delta = (b-a)/n$, satisfies $\int_a^b f - M_n = \dfrac{(b-a)\Delta^2}{24} f''(\xi)$ for some $\xi \in [a,b]$. ⋆⋆⋆ Hint · Answer · Solution

9.7.6 Prove that $S_{2m} = \dfrac{4T_{2m} - T_m}{3}$ for every integer $m \geq 1$ and every continuous f. Explain with Proposition 9.7.10 why this combination improves on T_{2m}. ⋆⋆ Hint · Answer · Solution

9.7.7 For $\displaystyle\int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}$ compute T_1, T_2 and T_4 and the ratios of successive errors. Why does Theorem 9.7.3 not predict the ratio 4 here? ⋆⋆ Hint · Answer · Solution

9.7.8 Prove that $T_n(f) \to \int_a^b f(x)\,\mathrm{d}x$ as $n \to \infty$ for every continuous f on $[a,b]$, with no hypothesis on derivatives. ⋆⋆ Hint · Answer · Solution

9.8 Improper Integrals

Overview

Objectives: Define the integral of a 1-form over an unbounded interval and the integral of an integrand unbounded near an endpoint as limits of integrals over closed bounded intervals (Definitions 9.8.2 and 9.8.3), and prove that a split point does not matter (Lemma 9.8.4). Prove the p-integral benchmarks (Theorem 9.8.7), the comparison and limit comparison tests with counterexamples for their hypotheses (Theorems 9.8.9 and 9.8.10), and the convergence of an absolutely convergent integral (Theorem 9.8.13). Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges conditionally (Proposition 9.8.14).
Prerequisites: The integral of a 1-form and its properties (Definition 8.2.2 and Theorem 8.2.15); the Fundamental Theorem (Theorem 8.3.5); one-sided limits, limits at infinity and monotone limits (Definition 5.2.10 and Proposition 5.2.23); integration by parts (Theorem 9.2.1).
Key results: $\int_1^\infty x^{-p}\,dx$ converges exactly when $p > 1$ and $\int_0^1 x^{-p}\,dx$ exactly when $p < 1$. For $0 \le f \le g$, convergence of $\int g$ implies convergence of $\int f$. Absolute convergence implies convergence. The converse is false.

Notation

β	the improper endpoint: ∞ or a finite point near which f is unbounded
$F(b)$	the proper integral $\int_a^b f(x)\,dx$ for $a \le b < \beta$
$\int_a^\beta f$	the improper integral $\lim_{b\to\beta^-} F(b)$
p	the exponent in the benchmark integrand x^{-p}

The integral of a 1-form (Definition 8.2.2) is defined over a closed bounded interval. Theorem 8.2.13 guarantees it for an integrand continuous there. Neither $\int_1^\infty dx/x^2$ nor $\int_0^1 dx/\sqrt{x}$ meets these conditions. The first interval is unbounded. The second integrand is unbounded near 0. No Riemann sum with a sample point near 0 is controlled. Each receives a value when it exists as a limit of integrals of $f(x)\,dx$ over closed bounded intervals that exhaust the interval.

9.8.1 Definitions

Definition 9.8.2 (Integral over an unbounded interval)**.** Let f be continuous on $[a, \infty)$. The integral of $f(x)\,dx$ over $[a, \infty)$ is

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx. \tag{9.8.1}$$

It *converges* when the limit exists as a real number and *diverges* otherwise. For f continuous on $(-\infty, b]$,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx. \tag{9.8.2}$$

For f continuous on $\mathbb{R}$ and a real c,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \tag{9.8.3}$$

which converges when *both* integrals on the right converge.

Definition 9.8.3 (Integral of an unbounded integrand)**.** Let f be continuous on $[a, b)$. The improper integral is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b^-} \int_a^c f(x)\,\mathrm{d}x, \tag{9.8.4}$$

and for f continuous on $(a, b]$

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to a^+} \int_c^b f(x)\,\mathrm{d}x. \tag{9.8.5}$$

For f continuous on $[a, b]$ except at a point d with $a < d < b$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^d f(x)\,\mathrm{d}x + \int_d^b f(x)\,\mathrm{d}x, \tag{9.8.6}$$

which converges when both integrals on the right converge. Convergence and divergence are defined as in Definition 9.8.2.

If f is continuous on the closed interval $[a, b]$, the limit Eq. (9.8.4) equals the proper integral because $c \mapsto \int_a^c f$ is continuous (Theorem 8.3.4). The definitions therefore extend the integral of Chapter 8. By the next lemma the split points in Eqs. (9.8.3) and (9.8.6) are arbitrary.

Lemma 9.8.4 (Independence of the split point of an improper integral)**.** *Let f be continuous on the open interval with endpoints $\alpha < \beta$. Either endpoint may be infinite or a point where f is undefined. Let c, c' lie strictly between them. Then $\int_c^\beta f$ converges exactly when $\int_{c'}^\beta f$ does. In that case $\int_c^\beta f = \int_c^{c'} f + \int_{c'}^\beta f$. The same holds at α. Hence the convergence and the value of $\int_\alpha^c f + \int_c^\beta f$ do not depend on c.*

Proof. For b between c' and β, additivity (Theorem 8.2.15) gives $\int_c^b f = \int_c^{c'} f + \int_{c'}^b f$. The first term on the right is a fixed number. As $b \to \beta^-$, the left side has a limit exactly when $\int_{c'}^b f$ has one. The two limits then differ by $\int_c^{c'} f$ (Theorem 5.2.13). The argument at α is the same. In the sum of the two identities the terms $\int_c^{c'} f$ and $\int_{c'}^c f$ cancel. ∎

The two limits in Eq. (9.8.3) are independent. A symmetric limit is a different quantity. For $f(x) = x$, $\int_{-R}^R x\,\mathrm{d}x = 0$ for every R. The integral $\int_0^\infty x\,\mathrm{d}x$ diverges and so does $\int_{-\infty}^\infty x\,\mathrm{d}x$.

Example 9.8.5 (Interior singularity)**.** The function $1/x$ has no value at 0. The integral $\int_{-1}^1 \mathrm{d}x/x$ is therefore improper there. By Eq. (9.8.5) $\int_0^1 \mathrm{d}x/x = \lim_{c\to 0^+}(-\ln c) = \infty$. The integral therefore diverges. The computation $\left[\ln|x|\right]_{-1}^1 = 0$ applies Theorem 8.3.5 to a function that is not an antiderivative on $[-1, 1]$.

9.8.6 Benchmarks and comparison

The rest of the section treats an integral improper only at its upper end β. Either $\beta = \infty$ or β is a finite point near which f is unbounded. Write $\int_a^\beta f = \lim_{b\to\beta^-} \int_a^b f$. Every statement has a mirror image at a lower end with the same proof.

Theorem 9.8.7 (p-integral benchmarks)**.** *For a real number p,*

$$\begin{aligned} &\int_1^\infty \frac{\mathrm{d}x}{x^p} \text{ converges for } p > 1 \text{ and diverges for } p \le 1, \\ &\int_0^1 \frac{\mathrm{d}x}{x^p} \text{ converges for } p < 1 \text{ and diverges for } p \ge 1. \end{aligned} \tag{9.8.7}$$

For $p > 1$ the first integral is $\dfrac{1}{p-1}$. For $p < 1$ the second is $\dfrac{1}{1-p}$.

Proof. For $p \neq 1$ the power rule gives $\int_1^b x^{-p}\,\mathrm{d}x = \dfrac{b^{1-p}-1}{1-p}$. If $p > 1$, then $b^{1-p} \to 0$ as $b \to \infty$ and the limit is $\dfrac{1}{p-1}$. If $p < 1$, then $b^{1-p} \to \infty$ and the integral diverges. Similarly $\int_a^1 x^{-p}\,\mathrm{d}x = \dfrac{1-a^{1-p}}{1-p}$. As $a \to 0^+$ this tends to $\dfrac{1}{1-p}$ when $p < 1$ and to ∞ when $p > 1$. For $p = 1$ the antiderivative is $\ln x$. As $b \to \infty$, $\ln b \to \infty$. As $a \to 0^+$, $-\ln a \to \infty$. Both integrals diverge. ∎

Lemma 9.8.8 (Boundedness criterion for improper integrals of non-negative functions)**.** *Let $f \ge 0$ be continuous on $[a, \beta)$. Then $\int_a^\beta f$ converges exactly when $F(b) = \int_a^b f$ is bounded above on $[a, \beta)$, and then $\int_a^\beta f = \sup_b F(b)$.*

Proof. For $a \le b < b' < \beta$, additivity and the order property give $F(b') - F(b) = \int_b^{b'} f \ge 0$. Hence F is nondecreasing. If F is bounded above, Proposition 5.2.23 gives the limit $\sup F$. If F is unbounded above, then for every K some b_0 has $F(b_0) > K$. Then $F(b) \ge F(b_0) > K$ for all $b \ge b_0$. The limit is then ∞. ∎

Theorem 9.8.9 (Direct comparison test)**.** *Let f and g be continuous on $[a, \beta)$ with*

$$0 \le f(x) \le g(x), \qquad a \le x < \beta. \tag{9.8.8}$$

If $\int_a^\beta g(x)\,\mathrm{d}x$ converges, then $\int_a^\beta f(x)\,\mathrm{d}x$ converges. If $\int_a^\beta f(x)\,\mathrm{d}x$ diverges, then $\int_a^\beta g(x)\,\mathrm{d}x$ diverges.

Proof. By the order property $\int_a^b f \le \int_a^b g \le \int_a^\beta g$ for every $b < \beta$, the last step by Lemma 9.8.8. The integrals of f are bounded above and Lemma 9.8.8 gives convergence. The second statement is the contrapositive of the first. ∎

Both inequalities are needed. Without $f \le g$ the pair $f = 1/x$ and $g = 1/x^2$ on $[1, \infty)$ has $0 \le f$ with $\int g$ convergent and $\int f$ divergent. Without $0 \le f$ the pair $f = -1$ and $g = 0$ on $[1, \infty)$ has $f \le g$ with $\int g$ convergent and $\int f$ divergent. The inequality need only hold on some $[x_0, \beta)$. The integral $\int_a^{x_0}$ is proper and Lemma 9.8.4 applies.

Theorem 9.8.10 (Limit comparison test)**.** *Let f and g be positive and continuous on $[a, \beta)$ with*

$$\lim_{x \to \beta^-} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty. \tag{9.8.9}$$

Then $\int_a^\beta f(x)\,\mathrm{d}x$ and $\int_a^\beta g(x)\,\mathrm{d}x$ either both converge or both diverge.

Proof. There is $x_0 \in [a, \beta)$ with $\frac{L}{2} < f(x)/g(x) < \frac{3L}{2}$ for $x_0 \le x < \beta$, that is, $\frac{L}{2}g < f < \frac{3L}{2}g$ there. If $\int_a^\beta g$ converges, so does $\int_a^\beta \frac{3L}{2}g$. Theorem 9.8.9 on $[x_0, \beta)$ then makes $\int_a^\beta f$ converge. If $\int_a^\beta g$ diverges, so does $\int_a^\beta \frac{L}{2}g$. Theorem 9.8.9 then makes $\int_a^\beta f$ diverge. ∎

The condition $L > 0$ is needed. For $f = x^{-2}$ and $g = x^{-1}$ on $[1, \infty)$ the ratio tends to 0, the integral of f converges and that of g diverges. Positivity of f and g is needed as well (Problem 9.8.10).

Worked Example 9.8.1

Evaluate $\displaystyle\int_0^\infty e^{-x}\,\mathrm{d}x$.

By Eq. (9.8.1),

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}(1 - e^{-b}) = 1.$$

Worked Example 9.8.2

Evaluate $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}$.

The integrand is continuous on $[0, 1)$ and unbounded as $x \to 1^-$. By Eq. (9.8.4) and continuity of arcsin at 1,

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{c\to 1^-} \arcsin c = \frac{\pi}{2}.$$

This value is the length of the quarter circle of Theorem 8.6.6.

Worked Example 9.8.3

Evaluate $\displaystyle\int_{-1}^4 \frac{\mathrm{d}x}{\sqrt{|x|}}$.

The integrand is undefined at 0. Equation (9.8.6) splits the integral there. An antiderivative is $-2\sqrt{-x}$ on $[-1, 0)$ and $2\sqrt{x}$ on $(0, 4]$. Hence

$$\int_{-1}^0 \frac{\mathrm{d}x}{\sqrt{-x}} = \lim_{c\to 0^-}(2 - 2\sqrt{-c}) = 2, \qquad \int_0^4 \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{c\to 0^+}(4 - 2\sqrt{c}) = 4.$$

Both pieces converge. The integral is 6.

Worked Example 9.8.4

Decide whether $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^4+1}}$ converges.

For $x \ge 1$, $\sqrt{x^4+1} > x^2$ and $0 < 1/\sqrt{x^4+1} < 1/x^2$. The integral of $1/x^2$ over $[1, \infty)$ converges by Theorem 9.8.7, and Theorem 9.8.9 shows that the given integral converges.

9.8.11 Absolute and conditional convergence

Definition 9.8.12 (Absolute convergence)**.** Let f be continuous on $[a,\beta)$. The integral $\int_a^\beta f$ *converges absolutely* if $\int_a^\beta |f|$ converges. It *converges conditionally* if it converges and $\int_a^\beta |f|$ diverges.

Theorem 9.8.13 (Absolute convergence implies convergence)**.** *If $\int_a^\beta |f|$ converges, then $\int_a^\beta f$ converges and $\left|\int_a^\beta f\right| \le \int_a^\beta |f|$.*

Proof. The function $f+|f|$ is continuous with $0 \le f+|f| \le 2|f|$. By Theorem 9.8.9 its integral converges. For every $b<\beta$, $\int_a^b f = \int_a^b (f+|f|) - \int_a^b |f|$. Both terms on the right have limits as $b \to \beta^-$. So does their difference (Theorem 5.2.13). The inequality $\left|\int_a^b f\right| \le \int_a^b |f|$ of Theorem 8.2.15(iv) holds for every b and passes to the limit. ∎

The converse is false. The standard example is $\sin x/x$. It is continuous on $(0,\infty)$ with limit 1 at 0 (Theorem 5.2.18). Its value at 0 is defined to be 1.

Proposition 9.8.14 (Conditional convergence of $\int \sin x/x$)**.** *The integral $\int_0^\infty \dfrac{\sin x}{x}\,dx$ converges and $\int_0^\infty \dfrac{|\sin x|}{x}\,dx$ diverges.*

Proof. On $[0,1]$ the integrand is continuous and the integral is proper. By Lemma 9.8.4 it suffices to treat $[1,\infty)$. For $b>1$, Eq. (9.2.2) with $u=1/x$ and $v=-\cos x$ gives

$$\int_1^b \frac{\sin x}{x}\,dx = \cos 1 - \frac{\cos b}{b} - \int_1^b \frac{\cos x}{x^2}\,dx. \tag{9.8.10}$$

As $b\to\infty$, $|\cos b/b| \le 1/b \to 0$. Since $|\cos x/x^2| \le 1/x^2$, the last integral converges absolutely by Theorems 9.8.7 and 9.8.9 and converges by Theorem 9.8.13. Hence $\int_1^\infty \dfrac{\sin x}{x}\,dx$ converges.

For the second statement use $|\sin x| \ge \sin^2 x = \frac{1}{2}(1-\cos 2x)$. For $b>1$,

$$\int_1^b \frac{\sin^2 x}{x}\,dx = \frac{1}{2}\int_1^b \frac{dx}{x} - \frac{1}{2}\int_1^b \frac{\cos 2x}{x}\,dx.$$

Integration by parts as in (9.8.10) with $\cos 2x$ in place of $\sin x$ shows that the last integral has a limit as $b\to\infty$. The first tends to ∞. Hence $\int_1^\infty \dfrac{\sin^2 x}{x}\,dx$ diverges, and Theorem 9.8.9 with $0 \le \dfrac{\sin^2 x}{x} \le \dfrac{|\sin x|}{x}$ shows that $\int_1^\infty \dfrac{|\sin x|}{x}\,dx$ diverges. ∎

The positive and negative parts of $\sin x/x$ each have divergent integrals. The integral converges because they cancel in the limit. A comparison test cannot detect that cancellation. Every comparison test proves absolute convergence.

Common misconception

Convergence of $\int_a^\infty f$ does not force $f(x)\to 0$. The integrand may consist of narrower and narrower spikes of height 1 (Problem 9.8.11). For series the corresponding statement is true (Part IV).

Study guide

Before integrating, find every point of $[a,b]$ where the integrand is undefined or unbounded and note whether an endpoint is infinite. Split the integral so that each piece is improper at one end only. Treat each piece as a limit of proper integrals of $f(x)\,\mathrm{d}x$. The integral converges only if every piece converges. The definition excludes a symmetric limit.
Without an antiderivative, compare. For a non-negative integrand the integral converges exactly when its partial integrals are bounded. An upper bound by a convergent integral proves convergence. A lower bound by a divergent integral proves divergence. The reverse pairings prove nothing. For an integrand that changes sign, test $|f|$ first. If that fails, integrate by parts to exhibit the cancellation as for $\sin x/x$.

Problems

9.8.1 Investigate the convergence of $\displaystyle\int_0^2 \frac{\mathrm{d}x}{(x-1)^2}$. Why is the value $\left[-\frac{1}{x-1}\right]_0^2 = -2$ wrong?

⋆⋆ Hint · Answer · Solution

9.8.2 Investigate the convergence of $\displaystyle\int_{-\infty}^{\infty} e^{x}\,\mathrm{d}x$.

⋆ Hint · Answer · Solution

9.8.3 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{x+1}{x^3+3}\,\mathrm{d}x$.

⋆⋆ Hint · Answer · Solution

9.8.4 The inequality $\dfrac{1}{\sqrt{x^4+x}} \geq \dfrac{1}{\sqrt{2}\,x^2}$ holds for $x \geq 1$. Explain why it says nothing about $\displaystyle\int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{x^4+x}}$ and decide the convergence of the integral.

⋆⋆ Hint · Answer · Solution

9.8.5 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{\mathrm{d}x}{\ln(x+1)}$.

⋆⋆ Hint · Answer · Solution

9.8.6 Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,\mathrm{d}x$.

⋆⋆ Hint · Answer · Solution

9.8.7 Prove by induction that $\displaystyle\int_0^{\infty} x^n e^{-x}\,\mathrm{d}x = n!$ for every integer $n \geq 0$.

⋆⋆ Hint · Answer · Solution

9.8.8 Prove that $\displaystyle\int_1^{\infty} \frac{\sin x}{x^p}\,\mathrm{d}x$ converges for every $p > 0$ and that it converges absolutely exactly when $p > 1$.

⋆⋆ Hint · Answer · Solution

9.8.9 For $n \geq 1$ let f_n on $[0,\infty)$ be $\dfrac{1}{n}$ on $[0,n]$, $\dfrac{1}{n}(n+1-x)$ on $[n,n+1]$ and 0 on $[n+1,\infty)$. Show that $\|f_n\|_\infty = \dfrac{1}{n}$ and $\displaystyle\int_0^{\infty} f_n\,\mathrm{d}x = 1 + \frac{1}{2n}$. Conclude that no constant K satisfies $\left|\displaystyle\int_0^{\infty} f\right| \leq K\|f\|_\infty$ for every continuous f with a convergent integral.

⋆⋆ Hint · Answer · Solution

9.8.10 Let $g(x) = \dfrac{\sin x}{\sqrt{x}}$ and $f(x) = g(x) + \dfrac{\sin^2 x}{x}$ on $[1,\infty)$. Show that $f(x)/g(x) \to 1$ as $x \to \infty$ through the points where $g(x) \neq 0$, that $\displaystyle\int_1^{\infty} g$ converges and that $\displaystyle\int_1^{\infty} f$ diverges. Which hypothesis of Theorem 9.8.10 fails?

⋆⋆⋆ Hint · Answer · Solution

9.8.11 Construct a continuous $f \geq 0$ on $[1, \infty)$ with $f(n) = 1$ for every integer $n \geq 2$ and $\int_1^\infty f$ convergent. ⋆⋆⋆ Hint · Answer · Solution

Part IV

Infinite Series

Chapter 10

Sequences and Series

An infinite series is the limit of its sequence of partial sums. This chapter applies the theory of sequence limits from Section 5.1 to series of real numbers and more generally to series whose terms lie in a normed space (Definition 3.6.3). Section 10.1 extends convergence and the Cauchy criterion to normed spaces, defines completeness and proves that the continuous functions on a closed interval form a complete space under the sup norm. Section 10.2 defines the sum of a series, proves the Cauchy criterion for series and shows that absolute convergence implies convergence in every complete normed space. The integral, comparison, ratio, root and alternating series tests of Sections 10.3 to 10.6 decide the convergence of a real series without a formula for its partial sums.

10.1 Sequences and Complete Spaces

Overview

Objectives: Apply the limit theory of Section 5.1 to the sequences that recur in the study of series and use the six standard limits (Theorem 10.1.4); define the upper and lower limits of a real sequence (Definition 10.1.6) and use their characterisation (Lemma 10.1.9); define a complete normed space (Definition 10.1.13) and give an example and a non-example; read uniform convergence as convergence in the sup norm (Definition 10.1.15) and prove that $C([a,b])$ is complete under that norm (Theorem 10.1.17).

Prerequisites: Limits of sequences (Section 5.1), in particular Definition 5.1.3 and Theorems 5.1.9, 5.1.11, 5.1.18 and 5.1.28; sequences in a normed space (Definitions 5.1.30 and 5.1.34); the continuous function theorem (Theorem 5.3.5); the growth ranking (Theorem 5.2.26) and $e^x = \lim(1 + x/n)^n$ (Theorem 5.4.15); norms and normed spaces (Definitions 3.6.2 and 3.6.3), the norms on $\mathbb{R}^n$ (Definition 3.6.6 and Theorem 3.6.15) and the sup norm (Definition 3.6.9).

Key results: The six standard limits (Theorem 10.1.4); the characterisation of the upper limit (Lemma 10.1.9); limits in a normed space (Theorem 5.1.31); complete normed spaces (Definition 10.1.13); continuity of a uniform limit (Lemma 10.1.16); completeness of $C([a,b])$ under the sup norm (Theorem 10.1.17).

Notation

$\limsup a_n$	the upper limit of a real sequence (Definition 10.1.6)
$\liminf a_n$	the lower limit, $-\limsup(-a_n)$
$\|v\|$	the norm of a vector v in a normed space V
$C([a,b])$	the continuous real functions on $[a,b]$
$\|f\|_\infty$	the sup norm $\sup_x \|f(x)\|$ (Definition 3.6.9)

Every convergence question in Part IV is a question about a sequence. A series is handled in Section 10.2 through the sequence of its partial sums, a power series in Section 11.1 through a sequence of polynomials, and a Taylor series in Section 11.3 through a sequence of remainders. This section collects the real limits that the chapter uses and extends the definitions to sequences of vectors and of functions with a norm in place of the absolute value.

10.1.1 Arithmetic and geometric sequences

A sequence is *arithmetic* when each term exceeds its predecessor by a fixed *common difference* d and *geometric* when each term is a fixed *common ratio* r times its predecessor. Unwinding each recursion from the first term a gives

$$a_n = a + (n-1)d \qquad \text{(arithmetic)}, \tag{10.1.1}$$

$$a_n = a\, r^{n-1} \qquad \text{(geometric)}. \tag{10.1.2}$$

Theorem 10.1.2 (Arithmetic and geometric partial sums)**.** *Let $s_n = a_1 + a_2 + \cdots + a_n$. For an arithmetic sequence with first term a and common difference d,*

$$s_n = \frac{n}{2}(2a + (n-1)d) = n \cdot \frac{a_1 + a_n}{2}, \tag{10.1.3}$$

and for a geometric sequence with first term a and common ratio $r \neq 1$,

$$s_n = \frac{a(1 - r^n)}{1 - r}. \tag{10.1.4}$$

Proof. Write s_n forwards and backwards and add the two lines term by term:

$$\begin{aligned} s_n &= a + (a+d) + \cdots + (a + (n-1)d), \\ s_n &= (a + (n-1)d) + \cdots + (a+d) + a, \\ 2s_n &= n(2a + (n-1)d), \end{aligned}$$

since each vertical pair sums to $2a + (n-1)d$. Dividing by 2 gives (10.1.3). For (10.1.4), $s_n = a\sum_{k=0}^{n-1} r^k$ and the finite geometric sum Eq. (4.6.11) with $n-1$ in place of n gives the formula. ∎

An arithmetic sequence with $d \neq 0$ is unbounded and diverges by Theorem 5.1.7. A geometric sequence ar^{n-1} with $a \neq 0$ tends to 0 when $|r| < 1$, is constant when $r = 1$ and diverges when $|r| > 1$ (Proposition 5.1.14) or $r = -1$. For $r = -1$ the terms alternate between a and $-a$. The first result of Section 10.2 rests on this asymmetry.

10.1.3 Six standard limits

A sequence is the restriction of a function to the positive integers. If $f(n) = a_n$ for every n and $f(x) \to L$ as $x \to \infty$, then $a_n \to L$ (Definition 5.2.20) and the methods of the derivative chapters apply to the real extension f. The converse fails: $\sin(\pi n) = 0$ for every n while $\sin(\pi x)$ has no limit as $x \to \infty$.

Theorem 10.1.4 (Standard limits of sequences)**.** *For any real x,*

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0, \qquad \lim_{n\to\infty} \sqrt[n]{n} = 1, \qquad \lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0), \tag{10.1.5}$$

$$\lim_{n\to\infty} x^n = 0 \quad (|x| < 1), \qquad \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \qquad \lim_{n\to\infty} \frac{x^n}{n!} = 0. \tag{10.1.6}$$

Proof. The first is the growth ranking (Theorem 5.2.26) restricted to the integers. For the second, $\sqrt[n]{n} = e^{(\ln n)/n}$ with an exponent that tends to 0 by the first limit. Since the exponential is continuous at 0 (Theorem 5.3.10), Theorem 5.3.5 gives $\sqrt[n]{n} \to e^0 = 1$. The third is Proposition 5.1.15, the fourth is Proposition 5.1.14 and the fifth is Theorem 5.4.15.

For the last, fix an integer $K > 2|x|$. For $n > K$,

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^K}{K!} \cdot \prod_{j=K+1}^{n} \frac{|x|}{j} \le \frac{|x|^K}{K!} \left(\frac{1}{2}\right)^{n-K} \tag{10.1.7}$$

because each factor $|x|/j$ with $j > K > 2|x|$ is at most $\frac{1}{2}$. The right-hand side of (10.1.7) is a constant times a geometric sequence of ratio $\frac{1}{2}$ and tends to 0. Corollary 5.1.12 gives $x^n/n! \to 0$. ∎

Worked Example 10.1.1

Decide whether $a_n = \dfrac{n^2}{2^n}$ converges and find its limit if it does.

Pass to the real extension $f(x) = x^2/2^x = x^2 e^{-x \ln 2}$ with $x > 0$, where it is differentiable. Since both x^2 and 2^x tend to infinity, two applications of L'Hôpital's rule (Theorem 7.2.16) give

$$\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0,$$

each intermediate quotient again of the form ∞/∞. Since $f(n) = a_n$ and $f(x) \to 0$, the sequence converges to 0. Lemma 5.2.25 gives the same limit without derivatives.

Worked Example 10.1.2

The sequence $a_1 = 1$, $a_{n+1} = 1 + \dfrac{a_n}{2}$ is given by a recursion. Show that it converges and find its limit.

1. Upper bound. Since $a_1 \le 2$ and $a_n \le 2$ implies $a_{n+1} \le 1 + 1 = 2$, induction gives $a_n \le 2$ for every n.

2. Monotonicity. By the upper bound $a_n \le 2$, $a_{n+1} - a_n = 1 - a_n/2 \ge 0$.

By Theorem 5.1.18 the sequence converges to some L. Since the shifted sequence $\{a_{n+1}\}$ converges to L as well, the limit laws (Theorem 5.1.9) applied to the recursion give $L = 1 + L/2$, that is $L = 2$. The recursion names the limit only after convergence is proved: the

same algebra applied to the divergent recursion $a_{n+1} = 2a_n + 1$ produces the number -1 with equal ease.

10.1.5 Upper and lower limits

The root test of Section 10.5 asks how large $|a_n|^{1/n}$ is for late n. For the sequence equal to 2^{-n} at even n and 3^{-n} at odd n, the roots $|a_n|^{1/n}$ alternate between $\frac{1}{2}$ and $\frac{1}{3}$ and have no limit although every late term is at most $\frac{1}{2}$. The quantity that records this is the largest value that the terms approach infinitely often.

Definition 10.1.6 (Upper and lower limits)**.** Let $\{a_n\}$ be a real sequence. If it is not bounded above, its *upper limit* is $\limsup_{n\to\infty} a_n = +\infty$. Otherwise the numbers $U_n = \sup\{a_k : k \geq n\}$ form a nonincreasing sequence with

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} U_n, \tag{10.1.8}$$

a real number when $\{U_n\}$ is bounded below (Theorem 5.1.18) and $-\infty$ otherwise. The *lower limit* is $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$.

Example 10.1.7. For $a_n = (-1)^n(1 + \frac{1}{n})$ the supremum U_n is $1 + \frac{1}{k}$ for the least even $k \geq n$. Hence $U_n \to 1$ and $\limsup a_n = 1$. In the same way $\liminf a_n = -1$.

Example 10.1.8 (Supremum of the terms (non-example))**.** The upper limit is not the supremum of the terms. In the previous example $\sup_n a_n = a_2 = \frac{3}{2}$ while $\limsup a_n = 1$. A single large early term changes the supremum and leaves the upper limit untouched because the upper limit discards every finite set of terms.

Lemma 10.1.9 (Characterisation of the upper limit)**.** *Let $L = \limsup_{n\to\infty} a_n$.*

1. *If $r > L$, then $a_n < r$ for every n beyond some index.*
2. *If $r < L$, then $a_n > r$ for infinitely many n.*
3. *The sequence converges to a real number a if and only if $\limsup a_n = \liminf a_n = a$.*
4. *Let $\ell = \liminf a_n$. If $r < \ell$, then $a_n > r$ for every n beyond some index. If $r > \ell$, then $a_n < r$ for infinitely many n.*

Proof. (1) Since $L < +\infty$, the sequence is bounded above and $U_n \to L$ with L real or $-\infty$. Since $r > L$, some $U_N < r$ and every $k \geq N$ has $a_k \leq U_N < r$.

(2) If $L = +\infty$ and only finitely many terms exceeded r, the sequence would be bounded above by the largest of r and those terms. Otherwise U_n decreases to L and therefore $U_n \geq L > r$ for every n. Since r is then smaller than the supremum of $\{a_k : k \geq n\}$, some $k \geq n$ has $a_k > r$. As n is arbitrary, infinitely many terms exceed r.

(3) Suppose $a_n \to a$ and let $\varepsilon > 0$. Since every term beyond some index lies in $(a-\varepsilon, a+\varepsilon)$, $a - \varepsilon \leq U_n \leq a + \varepsilon$ for large n and $|L - a| \leq \varepsilon$ (Theorem 5.1.10). Hence $\limsup a_n = a$, with $\liminf a_n = a$ by the same argument for $-a_n$. Conversely, suppose both equal a. Part (1) with $r = a + \varepsilon$ gives $a_n < a + \varepsilon$ eventually and part (1) applied to $\{-a_n\}$ with $r = -a + \varepsilon$ gives $a_n > a - \varepsilon$ eventually. Hence $a_n \to a$.

(4) Since $\limsup(-a_n) = -\ell$, parts (1) and (2) applied to $\{-a_n\}$ with $-r$ in place of r give the two statements. ∎

By part (3) an upper limit replaces a limit that may fail to exist and agrees with it whenever it does exist. Part (4) is parts (1) and (2) for the lower limit. Parts (1) and (2) are the form in which the root test and the radius of convergence (Theorem 11.1.5) use it.

10.1.10 Sequences in a normed space

Section 5.1 states convergence (Definition 5.1.30) and the Cauchy property (Definition 5.1.34) for a sequence in a normed space, proves the limit laws there (Theorem 5.1.31) and shows that Cauchy sequences in $\mathbb{R}^m$ converge (Proposition 5.1.35) while a Cauchy sequence in the space c_{00} of finitely supported sequences need not (Example 5.1.36). This chapter uses those results for series and for sequences of functions.

Example 10.1.11 (Two norms on one space (non-example))**.** On $C([0,1])$ the integral $\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x$ is a norm (Problem 10.1.9). The functions $f_n(x) = x^n$ satisfy $\|f_n\|_1 = 1/(n+1) \to 0$ while $\|f_n\|_\infty = 1$ for every n. The same sequence converges to 0 in one norm and fails to converge to 0 in the other. On $C([0,1])$ the choice of norm decides which sequences converge and the conclusion of Theorem 3.6.15 fails there.

10.1.12 Complete normed spaces

Definition 10.1.13 (Complete normed space)**.** A normed space V is *complete* (or a *Banach space*) if every Cauchy sequence in V (Definition 5.1.34) converges to an element of V.

Section 5.1 has the first examples: $\mathbb{R}$ and $\mathbb{R}^m$ under every norm are complete and c_{00} is not (Theorem 5.1.28, Proposition 5.1.35, and Example 5.1.36). In particular $\mathbb{C}$ is complete.

10.1.14 Uniform convergence

A sequence of functions f_n on a set S converges *pointwise* to f if $f_n(x) \to f(x)$ for every $x \in S$. The index N in that statement may depend on x. For $f_n(x) = x^n$ on $[0,1]$ the pointwise limit is 0 on $[0,1)$ and 1 at $x = 1$. Since every f_n is continuous and the limit is not, pointwise convergence fails to preserve continuity. The remedy is to measure the distance between functions with the sup norm.

Definition 10.1.15 (Uniform convergence)**.** Let f_n and f be real functions on a set S. The sequence $\{f_n\}$ *converges uniformly* to f on S if for every $\varepsilon > 0$ one N serves every $x \in S$: $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and all $x \in S$. A series of functions converges uniformly when its partial sums do.

The definition asks for no bound on f_n or f. When every difference $f_n - f$ is bounded on S, uniform convergence is convergence in the sup norm of Definition 3.6.9:

$$f_n \to f \text{ uniformly on } S \iff \|f_n - f\|_\infty = \sup_{x \in S} |f_n(x) - f(x)| \to 0. \tag{10.1.9}$$

If $|f_n(x) - f(x)| < \varepsilon$ for all x, the supremum is at most ε. Conversely, if the supremum is below ε, so is every value. If $f_n - f$ is unbounded on S for infinitely many n, the convergence is not uniform: every N is exceeded by such an n and that f_n differs from f by more than 1 somewhere on S.

Uniform convergence implies pointwise convergence: the N that serves every x serves each x. The two definitions differ in the position of the quantifier over x. Figure 10.1.1 shows that the sequence x^n converges uniformly on $[0,r]$ for every $r < 1$ since $\|x^n\|_\infty = r^n \to 0$ there and fails to converge uniformly on $[0,1)$ since $\sup_{0 \le x < 1} x^n = 1$ for every n.

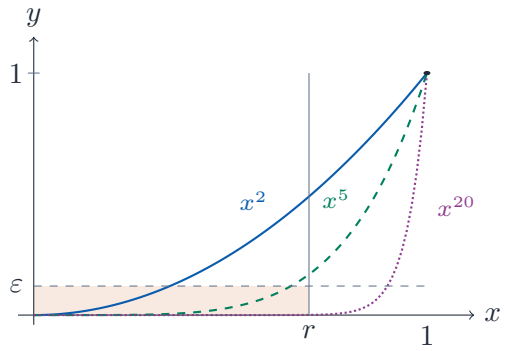


Figure 10.1.1: Graphs of x^2 (solid), x^5 (dashed) and x^{20} (dotted) on $[0,1]$. Since every late graph on $[0,r]$ lies in the shaded band of height ε above the limit 0, the convergence there is uniform. Since each graph climbs to 1 near $x=1$ and leaves the band, no single index serves all of $[0,1)$.

Lemma 10.1.16 (Continuity of a uniform limit)**.** *If every f_n is continuous on an interval I and $f_n \to f$ uniformly on I, then f is continuous on I.*

Proof. Let $x_0 \in I$ and $\varepsilon > 0$. Choose n with $|f_n(x) - f(x)| < \varepsilon/3$ for every $x \in I$ and then, by the continuity of f_n at x_0, a $\delta > 0$ with $|f_n(x) - f_n(x_0)| < \varepsilon/3$ for $x \in I$ and $|x - x_0| < \delta$. For such x,

$$|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon. \qquad \blacksquare$$

The sequence x^n on $[0,1]$ shows that Lemma 10.1.16 fails for pointwise convergence. The lemma is the first half of the following theorem.

Theorem 10.1.17 (Completeness of $C([a,b])$)**.** *The space $C([a,b])$ of continuous functions on a closed bounded interval, with the sup norm $\|f\|_\infty = \max_{a\le x\le b}|f(x)|$, is complete.*

Proof. Since every $f \in C([a,b])$ is bounded by the Extreme Value Theorem (Theorem 5.4.6), the sup norm is defined (Definition 3.6.9). Let $\{f_n\}$ be a Cauchy sequence in $C([a,b])$. For each fixed x,

$$|f_m(x) - f_n(x)| \le \|f_m - f_n\|_\infty. \tag{10.1.10}$$

By (10.1.10) the real sequence $\{f_n(x)\}$ is Cauchy and converges by Theorem 5.1.28. Call its limit $f(x)$.

Let $\varepsilon > 0$ and choose N with $\|f_m - f_n\|_\infty < \varepsilon$ for $m, n > N$. Fix $n > N$ and $x \in [a,b]$. For every $m > N$, $|f_n(x) - f_m(x)| < \varepsilon$. Letting $m \to \infty$, Theorem 5.1.10 gives $|f_n(x) - f(x)| \le \varepsilon$. Since x was arbitrary, $|f_n(x) - f(x)| \le \varepsilon < 2\varepsilon$ for every $x \in [a,b]$ and every $n > N$. Hence $f_n \to f$ uniformly (Definition 10.1.15). By Lemma 10.1.16 the function f is continuous. Hence $f \in C([a,b])$ and $\|f_n - f\|_\infty \le \varepsilon$ for $n > N$, that is $\|f_n - f\|_\infty \to 0$ by (10.1.9). $\blacksquare$

The norm matters. With the norm $\|\cdot\|_1$ in place of the sup norm, $C([0,1])$ is not complete (Problem 10.1.12). On a closed bounded interval every continuous function is bounded (Theorem 5.4.6) and has a sup norm. On the open interval $(0,1)$ the continuous function $1/x$ is unbounded and has no sup norm. The bounded continuous functions on $(0,1)$ form a complete space under the sup norm by the same proof.

Common misconception

A sequence of functions can converge at every point without converging uniformly. The test for uniform convergence is a single number, $\|f_n - f\|_\infty$, computed for each n by maximising $|f_n(x) - f(x)|$ over x. Showing that $f_n(x) \to f(x)$ for each x separately says nothing about that maximum.

Worked Example 10.1.3

Let $f_n(x) = \dfrac{x}{1+nx^2}$ on $\mathbb{R}$. Show that $f_n \to 0$ uniformly and that $f_n'(0)$ does not tend to the derivative of the limit.

For $x \neq 0$ the inequality $1 + nx^2 \geq 2\sqrt{n}\,|x|$ (equivalent to $(1 - \sqrt{n}|x|)^2 \geq 0$) gives $|f_n(x)| \leq 1/(2\sqrt{n})$. Equality holds at $x = 1/\sqrt{n}$ and therefore

$$\|f_n\|_\infty = \frac{1}{2\sqrt{n}} \to 0.$$

Hence $f_n \to 0$ uniformly on $\mathbb{R}$. By the quotient rule $f_n'(x) = (1 - nx^2)/(1 + nx^2)^2$ and hence $f_n'(0) = 1$ for every n while the limit function has derivative 0. Uniform convergence of f_n gives no control of f_n'. Theorem 11.1.2 differentiates a limit only under a hypothesis on the derivatives.

Study guide

To prove that a real sequence converges, reduce it to Theorem 10.1.4 and the limit laws, bound it and use Theorem 5.1.11, or show it monotone and bounded. Use the upper limit when a limit may fail to exist and only an eventual bound is needed. In a normed space every argument runs as on the line with $\|\cdot\|$ in place of $|\cdot|$, except the Cauchy criterion that holds exactly in the complete spaces.

For a sequence of functions, find the pointwise limit f first. Then compute $\|f_n - f\|_\infty$ by maximising over x and decide uniform convergence from whether that number tends to 0. A limit function that is discontinuous while every f_n is continuous rules uniform convergence out at once by Lemma 10.1.16.

Problems

10.1.1 Let the arithmetic sequence a_n have first term 2 and common difference 3 and the geometric sequence b_n have first term 2 and common ratio $\frac{1}{3}$. Find a_{10}, the sum of the first ten terms of $\{a_n\}$, and $\lim_{n\to\infty} b_n$. ⋆ Hint · Answer · Solution

10.1.2 For $a_n = (-1)^n(1 + \frac{1}{n})$ find $\sup_n a_n$, $\inf_n a_n$, $\limsup a_n$ and $\liminf a_n$. ⋆ Hint · Answer · Solution

10.1.3 Find $\displaystyle\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^n$ and $\displaystyle\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{2n}$. ⋆ Hint · Answer · Solution

10.1.4 Find $\displaystyle\lim_{n\to\infty}\frac{2^n}{n!}$ by a direct bound and explain why L'Hôpital's rule is unavailable here. ⋆⋆ Hint · Answer · Solution

10.1.5 Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that $\{a_n\}$ converges and find its limit. ⋆⋆ Hint · Answer · Solution

10.1.6 Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ for bounded real sequences and give an example in which the inequality is strict. ⋆⋆ Hint · Answer · Solution

10.1.7 Let V be a normed space and $\mathbf{v} \neq \mathbf{0}$ a vector in V. Show that $\mathbf{v}_n = (-1)^n \mathbf{v}$ has $\|\mathbf{v}_n\| \to \|\mathbf{v}\|$ and that $\{\mathbf{v}_n\}$ diverges. What does this say about the converse of the last statement of Theorem 5.1.31? ⋆ Hint · Answer · Solution

10.1.8 Let $f_n(x) = nxe^{-nx}$. Compute $\|f_n\|_\infty$ on $[0,1]$ and on $[\frac{1}{2},1]$ and decide on which interval $f_n \to 0$ uniformly. ⋆⋆ Hint · Answer · Solution

10.1.9 Show that $\|f\|_1 = \int_0^1 |f(x)|\,dx$ is a norm on $C([0,1])$. Compute $\|x^n\|_1$ and $\|x^n\|_\infty$ and explain why no constant K satisfies $\|f\|_\infty \leq K\|f\|_1$ for every $f \in C([0,1])$. ⋆⋆ Hint · Answer · Solution

10.1.10 Prove that a Cauchy sequence in a normed space is bounded and that a Cauchy sequence with a convergent subsequence converges to the limit of that subsequence. ⋆⋆ Hint · Answer · Solution

10.1.11 Let $f_n(x) = \sqrt{x^2 + 1/n}$ on $[-1,1]$. Show that $\|f_n - |x|\|_\infty = 1/\sqrt{n}$. Conclude that a uniform limit of differentiable functions need not be differentiable. ⋆⋆ Hint · Answer · Solution

10.1.12 For $n \geq 2$ let $g_n \in C([0,1])$ be 0 on $[0,\frac{1}{2}]$, 1 on $[\frac{1}{2}+\frac{1}{n},1]$ and linear in between. Show that $\{g_n\}$ is a Cauchy sequence for the norm $\|\cdot\|_1$ of Problem 10.1.9 and that it converges to no element of $C([0,1])$ in that norm. Conclude that $(C([0,1]), \|\cdot\|_1)$ is not complete. ⋆⋆⋆ Hint · Answer · Solution

10.2 Infinite Series

Overview

Objectives: Define the sum of a series in a normed space as the limit of its partial sums (Definition 10.2.1); use the nth-term test (Theorem 10.2.4) as a test for divergence only; sum a geometric series and a telescoping series; prove the Cauchy criterion for series (Theorem 10.2.12); prove that absolute convergence implies convergence in a complete normed space (Theorem 10.2.15) and show that completeness cannot be dropped; read the Weierstrass M-test (Theorem 10.2.18) as that theorem in $C([a,b])$ with the sup norm.
Prerequisites: Sequences in a normed space and completeness (Definitions 5.1.30 and 10.1.13 and Theorem 5.1.31); the completeness of $C([a,b])$ (Theorem 10.1.17); the Monotone Convergence Theorem (Theorem 5.1.18) and the Cauchy criterion (Theorem 5.1.28); reindexing and telescoping (Eqs. (4.6.3) and (4.6.10)); the geometric partial sum (Eq. (10.1.4)).
Key results: The sum of a series (Definition 10.2.1); the nth-term test (Theorem 10.2.4); the divergence of the harmonic series (Theorem 10.2.5); the geometric series (Theorem 10.2.7); the Cauchy criterion (Theorem 10.2.12); absolute convergence in a complete space (Theorem 10.2.15 and Corollary 10.2.16); the Weierstrass M-test (Theorem 10.2.18).

Notation

$\mathbf{s}_n$	the nth partial sum, $\mathbf{v}_1 + \cdots + \mathbf{v}_n$
$\sum \mathbf{v}_n$	the series and its sum when it converges
H_n	the nth partial sum of the harmonic series
M_n	a constant bounding $\|u_n\|_\infty$ in the M-test

A sum of infinitely many terms needs a definition. Add the terms one at a time and ask where the running total goes. The recipe uses only addition and a notion of distance and so serves for series of vectors and of functions.

Definition 10.2.1 (Partial sums and sum of a series)**.** Let $\{\mathbf{v}_n\}$ be a sequence in a normed space V. The *nth partial sum* is

$$\mathbf{s}_n = \sum_{k=1}^{n} \mathbf{v}_k = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n. \tag{10.2.1}$$

The *series* $\sum_{n=1}^{\infty} \mathbf{v}_n$ *converges* to the *sum* $\mathbf{S} \in V$ if $\mathbf{s}_n \to \mathbf{S}$ in V (Definition 5.1.30), written $\sum_{n=1}^{\infty} \mathbf{v}_n = \mathbf{S}$. If $\{\mathbf{s}_n\}$ has no limit in V, the series *diverges*.

For $V = \mathbb{R}$ this is the sum of a real series. Most of the chapter concerns that case. For $V = C([a,b])$ with the sup norm, a series of continuous functions converges exactly when its partial sums converge uniformly (Definition 10.1.15). The symbol $\sum \mathbf{v}_n$ names a limit and denotes an element of V only when that limit exists.

Example 10.2.2. The series $\sum_{n\geq 1} 2^{-n}$ has $s_n = 1 - 2^{-n} \to 1$ and converges to 1. The series $\sum_{n\geq 1} (-1)^n$ has partial sums alternating between -1 and 0 and diverges although every partial sum is bounded.

10.2.3 nth-term test

Theorem 10.2.4 (nth-term test for divergence)**.** *If $\sum \mathbf{v}_n$ converges in a normed space V, then $\mathbf{v}_n \to \mathbf{0}$. Equivalently, if $\mathbf{v}_n \not\to \mathbf{0}$, the series diverges.*

Proof. If $\mathbf{s}_n \to \mathbf{S}$, then $\mathbf{s}_{n-1} \to \mathbf{S}$ as well and by Theorem 5.1.31 $\mathbf{v}_n = \mathbf{s}_n - \mathbf{s}_{n-1} \to \mathbf{S} - \mathbf{S} = \mathbf{0}$. ∎

Common misconception

Theorem 10.2.4 applies in one direction only. Terms that fail to tend to zero prove divergence. Terms that do tend to zero decide nothing: $\sum 1/n^2$ converges (Example 10.2.8) and the harmonic series $\sum 1/n$ diverges (Theorem 10.2.5). Writing "$a_n \to 0$, hence the series converges" is the commonest error in this chapter.

Theorem 10.2.5 (Divergence of the harmonic series)**.** *The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges although its terms tend to 0.*

Proof. Group the terms in blocks whose lengths double:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{2 \text{ terms}} + \underbrace{\left(\frac{1}{5} + \cdots + \frac{1}{8}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{8 \text{ terms}} + \cdots \tag{10.2.2}$$

The block of 2^{k-1} terms ending at $1/2^k$ sums to at least $2^{k-1} \cdot 2^{-k} = \frac{1}{2}$. Hence (10.2.2) gives $s_{2^m} \geq 1 + m/2$, the partial sums are unbounded, and the series diverges by Theorem 5.1.7. ∎

10.2.6 Geometric and telescoping series

Theorem 10.2.7 (Geometric series)**.** *For real $a \neq 0$,*

$$\sum_{n=1}^{\infty} a\, r^{n-1} = a + ar + ar^2 + \cdots = \frac{a}{1-r} \qquad \text{when } |r| < 1, \tag{10.2.3}$$

and the series diverges when $|r| \geq 1$.

Proof. For $r \neq 1$ the partial sum is $s_n = a(1 - r^n)/(1 - r)$ by (10.1.4). When $|r| < 1$, $r^n \to 0$ by Theorem 10.1.4 and $s_n \to a/(1-r)$ by Theorem 5.1.9. When $|r| \geq 1$ the terms ar^{n-1} do not tend to 0 and the series diverges by the nth-term test (Theorem 10.2.4). ∎

In (10.2.3) the number a is the first term the series has. For a series that starts at $n = 3$ the numerator is the term of index 3. Write out the first two terms, take a to be the first and r the ratio of the second to the first, and check $|r| < 1$ before quoting the formula.

When $\mathbf{v}_n = \mathbf{b}_n - \mathbf{b}_{n+1}$ for a sequence $\{\mathbf{b}_n\}$ in V, the partial sum collapses by Eq. (4.6.10) to $\mathbf{s}_n = \mathbf{b}_1 - \mathbf{b}_{n+1}$. The series therefore converges exactly when $\{\mathbf{b}_n\}$ converges and then

$$\sum_{n=1}^{\infty} (\mathbf{b}_n - \mathbf{b}_{n+1}) = \mathbf{b}_1 - \lim_{n \to \infty} \mathbf{b}_n. \tag{10.2.4}$$

A real general term is put in this shape by partial fractions (Theorem 9.5.2): $1/(n(n+1)) = 1/n - 1/(n+1)$.

Example 10.2.8 (Sum of reciprocal squares)**.** The series $\sum_{n\geq 1} 1/n^2$ converges. For $k \geq 2$, $1/k^2 \leq 1/(k(k-1)) = 1/(k-1) - 1/k$. By Eq. (4.6.10)

$$s_n \leq 1 + \sum_{k=2}^{n} \Big(\frac{1}{k-1} - \frac{1}{k} \Big) = 2 - \frac{1}{n} < 2.$$

Since the partial sums increase and are bounded above by 2, Theorem 5.1.18 gives convergence.

10.2.9 Linearity of series

Theorem 10.2.10 (Linearity of series)**.** *If $\sum \mathbf{v}_n = \mathbf{A}$ and $\sum \mathbf{w}_n = \mathbf{B}$ converge in a normed space V and c is a real constant, then*

$$\sum_{n=1}^{\infty} (\mathbf{v}_n + \mathbf{w}_n) = \mathbf{A} + \mathbf{B}, \qquad \sum_{n=1}^{\infty} c\,\mathbf{v}_n = c\mathbf{A}. \tag{10.2.5}$$

Adding, deleting or altering finitely many terms changes the sum by a fixed vector and leaves the convergence or divergence of the series unchanged.

Proof. Since each partial sum of the combined series is the same combination of partial sums (Theorem 4.6.5), the limit passes through it by Theorem 5.1.31. If two series differ only in the terms of index at most K, their partial sums with $n \geq K$ differ by the fixed vector $\sum_{k\leq K}(\mathbf{v}_k - \mathbf{v}'_k)$ (Eq. (4.6.5)). Adding or deleting j terms shifts the index of every later partial sum by j (Eq. (4.6.3)) and leaves its limit unchanged. ∎

The converse of the first identity fails: $\sum(1 + (-1))$ converges to 0 while $\sum 1$ and $\sum(-1)$ both diverge. A convergent series of sums splits into two series only when each of them converges.

10.2.11 Cauchy criterion

The Cauchy criterion settles convergence without naming the sum.

Theorem 10.2.12 (Cauchy criterion for series)**.** *Let V be a complete normed space. The series $\sum \mathbf{v}_n$ converges if and only if for every $\varepsilon > 0$ there is an N such that*

$$\Big\| \sum_{k=n+1}^{m} \mathbf{v}_k \Big\| < \varepsilon \qquad \text{whenever } m > n > N. \tag{10.2.6}$$

In every normed space the condition is necessary for convergence.

Proof. Since the block sum is $\mathbf{s}_m - \mathbf{s}_n$ for $m > n$, (10.2.6) states that $\{\mathbf{s}_n\}$ is a Cauchy sequence (Definition 5.1.34). A convergent sequence is Cauchy in every normed space. In a complete space every Cauchy sequence converges (Definition 10.1.13). ∎

By (10.2.6) a series in a complete space converges exactly when every sufficiently late block of consecutive terms is small however long the block. Blocks of length one ($m = n + 1$) recover Theorem 10.2.4. The harmonic series has terms tending to 0 and late blocks $\dfrac{1}{n+1} + \cdots + \dfrac{1}{2n} \geq \frac{1}{2}$ that never become small (Problem 10.2.6).

10.2.13 Absolute convergence

Definition 10.2.14 (Absolute convergence)**.** A series $\sum \mathbf{v}_n$ in a normed space *converges absolutely* if the real series $\sum \|\mathbf{v}_n\|$ converges.

Since $\sum\|\mathbf{v}_n\|$ has nonnegative terms, its partial sums increase and it converges exactly when they are bounded by Theorem 5.1.18. Absolute convergence is therefore a question about real numbers that the tests of Sections 10.3 to 10.5 answer.

Theorem 10.2.15 (Absolute convergence in a complete space)**.** *Let V be a complete normed space. If $\sum\|\mathbf{v}_n\|$ converges, then $\sum\mathbf{v}_n$ converges in V and*

$$\Big\|\sum_{n=1}^{\infty}\mathbf{v}_n\Big\| \le \sum_{n=1}^{\infty}\|\mathbf{v}_n\|. \tag{10.2.7}$$

Proof. Let $\varepsilon > 0$. Since the real series $\sum\|\mathbf{v}_n\|$ converges, Theorem 10.2.12 in the complete space $\mathbb{R}$ gives N with $\sum_{k=n+1}^{m}\|\mathbf{v}_k\| < \varepsilon$ for $m > n > N$. The triangle inequality applied to a finite sum gives

$$\Big\|\sum_{k=n+1}^{m}\mathbf{v}_k\Big\| \le \sum_{k=n+1}^{m}\|\mathbf{v}_k\| < \varepsilon. \tag{10.2.8}$$

Hence (10.2.8) shows that $\sum\mathbf{v}_n$ satisfies (10.2.6) and converges by Theorem 10.2.12 because V is complete. For the bound, $\|\mathbf{s}_n\| \le \sum_{k\le n}\|\mathbf{v}_k\| \le \sum_{k\ge 1}\|\mathbf{v}_k\|$ for every n and $\|\mathbf{s}_n\| \to \|\mathbf{S}\|$ by Theorem 5.1.31. Theorem 5.1.10 gives (10.2.7). ∎

Corollary 10.2.16 (Absolute convergence of real series)**.** *If $\sum|a_n|$ converges, then $\sum a_n$ converges and $\left|\sum a_n\right| \le \sum|a_n|$.*

Proof. The line $\mathbb{R}$ with the absolute value is complete (Theorem 5.1.28). Apply Theorem 10.2.15. ∎

Completeness cannot be dropped from Theorem 10.2.15. In the space c_{00} of Example 5.1.36, let $\mathbf{e}_k$ have 1 in place k and 0 elsewhere. The series $\sum_{k\ge 1}2^{-k}\mathbf{e}_k$ has $\sum\|2^{-k}\mathbf{e}_k\|_\infty = \sum 2^{-k} = 1$. Its partial sums are the sequence $\{\mathbf{s}_n\}$ of Example 5.1.36 that has no limit in c_{00}. The converse of Theorem 10.2.15 also fails: the alternating harmonic series converges and is not absolutely convergent (Section 10.6). In the other direction, a normed space in which every absolutely convergent series converges is complete (Problem 10.2.11). Hence Theorem 10.2.15 characterises completeness.

10.2.17 Weierstrass M-test

In $C([a,b])$ the norm of a term is $\|u_n\|_\infty$. Hence Theorem 10.2.15 becomes a test for uniform convergence of a series of functions.

Theorem 10.2.18 (Weierstrass M-test)**.** *Let $u_n \in C([a,b])$ and let M_n be constants with $|u_n(x)| \le M_n$ for every $x \in [a,b]$ and every n. If $\sum M_n$ converges, then $\sum u_n$ converges uniformly on $[a,b]$ to a continuous function f and $\sum u_n(x)$ converges absolutely at every $x \in [a,b]$.*

Proof. Since the partial sums of $\sum\|u_n\|_\infty$ increase and are bounded by $\sum M_n$, the series converges by Theorem 5.1.18. Since the space $C([a,b])$ is complete under the sup norm (Theorem 10.1.17), Theorem 10.2.15 gives a sum $f \in C([a,b])$ with partial sums converging uniformly to f because convergence in the sup norm is uniform convergence. At each x the same monotone argument applied to $\sum|u_n(x)| \le \sum M_n$ gives absolute convergence. ∎

Example 10.2.19. The series $\sum_{n\ge 1}\sin(nx)/n^2$ converges uniformly on every interval $[a,b]$ to a continuous function: $|\sin(nx)/n^2| \le 1/n^2$ and $\sum 1/n^2$ converges by Example 10.2.8.

Example 10.2.20 (Uniform convergence without the M-test (non-example))**.** The M-test gives a sufficient condition only. Let u_n be the tent function of height $1/n$ supported on $[\frac{1}{n+1}, \frac{1}{n}]$ and zero elsewhere on $[0,1]$. Since the supports overlap only at endpoints where the tents vanish, at each x at most one term is nonzero and the tail after N terms has sup norm $1/(N+1) \to 0$. The series converges uniformly while $\sum \|u_n\|_\infty = \sum 1/n$ diverges and no admissible M_n exists.

Study guide

Faced with a real series, check the two closed forms first: a geometric series is settled by (10.2.3) and a telescoping series by (10.2.4). Otherwise apply Theorem 10.2.4 for a quick divergence check and then a test from Sections 10.3 to 10.6. For a series in a complete space such as a series of functions in the sup norm, try absolute convergence first: Theorem 10.2.15 reduces the question to a real series of nonnegative terms.

Marks are lost in two places. The first is reading the nth-term test backwards. The second is quoting $a/(1-r)$ with the value of the general term at $n = 1$ when the series starts elsewhere. Write out the first term of the series as printed and use it.

Worked Example 10.2.1

Evaluate $\displaystyle\sum_{n=2}^{\infty} \frac{4}{3^n}$.

The series starts at $n = 2$ with first two terms $\frac{4}{9}$ and $\frac{4}{27}$. Hence $a = \frac{4}{9}$ and $r = \frac{4/27}{4/9} = \frac{1}{3}$, with $|r| < 1$. By Theorem 10.2.7

$$\sum_{n=2}^{\infty} \frac{4}{3^n} = \frac{4/9}{1 - \frac{1}{3}} = \frac{2}{3}.$$

The value $4/3$ of the general term at $n = 1$ would give 2, the sum of the series that starts one term earlier. The two sums differ by exactly that term.

Worked Example 10.2.2

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

Partial fractions give $\dfrac{1}{n(n+2)} = \dfrac{1}{2}\Big(\dfrac{1}{n} - \dfrac{1}{n+2}\Big)$. Each $-1/k$ with $3 \le k \le n$ cancels against the $+1/k$ two brackets later. Hence two terms remain at each end:

$$s_n = \frac{1}{2}\Big(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\Big).$$

Since both $1/(n+1)$ and $1/(n+2)$ tend to 0, the sum is $\frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4}$.

Problems

10.2.1 Evaluate $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{2}{5}\right)^n$. ⋆ Hint · Answer · Solution

10.2.2 Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. ⋆ Hint · Answer · Solution

10.2.3 Evaluate $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2^n}-\frac{2}{3^n}\right)$. ⋆ Hint · Answer · Solution

10.2.4 For which real x does $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n}$ converge and what is its sum there?

⋆⋆ Hint · Answer · Solution

10.2.5 Show that $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ converges uniformly on $[-1,1]$, find f, and find the least N for which the partial sum $s_N=\sum_{n=0}^{N}x^n/2^n$ satisfies $\|f-s_N\|_\infty<10^{-3}$ on $[-1,1]$.

⋆⋆ Hint · Answer · Solution

10.2.6 Show that $\sum_{k=n+1}^{2n}\dfrac{1}{k}\geq\tfrac{1}{2}$ for every n and deduce the divergence of the harmonic series from Theorem 10.2.12. ⋆ Hint · Answer · Solution

10.2.7 Let $\sum\mathbf{v}_n$ converge to $\mathbf{S}$ in a normed space. Show that each tail $\mathbf{r}_n=\sum_{k=n+1}^{\infty}\mathbf{v}_k$ converges, that $\mathbf{r}_n=\mathbf{S}-\mathbf{s}_n$, and that $\mathbf{r}_n\to\mathbf{0}$. ⋆⋆ Hint · Answer · Solution

10.2.8 Let z be a complex number. Show that $\sum_{n\geq0}z^n/n!$ converges absolutely in $\mathbb{C}$ and hence converges. ⋆⋆ Hint · Answer · Solution

10.2.9 Show that $\sum_{n\geq0}x^n$ converges at every point of $(-1,1)$ and converges uniformly on $[-r,r]$ for every $r<1$. Show that no partial sum s_N is within 1 of the sum at every point of $(-1,1)$ and conclude that the convergence on $(-1,1)$ is not uniform.

⋆⋆ Hint · Answer · Solution

10.2.10 Construct a divergent real series whose terms tend to 0 and whose partial sums are bounded. ⋆⋆ Hint · Answer · Solution

10.2.11 Let V be a normed space in which every absolutely convergent series converges. Prove that V is complete. ⋆⋆⋆ Hint · Answer · Solution

10.3 Integral Test

Overview

Objectives: Use the bounded-partial-sums criterion (Theorem 10.3.1) as the basis of every test for series of nonnegative terms; state and prove the integral test (Theorem 10.3.2) and show by counterexample that its monotonicity hypothesis cannot be dropped; derive the p-series (Theorem 10.3.3); bound the tail of a convergent series by an improper integral (Theorem 10.3.5) and turn the bound into a number of terms.
Prerequisites: Partial sums and convergence (Definition 10.2.1); the Monotone Convergence Theorem (Theorem 5.1.18); integrability and the order property of the integral (Theorems 8.2.13 and 8.2.15); improper integrals of Type I (Definition 9.8.2) and the p-integral benchmarks (Theorem 9.8.7 and Eq. (9.8.7)).
Key results: The bounded-partial-sums criterion (Theorem 10.3.1); the integral test (Theorem 10.3.2); the p-series (Theorem 10.3.3 and Eq. (10.3.4)); the tail bound (Theorem 10.3.5 and Eq. (10.3.5)).

Notation

f	a continuous nonincreasing function with $f(n) = a_n$
p	the exponent in the term $1/n^p$
R_n	the tail $S - s_n$, the error in stopping at n terms

The tests of this section and the next two concern real series with nonnegative terms. Since in such a series each partial sum is at least the one before it, Theorem 5.1.18 applies to the nondecreasing sequence $\{s_n\}$.

Theorem 10.3.1 (Bounded partial sums)**.** *A series $\sum a_n$ with $a_n \geq 0$ for every n converges if and only if its partial sums are bounded above. If they are unbounded, $s_n \to \infty$.*

Proof. Since $s_{n+1} - s_n = a_{n+1} \geq 0$, the sequence $\{s_n\}$ is nondecreasing. If it is bounded above, it converges by Theorem 5.1.18. A convergent sequence is bounded (Theorem 5.1.7). If $\{s_n\}$ is unbounded, then for every M some s_N exceeds M and every later partial sum does too. ∎

A nonnegative series therefore either has a finite sum or diverges to infinity. The integral test decides between the two by bounding the partial sums with an area.

Theorem 10.3.2 (Integral test)**.** *Let N be a positive integer and let f be continuous, nonnegative and nonincreasing on $[N, \infty)$, with $f(n) = a_n$ for every integer $n \geq N$. Then*

$$\sum_{n=N}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,\mathrm{d}x \tag{10.3.1}$$

both converge or both diverge.

Proof. Since by Theorem 10.2.10 the terms before N affect no convergence question, we take $N = 1$. Every integral below exists by Theorem 8.2.13. On $[k, k+1]$ the function satisfies $a_{k+1} = f(k+1) \leq f(x) \leq f(k) = a_k$ because it is nonincreasing and the order property of Theorem 8.2.15 gives

$$a_{k+1} \leq \int_k^{k+1} f(x)\,\mathrm{d}x \leq a_k. \tag{10.3.2}$$

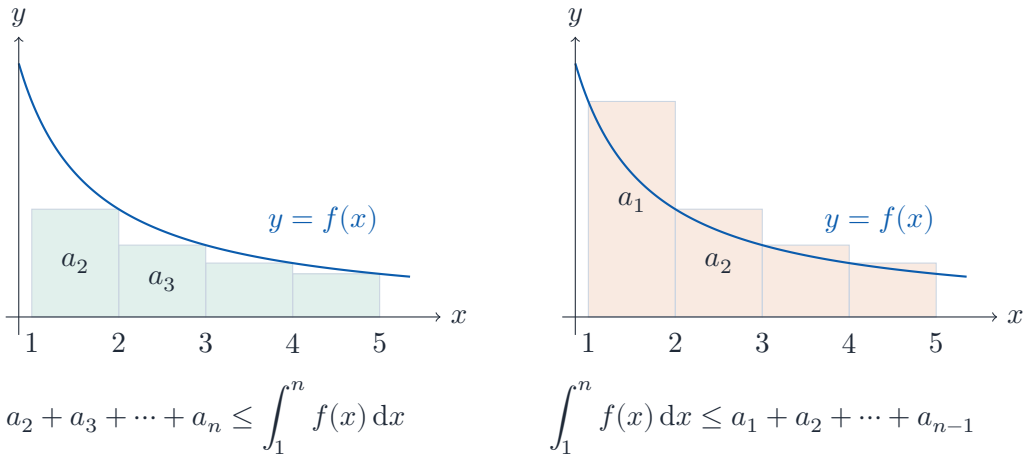


Figure 10.3.1: Rectangle families of Theorem 10.3.2 drawn at $N = 1$. On the left each rectangle takes its height from the right end of its base and lies under the nonincreasing curve. On the right each height comes from the left end of the base and every rectangle reaches above the curve. The two inequalities trap the partial sums between two copies of one integral.

Summing (10.3.2) over $k = 1, \dots, n-1$ with additivity,

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx \le a_1 + a_2 + \cdots + a_{n-1}. \tag{10.3.3}$$

Figure 10.3.1 draws both inequalities. Since $f \ge 0$, the function $b \mapsto \int_1^b f$ is nondecreasing and the improper integral converges exactly when $\int_1^n f$ is bounded in n (Proposition 5.2.23).

If $\int_1^\infty f = I$ converges, the left inequality of (10.3.3) gives $s_n \le a_1 + I$ and Theorem 10.3.1 gives convergence of the series. If the integral diverges, then $\int_1^n f \to \infty$ and the right inequality of (10.3.3) gives $s_{n-1} \to \infty$. ∎

The proof uses the monotonicity of f to compare its integral over $[k, k+1]$ with its values at the two ends. Without it the conclusion fails in both directions. The continuous positive function $f(x) = 1/x^2 + \sin^2(\pi x)$ has $f(n) = 1/n^2$. Hence $\sum f(n)$ converges while $\int_1^\infty f$ diverges because $\int_k^{k+1} \sin^2(\pi x)\,dx = \frac{1}{2}$ for every k. Problem 10.3.8 gives a positive f with a convergent integral and a divergent series. Continuity is used only to make the integrals exist in the sense of Definition 8.2.4. The sign hypothesis can be dropped (Problem 10.3.7).

The integral in Theorem 10.3.2 is evaluated as $\lim_{b\to\infty} \int_N^b f$ (Eq. (9.8.1)) only to decide whether that limit is finite. Its value is almost never the sum of the series. The inequalities (10.3.3) bound the difference.

Theorem 10.3.3 (p-series)**.** *For a real number p,*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \textit{converges for } p > 1 \quad \textit{and diverges for } p \le 1. \tag{10.3.4}$$

Proof. For $p \le 0$ the terms do not tend to 0 and Theorem 10.2.4 gives divergence. For $p > 0$ the function x^{-p} is continuous, positive and decreasing on $[1, \infty)$ and Theorem 10.3.2 reduces the question to $\int_1^\infty x^{-p}\,dx$. That integral converges for $p > 1$ and diverges for $p \le 1$ by Eq. (9.8.7). ∎

Since the p-series at $p = 1$ is the harmonic series, Theorem 10.3.3 recovers Theorem 10.2.5 from an area comparison. The boundary is sharp: $\sum 1/n$ diverges while $\sum 1/n^{1.01}$ converges.

10.3.4 Remainder after n terms

Theorem 10.3.5 (Integral bound for the tail of a series)**.** *Under the hypotheses of Theorem 10.3.2, suppose that the series converges with sum S and let $R_n = S - s_n$. Then for every $n \geq N$*

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \;\leq\; R_n \;\leq\; \int_{n}^{\infty} f(x)\,\mathrm{d}x. \tag{10.3.5}$$

Proof. The tail is $R_n = \sum_{k>n} a_k$ (Problem 10.2.7). Summing $a_{k+1} \leq \int_k^{k+1} f$ over $n \leq k < m$ and letting $m \to \infty$ gives $R_n \leq \int_n^{\infty} f$. Summing $\int_k^{k+1} f \leq a_k$ over $n+1 \leq k < m$ and letting $m \to \infty$ gives $\int_{n+1}^{\infty} f \leq R_n$. Both limits pass through the inequalities by Theorem 5.1.10. ∎

Study guide

Before writing an integral, check the hypotheses: continuous, nonnegative and nonincreasing on $[N, \infty)$. Monotonicity is the one that fails in practice. Check it by finding where $f' \leq 0$ since that may force a start at $N = 2$ or later. Then evaluate the improper integral with the limit written out.

Marks are lost in two places. The first is reporting the value of the integral as the sum of the series. The second is applying the test to terms that are not eventually nonincreasing. In that case the counterexample after the proof shows that no conclusion follows.

Worked Example 10.3.1

Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges.

The function $f(x) = x/(x^2+1)$ is continuous and positive on $[1, \infty)$ and satisfies

$$f'(x) = \frac{1 - x^2}{(x^2+1)^2} \leq 0 \qquad (x \geq 1).$$

Hence f is nonincreasing there. With $u = g(x) = x^2 + 1$ the 1-form $\dfrac{x}{x^2+1}\,\mathrm{d}x$ is $\frac{1}{2}g^*(\mathrm{d}u/u)$ (Definition 6.2.11) and the differential of $\frac{1}{2}\ln(x^2+1)$ by Theorem 6.3.5. Hence Eq. (8.3.6) gives

$$\int_1^b \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2 \;\xrightarrow[b\to\infty]{}\; \infty.$$

The integral diverges and so does the series by Theorem 10.3.2.

Worked Example 10.3.2

Decide whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

On $[2,\infty)$ the function $f(x)=1/(x\ln x)$ is continuous and positive and decreases because $x\ln x$ is a product of positive increasing factors. With $u=\ln x$ the 1-form $\mathrm{d}x/(x\ln x)$ is the pullback of $\mathrm{d}u/u$ and Theorem 8.5.2 gives

$$\int_2^b \frac{\mathrm{d}x}{x\ln x}=\int_{\ln 2}^{\ln b}\frac{\mathrm{d}u}{u}=\ln(\ln b)-\ln(\ln 2)\ \xrightarrow[b\to\infty]{}\ \infty.$$

The series diverges. By (10.3.3) its partial sums grow like $\ln(\ln n)$: with $a_2=1/(2\ln 2)$ the trap gives $s_n\le \ln(\ln n)+\frac{1}{2\ln 2}-\ln(\ln 2)<\ln(\ln n)+1.09$. A partial sum of 10 therefore needs more than $e^{e^{8.9}}$ terms. That number has over three thousand digits. A computer summing the series sees the partial sums stall at small values and suggests convergence. Only the proof settles the question.

Problems

10.3.1 Decide whether each series converges: (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}$.

⋆ Hint · Answer · Solution

10.3.2 Use the integral test to decide whether $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges and bound its sum above and below.

⋆ Hint · Answer · Solution

10.3.3 For which real p does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converge?

⋆⋆ Hint · Answer · Solution

10.3.4 For $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$, bound the tail R_4 left after four terms, above and below.

⋆⋆ Hint · Answer · Solution

10.3.5 How many terms of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ suffice for the partial sum to be within 0.001 of the total? What does the bound give for an accuracy of 10^{-6}?

⋆⋆ Hint · Answer · Solution

10.3.6 Let $H_n=\sum_{k=1}^{n}1/k$ and $d_n=H_n-\ln n$. Prove that $\{d_n\}$ is nonincreasing and that $0\le d_n\le 1$. Conclude that d_n converges. Its limit is *Euler's constant* $\gamma\approx 0.5772$.

⋆⋆ Hint · Answer · Solution

10.3.7 Let f be continuous and nonincreasing on $[1,\infty)$ with no sign condition. Show that $\sum f(n)$ and $\int_1^{\infty} f$ still both converge or both diverge.

⋆⋆ Hint · Answer · Solution

10.3.8 Construct a continuous function $f>0$ on $[1,\infty)$ for which $\int_1^{\infty} f$ converges and $\sum f(n)$ diverges. Which hypothesis of Theorem 10.3.2 fails?

⋆⋆ Hint · Answer · Solution

10.3.9 For $f(x)=1/x^2$ the integral $\int_1^{\infty} f$ equals 1. Use (10.3.3) to locate $\sum_{n\ge 1}1/n^2$ in an interval of length 1 and explain why no choice of f makes the integral equal to the sum in general.

⋆⋆ Hint · Answer · Solution

10.3.10 Let $a_n\ge 0$ be nonincreasing. Prove that $\sum a_n$ converges if and only if $\sum_{k\ge 0}2^k a_{2^k}$ converges and recover Theorem 10.3.3 without integrals.

⋆⋆⋆ Hint · Answer · Solution

10.4 Comparison Tests

Overview

Objectives: Compare a series of nonnegative terms with one whose behaviour is already known, either termwise (Theorem 10.4.1) or through the limit of the ratio of general terms (Theorem 10.4.2); choose the comparison series by reading the dominant power, exponential or factorial off the general term with the growth ordering of Section 4.6; recognise when a direct comparison points the wrong way and switch to a limit comparison; give counterexamples showing that the sign hypotheses of both tests are needed.
Prerequisites: The bounded-partial-sums criterion (Theorem 10.3.1); the p-series (Theorem 10.3.3) and the geometric series (Theorem 10.2.7), the sources of almost every comparison target; the growth ordering $n^p \ll r^n \ll n!$ and the closed forms of Section 4.6; limits of sequences (Theorem 5.1.9).
Key results: The direct comparison test (Theorem 10.4.1); the limit comparison test (Theorem 10.4.2); the catalogue of comparison targets in Table 10.4.1.

Notation

a_n	the general term of the series under test
b_n	the general term of the comparison series
c	the finite positive limit of a_n/b_n
N	the index beyond which the comparison inequality is satisfied

A series of nonnegative terms is settled by comparison with a series of known behaviour, usually a p-series (Theorem 10.3.3) or a geometric series (Theorem 10.2.7). The two tests below are the series counterparts of Theorems 9.8.9 and 9.8.10 for improper integrals and rest on Theorem 10.3.1.

Theorem 10.4.1 (Direct comparison test)**.** *Suppose $0 \le a_n \le b_n$ for every n beyond some index N.*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*
2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

Proof. Since by Theorem 10.2.10 the finitely many terms before N change no convergence question, take $N = 1$. Write s_n and t_n for the partial sums of $\sum a_n$ and $\sum b_n$. Termwise, $a_k \le b_k$ gives $s_n \le t_n$ for every n.

For (1), suppose $\sum b_n$ converges to T. Because $\{t_n\}$ is nondecreasing, $t_n \le T$ for every n. Hence $s_n \le T$ and the partial sums of $\sum a_n$ are bounded above. Theorem 10.3.1 gives convergence.

Statement (2) is the contrapositive of (1): were $\sum b_n$ to converge, $\sum a_n$ would converge by (1). ∎

A convergent upper bound and a divergent lower bound each settle a series. No conclusion follows from a convergent lower bound or a divergent upper bound.

The hypothesis $a_n \ge 0$ cannot be dropped. The constant sequence $a_n = -1$ satisfies $a_n \le b_n = 1/n^2$. Here $\sum b_n$ converges and $\sum a_n$ diverges. The pair defeats both parts. The inequality need hold only eventually by Theorem 10.2.10.

Theorem 10.4.2 (Limit comparison test)**.** *Suppose $a_n > 0$ and $b_n > 0$ for every n beyond some index and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c. \tag{10.4.1}$$

1. *If $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.*
2. *If $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*
3. *If $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Proof. For (1), let $\varepsilon = c/2 > 0$. By Definition 5.1.3 and (10.4.1) there is an N with $|a_n/b_n - c| < c/2$ for $n > N$, that is,

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \implies \frac{c}{2}\, b_n < a_n < \frac{3c}{2}\, b_n, \tag{10.4.2}$$

multiplying through by the positive b_n. If $\sum b_n$ converges, then so does $\sum \frac{3c}{2} b_n$ by Theorem 10.2.10. The right inequality of (10.4.2) and Theorem 10.4.1 then give convergence of $\sum a_n$. If $\sum b_n$ diverges, the left inequality of (10.4.2) forces $\sum a_n$ to diverge because $\sum \frac{c}{2} b_n$ diverges as well.

For (2), take $\varepsilon = 1$. Beyond some index $a_n/b_n < 1$ and hence $a_n < b_n$ to which Theorem 10.4.1 applies. For (3), beyond some index $a_n/b_n > 1$ and hence $a_n > b_n$ to which the divergence half of Theorem 10.4.1 applies. ∎

Part (1) needs no inequality and no guessed direction. In return c must be shown finite and nonzero by a limit computation.

Positivity cannot be dropped from Theorem 10.4.2. Example 10.6.9 gives two series with terms of both signs whose ratio of terms tends to 1 while one converges and the other diverges. The counterexample needs the alternating series test of Section 10.6. Part (2) cannot be strengthened to an equivalence (Problem 10.4.8). Neither can part (3): against $b_n = 1/n^2$ the divergent $\sum 1/n$ and the convergent $\sum 1/n^{3/2}$ both give $c = \infty$.

10.4.3 Choosing the comparison series

The comparison series is read off the general term: keep the part that dominates as n grows and discard everything else. The growth ordering of Section 4.6 ranks the candidates: every positive power outgrows a logarithm, every exponential with ratio above 1 outgrows a power, and the factorial outgrows an exponential.

A factorial outgrows every geometric sequence and no p-series matches it. A comparison series for a term with a factorial has to be built by hand, as in the proof of Theorem 10.1.4. The ratio test of Section 10.5 settles such a series in two lines.

Study guide

Choose the comparison series first from Table 10.4.1 and write down whether it converges before touching the series in front of you. Then pick the test: use Theorem 10.4.1 when a clean inequality is visible in one step and Theorem 10.4.2 otherwise. Most of the direct comparisons in this book come from two observations: a positive quantity added to a denominator makes a fraction smaller and one added to a numerator makes it larger.

Marks are lost in two places. The first is using an inequality that points the wrong way. The commonest case is "$a_n \geq b_n$ and $\sum b_n$ converges" from which nothing follows. The second is reporting $c = 0$ or $c = \infty$ from a limit comparison and drawing a conclusion that parts (2) and (3) do not allow. Each of those parts requires a hypothesis about $\sum b_n$ without which the test is inconclusive.

Table 10.4.1: Comparison targets by the shape of the general term. Each choice keeps the dominant behaviour of the numerator and of the denominator and discards the rest by the growth ordering of Section 4.6.

Shape of a_n	**Compare with**	**Decided by**
Quotient of polynomials in n, of degrees j over k	$1/n^{k-j}$	Theorem 10.3.3
Polynomial or root over a root, as in $\sqrt{n}/(n^2+1)$	$1/n^p$ with p the difference of the powers	Theorem 10.3.3
A polynomial over R^n with $R>1$, as in $(n^2+1)/3^n$	a geometric series $(1/q)^n$ with $1<q<R$, here $(1/2)^n$	Theorem 10.2.7
Quotient of two exponentials, as in $(3^n+n)/(4^n+n^2)$	the quotient of the dominant bases, here $(3/4)^n$	Theorem 10.2.7
A bounded factor such as $\sin^2 n$ or $2+\cos n$ multiplying something	the something, with the factor replaced by its bound	Theorem 10.4.1
$\ln n$ in the numerator	$1/n^p$ with p a little below the power present since $\ln n \ll n^{\varepsilon}$	Theorem 10.3.3
A factorial anywhere	a geometric series built by hand, or the ratio test	Theorem 10.5.2

Worked Example 10.4.1

Decide whether $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n+n}$ converges.

Since the exponential 2^n dominates the polynomial term in the denominator, Table 10.4.1 suggests the convergent geometric series $\sum 1/2^n$ with ratio $\frac{1}{2}$.

Adding $n>0$ to the denominator makes the fraction smaller:

$$0<\frac{1}{2^n+n}<\frac{1}{2^n}\qquad\text{for every } n\geq 1.$$

Since $\sum 1/2^n$ converges by Theorem 10.2.7, the given series converges by part (1) of Theorem 10.4.1.

The test settles convergence only: the sum of the given series is about 0.69728 and differs from the sum 1 of the comparison series.

Worked Example 10.4.2

Decide whether $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+3n}{5n^4+n+1}$ converges.

Since the general term is a quotient of polynomials of degree 2 over degree 4, Table 10.4.1 calls for $b_n=1/n^{4-2}=1/n^2$, convergent by Theorem 10.3.3 at $p=2$.

A direct comparison would require an awkward inequality between two quartics. The limit comparison requires none. Both terms are positive for $n \geq 1$, with ratio

$$\frac{a_n}{b_n} = \frac{2n^2+3n}{5n^4+n+1} \cdot \frac{n^2}{1} = \frac{2n^4+3n^3}{5n^4+n+1} = \frac{2+\dfrac{3}{n}}{5+\dfrac{1}{n^3}+\dfrac{1}{n^4}} \xrightarrow[n\to\infty]{} \frac{2}{5},$$

dividing above and below by n^4 at the third step. Since the limit $c = \frac{2}{5}$ is finite and nonzero, part (1) of Theorem 10.4.2 applies and the two series behave alike. The given series therefore converges with $\sum 1/n^2$.
Only the finiteness and positivity of c matter. Taking $b_n = 3/n^2$ gives $c = \frac{2}{15}$ and the same answer.

Problems

10.4.1 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges. ⋆ Hint · Answer · Solution

10.4.2 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ converges. ⋆ Hint · Answer · Solution

10.4.3 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges. ⋆ Hint · Answer · Solution

10.4.4 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converges. ⋆⋆ Hint · Answer · Solution

10.4.5 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+n^2}{4^n+n^5}$ converges and say on which terms of the numerator and the denominator the answer depends. ⋆⋆ Hint · Answer · Solution

10.4.6 Let $\sum a_n^2$ and $\sum b_n^2$ converge. Prove that $\sum a_n b_n$ converges absolutely and deduce that $\sum a_n/n$ converges absolutely whenever $\sum a_n^2$ converges. ⋆⋆ Hint · Answer · Solution

10.4.7 Let $a_n > 0$. Prove that $\sum a_n$ converges if and only if $\sum a_n/(1+a_n)$ converges. ⋆⋆ Hint · Answer · Solution

10.4.8 Take $b_n = 1/n$. Find positive sequences a_n and a'_n with $a_n/b_n \to 0$ and $a'_n/b_n \to 0$ such that $\sum a_n$ converges and $\sum a'_n$ diverges. What does this show about part (2) of Theorem 10.4.2 when $\sum b_n$ diverges? ⋆⋆ Hint · Answer · Solution

10.4.9 Let $b_n > 0$ and let $\sum b_n$ converge. Prove that there is a convergent series $\sum a_n$ of positive terms with $a_n/b_n \to \infty$. Conclude that no single convergent series is a comparison series for every convergent series. ⋆⋆⋆ Hint · Answer · Solution

10.5 Ratio and Root Tests

Overview

Objectives: State and prove the root test in upper-limit form (Theorem 10.5.1) and the ratio test (Theorem 10.5.2); prove that every convergence the ratio test proves the root test proves as well (Lemma 10.5.4) and exhibit a series that the root test settles and the ratio test does not; recognise the inconclusive case and the hypotheses that cannot be weakened; cancel factorials with Eq. (4.6.14) in a ratio computation.
Prerequisites: Upper and lower limits (Definition 10.1.6 and Lemma 10.1.9); the geometric series (Theorem 10.2.7) and the direct comparison test (Theorem 10.4.1); absolute convergence (Corollary 10.2.16); the standard limits (Theorem 10.1.4); the factorial recursion and ratio (Eqs. (4.6.13) and (4.6.14)).
Key results: The root test (Theorem 10.5.1 and Eq. (10.5.1)); the ratio test (Theorem 10.5.2 and Eq. (10.5.2)); the inequality between ratios and roots (Lemma 10.5.4).

Notation

ρ	the upper limit of $\lvert a_n\rvert^{1/n}$, or the limit of $\lvert a_{n+1}/a_n\rvert$
r	a number strictly between ρ and 1, used in both proofs

The comparison tests of Section 10.4 need a comparison series chosen from outside. The two tests here compare a series with a geometric series built from its own terms. They apply to terms of any sign through $\sum\lvert a_n\rvert$ and Corollary 10.2.16 and are the tests that determine the radius of convergence of a power series in Section 11.1.

Theorem 10.5.1 (Root test)**.** *Let $\{a_n\}$ be a real or complex sequence and let*

$$\rho = \limsup_{n\to\infty} \lvert a_n\rvert^{1/n}. \tag{10.5.1}$$

If $\rho < 1$, the series $\sum a_n$ converges absolutely. If $\rho > 1$, it diverges. If $\rho = 1$, the test is inconclusive.

Proof. Suppose $\rho < 1$ and choose r with $\rho < r < 1$. By part (1) of Lemma 10.1.9 there is an N with $\lvert a_n\rvert^{1/n} < r$, that is $\lvert a_n\rvert < r^n$, for $n > N$. Since $\sum r^n$ converges (Theorem 10.2.7), Theorem 10.4.1 gives convergence of $\sum\lvert a_n\rvert$ and hence, by Theorem 10.2.15 in $\mathbb{R}$ or $\mathbb{C}$, of $\sum a_n$. Suppose $\rho > 1$. By part (2) of Lemma 10.1.9, $\lvert a_n\rvert^{1/n} > 1$ and hence $\lvert a_n\rvert > 1$ for infinitely many n. The terms do not tend to 0 and Theorem 10.2.4 gives divergence. The case $\rho = 1$ is treated after Lemma 10.5.4. ∎

Theorem 10.5.2 (Ratio test)**.** *Let $\{a_n\}$ be a real or complex sequence with $a_n \neq 0$ for every n.*

1. *If $\limsup\lvert a_{n+1}/a_n\rvert < 1$, then $\sum a_n$ converges absolutely.*
2. *If $\lvert a_{n+1}/a_n\rvert \geq 1$ for every n beyond some index, then $\sum a_n$ diverges.*

In particular, if

$$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \rho, \tag{10.5.2}$$

the series converges absolutely when $\rho < 1$, diverges when $\rho > 1$ or $\rho = \infty$, and the test is inconclusive when $\rho = 1$.

Proof. (1) Let $L = \limsup|a_{n+1}/a_n| < 1$ and choose r with $L < r < 1$. By Lemma 10.1.9 there is an N with $|a_{n+1}| < r|a_n|$ for $n \geq N$. Induction gives $|a_{N+k}| < r^k|a_N|$. Since the tail of $\sum|a_n|$ is dominated by the convergent geometric series $|a_N|\sum r^k$, Theorem 10.4.1 gives convergence of $\sum|a_n|$ and Theorem 10.2.15 in $\mathbb{R}$ or $\mathbb{C}$ gives convergence of $\sum a_n$.

(2) Beyond some N, $|a_{n+1}| \geq |a_n|$. Hence $|a_n| \geq |a_N| > 0$ for $n \geq N$ and the terms do not tend to 0.

If the limit (10.5.2) is a real number ρ, it equals the upper limit by part (3) of Lemma 10.1.9. For $\rho < 1$ part (1) applies. For $\rho > 1$ the ratio exceeds 1 eventually by Definition 5.1.3 with $\varepsilon = \rho - 1$, and part (2) applies. If $\rho = \infty$, the ratio exceeds 1 eventually by the definition of a sequence tending to ∞, and part (2) applies again. ∎

The hypothesis of part (1) is an upper limit below 1 that cannot be weakened to $|a_{n+1}/a_n| < 1$ for every n. The harmonic series has $a_{n+1}/a_n = n/(n+1) < 1$ for every n and diverges. Since its ratios approach 1, no single $r < 1$ bounds them.

10.5.3 Ratios against roots

Both proofs of convergence find a geometric series of ratio $r < 1$ that eventually dominates the terms. The next lemma shows that for convergence the root test is the stronger of the two.

Lemma 10.5.4 (Ratio limits bound root limits)**.** *For a sequence of positive numbers a_n,*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{10.5.3}$$

In particular, if $a_{n+1}/a_n \to \rho$, then $a_n^{1/n} \to \rho$.

Proof. The middle inequality of (10.5.3) is true because every $\sup_{k\geq n}$ is at least the corresponding $\inf_{k\geq n}$. For the right inequality, let $L = \limsup a_{n+1}/a_n$, assume $L < \infty$ and let $r > L$. By Lemma 10.1.9 there is an N with $a_{n+1} < ra_n$ for $n \geq N$. Induction gives $a_n \leq Cr^n$ for $n \geq N$ with $C = a_N r^{-N}$. Then $a_n^{1/n} \leq C^{1/n}r$ with $C^{1/n} \to 1$ by Proposition 5.1.15. Every number $s > r$ therefore exceeds $a_n^{1/n}$ eventually, and part (2) of Lemma 10.1.9 gives $\limsup a_n^{1/n} \leq s$. Since $s > r > L$ were arbitrary, $\limsup a_n^{1/n} \leq L$.

For the left inequality, let $\ell = \liminf a_{n+1}/a_n$. If $\ell = 0$ there is nothing to prove since every root is positive. Otherwise let r be real with $0 < r < \ell$. By part (4) of Lemma 10.1.9 there is an N with $a_{n+1} > ra_n$ for $n \geq N$. Induction gives $a_n \geq Cr^n$ for $n \geq N$ with $C = a_N r^{-N} > 0$, and $a_n^{1/n} \geq C^{1/n}r$ with $C^{1/n} \to 1$. Every number $s < r$ is therefore below $a_n^{1/n}$ eventually, and part (4) of Lemma 10.1.9 gives $\liminf a_n^{1/n} \geq s$. Since $s < r < \ell$ were arbitrary, $\liminf a_n^{1/n} \geq \ell$.

If the ratios converge to a real ρ, the outer terms both equal ρ by part (3) of Lemma 10.1.9 and the inner ones do too. The roots therefore converge to ρ by part (3) again. ∎

Whenever the ratio test proves convergence, Lemma 10.5.4 gives $\limsup|a_n|^{1/n} < 1$ and the root test proves it as well. The converse fails. For divergence the comparison reverses: the constant sequence $a_n = 1$ has every ratio equal to 1 and part (2) of Theorem 10.5.2 proves divergence, while $|a_n|^{1/n} = 1$ and the root test is inconclusive. For $a_n = 2^{-n+(-1)^n}$ the ratios $a_{n+1}/a_n = 2^{-1-2(-1)^n}$ alternate between $\frac{1}{8}$ and 2 and neither part of Theorem 10.5.2 applies. The roots $a_n^{1/n} = 2^{-1+(-1)^n/n} \to \frac{1}{2}$ and the root test gives convergence (Problem 10.5.4). Because consecutive factorials cancel by Eq. (4.6.14), the ratio test remains the convenient one whenever the terms are built from factorials.

10.5.5 Inconclusive case

For $a_n = 1/n^p$,

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^p \to 1 \qquad \text{and} \qquad a_n^{1/n} = \left(\sqrt[n]{n}\right)^{-p} \to 1 \tag{10.5.4}$$

for every p, by Theorem 10.1.4. Since both tests give $\rho = 1$ for the divergent $\sum 1/n$ and for the convergent $\sum 1/n^2$, $\rho = 1$ contains no information about convergence. A general term that is a quotient of powers always gives $\rho = 1$ and is decided by Theorem 10.4.2 against the p-series of Table 10.4.1. Raabe's test (Problem 10.5.10) refines the ratio test at $\rho = 1$.

Study guide

Choose the test from the form of a_n. A factorial or a product of consecutive integers calls for the ratio test because Eq. (4.6.14) cancels it in one line. An nth power calls for the root test because the root removes the power. When ratios oscillate, compute roots. The root test proves convergence whenever the ratio test does.
Marks are lost in two places. The first is expanding factorials in place of cancelling them. The second is drawing a conclusion from $\rho = 1$. At that value another test is required.

Worked Example 10.5.1

Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$ converge.

For $a_n = 3^n/n!$,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \overset{(4.6.14)}{=\!=} \frac{3}{n+1} \xrightarrow[n\to\infty]{} 0.$$

The series converges. For $a_n = n!/10^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \overset{(4.6.14)}{=\!=} \frac{n+1}{10} \xrightarrow[n\to\infty]{} \infty.$$

The series diverges. Because $n!$ outgrows every exponential, an exponential over a factorial gives a convergent series and a factorial over an exponential a divergent one.

Worked Example 10.5.2

Decide whether $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+1}\right)^n$ converges.

Since the general term is an nth power of a positive number,

$$\sqrt[n]{a_n} = \frac{2n+3}{3n+1} = \frac{2+3/n}{3+1/n} \xrightarrow[n\to\infty]{} \frac{2}{3}.$$

Since $\rho = \frac{2}{3} < 1$, the series converges by Theorem 10.5.1. The ratios a_{n+1}/a_n also tend to $\frac{2}{3}$, but only after a proof that $\left(1 - \frac{7}{6n^2+17n+12}\right)^n \to 1$. Once the ratios are known to converge, Lemma 10.5.4 allows no limit other than $\frac{2}{3}$.

Problems

10.5.1 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges. ⋆ Hint · Answer · Solution

10.5.2 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges. ⋆ Hint · Answer · Solution

10.5.3 Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}$ converges. ⋆⋆ Hint · Answer · Solution

10.5.4 Let $a_n = 2^{-n+(-1)^n}$. Compute $\limsup a_{n+1}/a_n$, $\liminf a_{n+1}/a_n$ and $\lim a_n^{1/n}$. Decide convergence and find the sum. ⋆⋆ Hint · Answer · Solution

10.5.5 Prove that $\displaystyle\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$. ⋆⋆ Hint · Answer · Solution

10.5.6 For which positive real numbers x does $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{n^n}$ converge?
⋆⋆ Hint · Answer · Solution

10.5.7 Show that the ratio test and the root test are inconclusive for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and settle the series by another test. ⋆⋆ Hint · Answer · Solution

10.5.8 Let $a_n > 0$. Show that $a_{n+1}/a_n \le r$ for all n with a fixed $r < 1$ implies convergence of $\sum a_n$ and that $a_{n+1}/a_n < 1$ for every n does not. ⋆ Hint · Answer · Solution

10.5.9 Let $a_n \ge 0$ and suppose the series $\sum_k a_{2k-1}$ and $\sum_k a_{2k}$ of odd-index and even-index terms converge to A and B. Prove that $\sum a_n$ converges to $A + B$.
⋆⋆ Hint · Answer · Solution

10.5.10 Let $a_n > 0$ and suppose $n(1 - a_{n+1}/a_n) \ge p$ for every n beyond some N, with a constant $p > 1$. Prove that $\sum a_n$ converges. Check that $a_n = 1/n^2$ satisfies the hypothesis although the ratio test is inconclusive. ⋆⋆⋆ Hint · Answer · Solution

10.6 Alternating Series and Rearrangement

Overview

Objectives: State and prove the alternating series test (Theorem 10.6.2) and give a counterexample for each of its hypotheses; bound the error in stopping an alternating series (Theorem 10.6.3); distinguish absolute from conditional convergence (Definitions 10.2.14 and 10.6.5); state the effect of rearranging an absolutely convergent series (Theorem 10.6.7) and a conditionally convergent one (Theorem 10.6.8), both proved in Appendix A.
Prerequisites: Sign bookkeeping with $(-1)^n$ (Eqs. (4.6.19) and (4.6.20)); the Monotone Convergence Theorem (Theorem 5.1.18); subsequences (Theorem 5.1.23); the harmonic and p-series (Theorems 10.2.5 and 10.3.3); absolute convergence (Definition 10.2.14 and Corollary 10.2.16).
Key results: The alternating series test (Theorem 10.6.2); the alternating series estimate (Theorem 10.6.3 and Eq. (10.6.4)); the two rearrangement theorems (Theorems 10.6.7 and 10.6.8).

Notation

u_n	the size of the nth term, with $a_n = (-1)^{n+1}u_n$
L	the sum of a convergent alternating series
σ	a rearrangement, a bijection of the positive integers

The tests of Sections 10.3 to 10.5 concern nonnegative terms, or absolute values through Corollary 10.2.16. A series that converges only because its terms cancel is beyond their reach. The simplest such series alternate in sign and admit a test that uses the cancellation.

Definition 10.6.1 (Alternating series)**.** An *alternating series* is a series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots \qquad \text{with } u_n > 0 \text{ for every } n. \tag{10.6.1}$$

The form $(-1)^n u_n$ opens with $-u_1$ and differs from (10.6.1) by the factor -1 that Theorem 10.2.10 passes through the sum.

Theorem 10.6.2 (Alternating series test)**.** *Let $u_n > 0$. If $u_n \geq u_{n+1}$ for every n beyond some index and $u_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ converges.*

Proof. Deleting the first K terms changes no convergence question (Theorem 10.2.10). If K is odd, the remaining series has the form $(-1)^n u_n$ and is the series (10.6.1) times -1, which Theorem 10.2.10 passes through the sum. Hence assume the sizes are nonincreasing from $n = 1$. The even partial sums

$$s_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \tag{10.6.2}$$

form a nondecreasing sequence since every bracket is nonnegative. Regrouped,

$$s_{2m} = u_1 - (u_2 - u_3) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m} \leq u_1. \tag{10.6.3}$$

By (10.6.2), (10.6.3) and Theorem 5.1.18, $s_{2m} \to L$ for some L. The odd partial sums are $s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0$. Given $\varepsilon > 0$, both subsequences lie within ε of L beyond some

index. Every partial sum beyond the larger of the two indices therefore lies within ε of L and $s_n \to L$. ∎

Each hypothesis is needed.

- Without $u_n \to 0$ the series diverges by Theorem 10.2.4, as for $\sum(-1)^{n+1}(1+1/n)$ with sizes decreasing to 1.
- Without monotonicity the conclusion fails although $u_n > 0$ and $u_n \to 0$. Take $u_n = 1/n$ for odd n and $u_n = 1/n^2$ for even n. The partial sums s_{2m} are $\sum_{k\le m} 1/(2k-1)$ minus a quantity bounded by the convergence of $\sum 1/(2k)^2$. The first sum is at least $\frac{1}{2}\sum_{k\le m} 1/k$ and is unbounded. The series diverges.

Figure 10.6.1 shows the partial sums of $\sum(-1)^{n+1}/n$ approaching their limit from alternate sides.

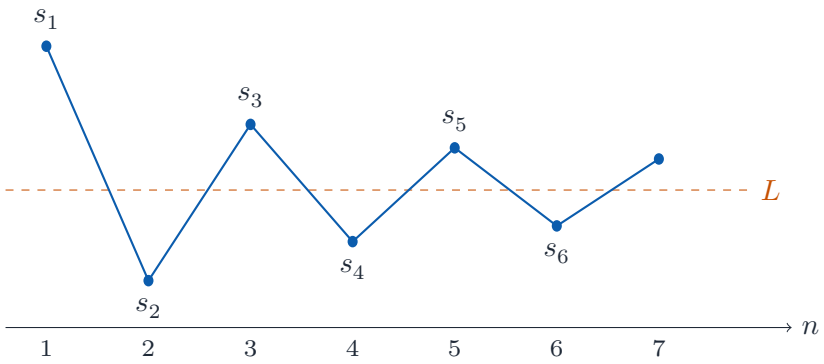


Figure 10.6.1: Partial sums of $\sum(-1)^{n+1}/n$. Each step reverses direction and is no longer than the one before it. Since the odd-indexed sums stay above the limit L and the even-indexed ones below, L lies between any two consecutive partial sums (Theorem 10.6.3).

Theorem 10.6.3 (Alternating series estimate)**.** *Under the hypotheses of Theorem 10.6.2 with $u_n \ge u_{n+1}$ for every n, the sum L satisfies*

$$|L - s_n| \le u_{n+1} \tag{10.6.4}$$

for every n and $L - s_n$ has the sign of the first omitted term $(-1)^{n+2}u_{n+1}$ or is zero.

Proof. The proof of Theorem 10.6.2 shows that s_{2m} increases to L. The odd partial sums $s_{2m+1} = u_1 - (u_2 - u_3) - \cdots - (u_{2m} - u_{2m+1})$ decrease and also tend to L. Hence $s_{2m} \le L \le s_{2m+1}$ and $s_{2m+2} \le L \le s_{2m+1}$ for every m. The limit L therefore lies between any two consecutive partial sums s_n and s_{n+1}. Since these differ by u_{n+1}, (10.6.4) follows. The point L lies on the side of s_n towards s_{n+1}. ∎

10.6.4 Absolute and conditional convergence

Definition 10.6.5 (Conditional convergence)**.** A real series *converges conditionally* if it converges and does not converge absolutely (Definition 10.2.14).

The alternating harmonic series $\sum(-1)^{n+1}/n$ converges by Theorem 10.6.2 and its series of absolute values is the harmonic series. It converges conditionally. Its sum is $\ln 2$ (Eq. (11.1.15)). A series with terms of mixed sign is examined in two steps: test $\sum|a_n|$ with the tests for nonnegative terms and ask whether cancellation makes $\sum a_n$ converge only if that series diverges.

For a real number a write $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$, with $a = a^+ - a^-$ and $|a| = a^+ + a^-$. A series converges absolutely exactly when $\sum a_n^+$ and $\sum a_n^-$ both converge. A conditionally convergent series has $\sum a_n^+ = \sum a_n^- = \infty$ (Problem 10.6.6). Its convergence comes from the order in which the positive and negative terms alternate.

10.6.6 Rearrangement

A *rearrangement* of $\sum a_n$ is a series $\sum a_{\sigma(n)}$ where σ is a bijection of the positive integers to themselves: the same terms, each used once, in another order. A finite sum is unchanged by reordering. For an infinite series the answer depends on the kind of convergence.

Theorem 10.6.7 (Rearrangement of absolutely convergent series)**.** *If $\sum a_n$ converges absolutely, every rearrangement $\sum a_{\sigma(n)}$ converges absolutely to the same sum.*

Theorem 10.6.8 (Riemann rearrangement theorem)**.** *Let $\sum a_n$ converge conditionally. For every real L there is a rearrangement with sum L. There are also rearrangements diverging to $+\infty$, to $-\infty$, and without any limit.*

Appendices A.5 and A.6 prove the two theorems. For the first, all but a small tail of $\sum|a_n|$ is used up by some finite stage of the rearranged series. The terms that remain have small total size. For the second, the positive terms and the negative terms each sum to infinity and tend to 0. Taking positive terms until the running sum exceeds L, then negative terms until it falls below L, and repeating, uses every term once. Each overshoot of L is at most the size of the last term taken and therefore tends to 0.

Worked Example 10.6.1

Show that the rearrangement

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

of the alternating harmonic series, one odd reciprocal followed by two even ones, converges to half the sum of the original series.

Group the terms in threes. The kth group is

$$\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} = \frac{1}{4k-2} - \frac{1}{4k} = \frac{1}{2}\Big(\frac{1}{2k-1} - \frac{1}{2k}\Big).$$

The partial sum T_{3m} of the rearranged series is therefore half a partial sum $\frac{1}{2}s_{2m}$ of the alternating harmonic series and tends to $\frac{1}{2}L$. Since the partial sums T_{3m+1} and T_{3m+2} differ from T_{3m} by at most two terms, each tending to 0, every partial sum tends to $\frac{1}{2}L = \frac{1}{2}\ln 2$. The same terms in another order have a different sum. Theorem 10.6.8 states that every real number is the sum of some rearrangement.

Study guide

For a series with mixed signs, test $\sum|a_n|$ first. If it converges, the series converges absolutely and may be rearranged freely. If it diverges, check the two hypotheses of Theorem 10.6.2 on $u_n = |a_n|$. State which kind of convergence you have found.

Marks are lost in two places. The first is applying Theorem 10.6.2 without checking that the sizes decrease. The second is quoting (10.6.4) with u_n in place of u_{n+1}: the bound is the first term omitted.

Worked Example 10.6.2

Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ converges absolutely and that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converges conditionally.

The first series converges absolutely because $\sum 1/n^2$ converges by Theorem 10.3.3. For the second, absolute convergence fails because $\sum 1/\sqrt{n}$ diverges by Theorem 10.3.3. Since the sizes $1/\sqrt{n}$ are positive, decreasing and tend to 0, Theorem 10.6.2 gives convergence. The convergence is conditional.

Example 10.6.9 (Limit comparison without positivity (non-example))**.** Positivity cannot be dropped from Theorem 10.4.2. Let $a_n = (-1)^n/\sqrt{n}$ and $b_n = a_n + 1/n$. Both are negative for odd $n \geq 3$. The ratio $b_n/a_n = 1 + (-1)^n/\sqrt{n}$ tends to 1. The series $\sum a_n$ is -1 times the second series of Worked Example 10.6.2 and converges by Theorem 10.2.10. If $\sum b_n$ converged, the harmonic series $\sum(b_n - a_n)$ would converge by Theorem 10.2.10. Hence $\sum b_n$ diverges (Theorem 10.2.5).

Worked Example 10.6.3

Estimate $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ by its first four terms and bound the error.

The partial sum is $s_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} = \frac{115}{144} \approx 0.79861$. By Theorem 10.6.3 the error is at most $u_5 = \frac{1}{25} = 0.04$ and has the sign of $+u_5$. Hence $0.79861 < L < 0.83861$. The sum is $\pi^2/12 \approx 0.82247$.

Problems

10.6.1 Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ converges and say whether the convergence is absolute or conditional. ⋆ Hint · Answer · Solution

10.6.2 Decide whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,2^n}{n!}$ converges absolutely. ⋆ Hint · Answer · Solution

10.6.3 For which real p does $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$ converge absolutely, converge conditionally, or diverge? ⋆⋆ Hint · Answer · Solution

10.6.4 Given that $1/e = \sum_{n\geq 0}(-1)^n/n!$ (Theorem 11.3.2), how many terms compute $1/e$ to within 0.001? Compare with Problem 10.3.5. ⋆⋆ Hint · Answer · Solution

10.6.5 Carry out the construction after Theorem 10.6.8 for the alternating harmonic series with target $L = 0$ and positive terms first. Write down the first eight terms of the rearrangement and its eighth partial sum. ⋆⋆ Hint · Answer · Solution

10.6.6 Prove that $\sum a_n$ converges absolutely if and only if $\sum a_n^+$ and $\sum a_n^-$ both converge and that if $\sum a_n$ converges conditionally, both diverge. ⋆⋆ Hint · Answer · Solution

10.6.7 Prove that a conditionally convergent series has infinitely many positive terms and infinitely many negative terms. ⋆ Hint · Answer · Solution

10.6.8 Given $\sum_{n\ge1} 1/n^2 = \pi^2/6$, justify the computation

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12},$$

and explain why the same split applied to $\sum(-1)^{n+1}/n$ proves nothing.

⋆⋆ Hint · Answer · Solution

10.6.9 Let $a_n \ge 0$ and let $\sum a_n$ converge. Prove that $\sum a_n^2$ converges. Show by an example that the conclusion fails without $a_n \ge 0$. ⋆⋆ Hint · Answer · Solution

10.6.10 Let $c_k = k^{-1/3}$ and form the series whose terms, in blocks of three, are $2c_k, -c_k, -c_k$ for $k = 1, 2, \ldots$. Show that it converges while the series of cubes of its terms diverges.

⋆⋆ Hint · Answer · Solution

10.6.11 Let the partial sums $B_n = \sum_{k\le n} b_k$ satisfy $|B_n| \le K$ for every n and let a_n decrease to 0. Prove *Dirichlet's test*: $\sum a_n b_n$ converges. Deduce Theorem 10.6.2.

⋆⋆⋆ Hint · Answer · Solution

Chapter 11

Power Series

A power series is a series whose terms are multiples of the powers of $x - a$. Its partial sums are polynomials with the sum as their limit in the sup norm on each closed interval inside the interval of convergence. Section 11.1 finds that interval with the root test, proves that the sum is continuous and differentiates and integrates it term by term. Section 11.2 builds the Taylor series of a function from its derivatives at one point. Section 11.3 decides with Taylor's theorem (Theorem 7.4.8) when that series converges to the function, separates analytic functions from smooth ones and derives Euler's formula. Section 11.4 develops the binomial series and its applications. Section 11.5 proves that the sum of a power series is analytic at every point of its open interval and identifies the exponential of the differentiation operator with translation.

11.1 Power Series

Overview

Objectives: Interchange a series of continuous functions with integration and, under a hypothesis on derivatives, with differentiation (Theorem 11.1.2); find the radius of convergence of a power series by the root test (Theorem 11.1.5) and by the ratio test as a special case (Corollary 11.1.7); prove uniform convergence on every closed subinterval (Theorem 11.1.11) and continuity of the sum; use Abel's theorem (Theorem 11.1.12) at an endpoint; differentiate and integrate a power series term by term (Theorem 11.1.15) and read the result as an identity of 1-forms; multiply two power series (Theorem 11.1.17).
Prerequisites: Uniform convergence and the completeness of $C([a,b])$ (Definition 10.1.15, Lemma 10.1.16, and Theorem 10.1.17); the M-test (Theorem 10.2.18); upper limits and the root and ratio tests (Lemmas 10.1.9 and 10.5.4 and Theorem 10.5.1); bounded linear maps (Definition 3.6.18); the Fundamental Theorem (Theorems 8.3.4 and 8.3.5) and the differential $\mathrm{d}F = F'(x)\,\mathrm{d}x$ (Eq. (6.2.6)).
Key results: Termwise operations on series of functions (Theorem 11.1.2); the Cauchy–Hadamard formula (Theorem 11.1.5 and Eq. (11.1.5)); uniform convergence inside the interval (Theorem 11.1.11); Abel's theorem (Theorem 11.1.12); termwise differentiation and integration (Theorem 11.1.15 and Eqs. (11.1.9) to (11.1.11)); the Cauchy product (Theorem 11.1.17 and Eq. (11.1.13)).

Notation

a	the centre of the power series
c_n	the coefficient of $(x-a)^n$
R	the radius of convergence, in $[0,\infty]$
f	the sum of the series on its interval of convergence

Problem 10.2.4 found a series in a variable that converges on an interval, diverges outside it and sums to $3/(5-x)$ there. Its partial sums are polynomials. A function defined as a limit of polynomials can be continuous, differentiable and integrable only if those properties pass to the limit. Section 10.1 showed that a pointwise limit of continuous functions can be discontinuous. This section proves that the partial sums of a power series converge in the sup norm on every closed interval inside the interval of convergence and that this is enough for every operation of calculus.

11.1.1 Series of functions

A linear map that is bounded (Definition 3.6.18) passes through a convergent series. If $T\colon V \to W$ has $\|T\mathbf{v}\| \le K\|\mathbf{v}\|$ and $\sum \mathbf{v}_n = \mathbf{S}$ in V, then the partial sums satisfy $\|T\mathbf{s}_n - T\mathbf{S}\| \le K\|\mathbf{s}_n - \mathbf{S}\| \to 0$ and hence

$$\sum_{n=1}^{\infty} T\mathbf{v}_n = T\Big(\sum_{n=1}^{\infty} \mathbf{v}_n\Big). \tag{11.1.1}$$

On $C([a,b])$ the integral $f \mapsto \int_a^b f(x)\,dx$ is such a map, with $\left|\int_a^b f\right| \le (b-a)\|f\|_\infty$ by the order property of Theorem 8.2.15.

Theorem 11.1.2 (Termwise operations on series of functions)**.** *Let $u_n \in C([\alpha,\beta])$ and let $\sum u_n$ converge uniformly on $[\alpha,\beta]$ to f.*

1. *The sum f is continuous on $[\alpha,\beta]$.*
2. *For every $x \in [\alpha,\beta]$, $\displaystyle\int_\alpha^x f(t)\,dt = \sum_{n=1}^{\infty} \int_\alpha^x u_n(t)\,dt$, and the series on the right converges uniformly in x.*
3. *Suppose instead that every u_n has a continuous derivative on $[\alpha,\beta]$, that $\sum u_n(\alpha)$ converges and that $\sum u_n'$ converges uniformly on $[\alpha,\beta]$ to g. Then $\sum u_n$ converges uniformly to a function f with $f' = g$ on $[\alpha,\beta]$, one-sided at the endpoints.*

Proof. (1) The partial sums are continuous and converge uniformly to f. By Lemma 10.1.16 the sum f is continuous on $[\alpha,\beta]$.

(2) Let $s_N = \sum_{n\le N} u_n$. For every $x \in [\alpha,\beta]$, linearity and the order property give

$$\Big|\int_\alpha^x f - \sum_{n\le N}\int_\alpha^x u_n\Big| = \Big|\int_\alpha^x (f - s_N)\Big| \le (\beta-\alpha)\|f - s_N\|_\infty. \tag{11.1.2}$$

The right side of (11.1.2) is a bound independent of x that tends to 0.

(3) By Theorem 8.3.5, $u_n(x) = u_n(\alpha) + \int_\alpha^x u_n'(t)\,dt$. Summing over $n \le N$,

$$s_N(x) = \sum_{n\le N} u_n(\alpha) + \int_\alpha^x \sum_{n\le N} u_n'(t)\,dt. \tag{11.1.3}$$

By part (2) applied to $\sum u_n'$, the right-hand side of (11.1.3) converges uniformly in x to $A + \int_\alpha^x g(t)\,dt$, where $A = \sum u_n(\alpha)$. Hence $\sum u_n$ converges uniformly to $f(x) = A + \int_\alpha^x g$. Since g is continuous by part (1), Theorem 8.3.4 gives $f' = g$. ∎

Theorem A.8.1 proves part (3) for terms that are differentiable with derivatives that need not be continuous.

Read through Eq. (6.2.6), part (3) states that the differential of the sum is the sum of the differentials, $\mathrm{d}(\sum u_n) = \sum u_n'(x)\,\mathrm{d}x$. Part (2) states that the integral of the 1-form $f(x)\,\mathrm{d}x$ over $[a, x]$ is the sum of the integrals of the 1-forms $u_n(x)\,\mathrm{d}x$.

Each hypothesis is needed. Pointwise convergence suffices for neither part (1) by the sequence x^n of Fig. 10.1.1 nor part (2) (Problem 11.1.10). Uniform convergence of $\sum u_n$ alone does not suffice for part (3): Worked Example 10.1.3 gives functions converging uniformly to 0 with derivatives at 0 equal to 1.

11.1.3 Radius of convergence

Definition 11.1.4 (Power series)**.** A *power series about* $x = a$ is a series of functions of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots, \tag{11.1.4}$$

with constant coefficients c_n. The number a is the *centre*. The same definition with complex c_n, a and x gives a complex power series.

A power series converges at its centre because every term after the first vanishes at $x = a$. The root test measures how far from a convergence extends.

Theorem 11.1.5 (Cauchy–Hadamard formula)**.** *Given a power series* $\sum c_n(x-a)^n$*, let*

$$\rho = \limsup_{n\to\infty} |c_n|^{1/n} \qquad \text{and} \qquad R = \frac{1}{\rho}, \tag{11.1.5}$$

with $R = \infty$ *when* $\rho = 0$ *and* $R = 0$ *when* $\rho = \infty$*. The series converges absolutely for* $|x-a| < R$ *and diverges for* $|x-a| > R$*. Exactly one of the following holds: the series converges only at* $x = a$*; it converges absolutely for every* x*; or there is a number* R *with* $0 < R < \infty$ *that separates convergence from divergence.*

Proof. For $x \neq a$ the root test (Theorem 10.5.1) applies to the terms $c_n(x-a)^n$, with

$$\limsup_{n\to\infty} \left|c_n(x-a)^n\right|^{1/n} = |x-a| \limsup_{n\to\infty} |c_n|^{1/n} = |x-a|\rho. \tag{11.1.6}$$

The middle equality of (11.1.6) follows because multiplying a sequence by a positive constant multiplies every $\sup_{k\geq n}$ and hence the upper limit by that constant. If $\rho = \infty$, the upper limit is ∞ for every $x \neq a$ and the series diverges. If $\rho = 0$, it is 0 and the series converges absolutely for every x. Otherwise the upper limit is below 1 exactly when $|x-a| < 1/\rho$ and above 1 exactly when $|x-a| > 1/\rho$. ∎

Definition 11.1.6 (Radius and interval of convergence)**.** The number $R \in [0, \infty]$ of Theorem 11.1.5 is the *radius of convergence*. The *interval of convergence* is the set of x at which the series converges. When $0 < R < \infty$ it is the interval from $a - R$ to $a + R$ with each endpoint included or not according to a separate test.

Corollary 11.1.7 (Radius from ratios)**.** *If* $c_n \neq 0$ *and* $|c_{n+1}/c_n| \to L \in [0, \infty]$*, then* $R = 1/L$ *and hence*

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| \tag{11.1.7}$$

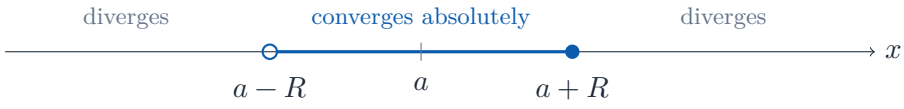
whenever this limit exists in $[0, \infty]$*.*

Proof. By Lemma 10.5.4, $|c_n|^{1/n} \to L$ and the upper limit equals this limit (Lemma 10.1.9). ∎

Example 11.1.8. For $\sum x^n/n!$ the ratio $|c_{n+1}/c_n| = 1/(n+1) \to 0$ and $R = \infty$. For $\sum n!\, x^n$ the ratio is $n+1 \to \infty$ and $R = 0$.

Example 11.1.9 (Ratios without a limit)**.** The series $\sum(2+(-1)^n)^n x^n$ has $|c_n|^{1/n}$ equal to 3 for even n and 1 for odd n. By Theorem 11.1.5, $R = \frac{1}{3}$. Since the ratios $|c_{n+1}/c_n|$ equal 3^{-n} for even n and 3^{n+1} for odd n and have no limit, Corollary 11.1.7 does not apply.

Figure 11.1.1 shows an interval of convergence with $0 < R < \infty$.



each endpoint tested on its own: here $a - R$ fails and $a + R$ passes

Figure 11.1.1: Interval of convergence when $0 < R < \infty$. The series converges absolutely inside the open interval and diverges outside it. Since the root test gives 1 and is inconclusive at each endpoint, each endpoint is settled by a test from Sections 10.3 to 10.6. Here $a - R$ is excluded (hollow dot) and $a + R$ is included (solid dot).

At $x = a \pm R$ the root test is inconclusive for every series. All four combinations of included and excluded endpoints occur: the geometric series $\sum x^n$ converges on $(-1, 1)$, Worked Example 11.1.1 finds $[1, 5)$, Problem 11.1.3 finds $(-1, 1]$ and Problem 11.1.9 finds $[-1, 1]$. To find an interval of convergence, compute R, write the open interval, substitute each endpoint and decide the resulting numerical series, and report the interval with the correct brackets.

11.1.10 Uniform convergence and continuity

Theorem 11.1.11 (Uniform convergence of a power series inside its interval)**.** *Let $\sum c_n(x-a)^n$ have radius $R > 0$ and let $0 < r < R$. The series converges uniformly on $[a - r, a + r]$ and its sum f is continuous on the open interval $(a - R, a + R)$.*

Proof. For $|x - a| \le r$, $|c_n(x-a)^n| \le |c_n| r^n = M_n$. Since $r < R$, the series $\sum M_n = \sum |c_n| r^n$ converges by Theorem 11.1.5 applied at $x = a + r$. The M-test (Theorem 10.2.18) gives uniform convergence on $[a - r, a + r]$ to a continuous function. Since every point of $(a - R, a + R)$ lies inside some such interval, f is continuous there. ∎

Uniform convergence on the whole open interval can fail: the geometric series converges on $(-1, 1)$ and not uniformly there (Problem 10.2.9). At an endpoint where the series converges, the next theorem extends continuity.

Theorem 11.1.12 (Abel's theorem)**.** *Let $\sum c_n(x-a)^n$ have radius R with $0 < R < \infty$ and converge at $x = a + R$. Then the series converges uniformly on $[a, a + R]$, its sum f is continuous on $(a - R, a + R]$, and*

$$\lim_{x \to (a+R)^-} f(x) = \sum_{n=0}^{\infty} c_n R^n. \tag{11.1.8}$$

The same holds at $a - R$ with the interval $[a - R, a]$.

Appendix A.7 proves Theorem 11.1.12 by summation by parts. An identity proved inside the interval reaches that endpoint only through the limit.

11.1.13 Differentiation and integration term by term

Lemma 11.1.14 (Radius of the derived series)**.** *The power series $\sum c_n(x-a)^n$, its derived series $\sum_{n\geq 1} nc_n(x-a)^{n-1}$ and its integrated series $\sum_{n\geq 0} \dfrac{c_n}{n+1}(x-a)^{n+1}$ have the same radius of convergence.*

Proof. For $x \neq a$ the series $\sum nc_n(x-a)^n$ equals $(x-a)$ times the derived series and converges exactly when it does. By Theorem 11.1.5 the radius of the former is $1/\limsup(n|c_n|)^{1/n}$. Since $n^{1/n} \to 1$ (Theorem 10.1.4), for every $\varepsilon > 0$ the factor $n^{1/n}$ lies in $[1, 1+\varepsilon)$ eventually. Hence $\limsup(n|c_n|)^{1/n}$ lies between ρ and $(1+\varepsilon)\rho$ for every ε and equals ρ. Since the integrated series has the original as its derived series, its radius is R as well. ∎

Theorem 11.1.15 (Term-by-term differentiation and integration)**.** *Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ have radius of convergence $R > 0$. Then f is differentiable on $(a-R, a+R)$ with*

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}, \tag{11.1.9}$$

and the function $F(x) = \sum_{n=0}^{\infty} \dfrac{c_n}{n+1}(x-a)^{n+1}$ satisfies $F(a) = 0$ and $\mathrm{d}F = f(x)\,\mathrm{d}x$ on $(a-R, a+R)$ and hence

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}, \qquad \int_a^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}. \tag{11.1.10}$$

Both series have radius R. Their behaviour at the endpoints may differ from that of the original series.

Proof. Let $|x_0 - a| < R$ and choose r with $|x_0 - a| < r < R$. By Lemma 11.1.14 the derived series has radius R. By Theorem 11.1.11 both the series and the derived series converge uniformly on $[a-r, a+r]$. The series converges at the left endpoint $a-r$ because $r < R$. Since the terms $c_n(x-a)^n$ are polynomials with continuous derivatives, part (3) of Theorem 11.1.2 on $[\alpha, \beta] = [a-r, a+r]$ gives (11.1.9) on $[a-r, a+r]$. The derivative at the interior point x_0 is two-sided. The same argument applied to F gives $F' = f$ because the derived series of F is the series of f. Equation (6.2.6) turns this into $\mathrm{d}F = f(x)\,\mathrm{d}x$. Since $F(a) = 0$, Theorem 8.3.5 gives the definite integral in (11.1.10) and Definition 8.1.4 the indefinite one. ∎

In the notation of 1-forms, Theorem 11.1.15 states that the differential passes through the sum inside the interval of convergence:

$$\mathrm{d}\Big(\sum_{n=0}^{\infty} c_n(x-a)^n\Big) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}\,\mathrm{d}x = \sum_{n=0}^{\infty} \mathrm{d}(c_n(x-a)^n). \tag{11.1.11}$$

Applying (11.1.9) repeatedly shows that f has derivatives of every order on $(a-R, a+R)$, each given by a power series of radius R. Theorem 11.2.1 reads off the coefficients from those derivatives.

Integration multiplies the nth coefficient by $1/(n+1)$ and can add an endpoint to the interval of convergence. Differentiation multiplies it by n and can remove one (Problem 11.1.9).

11.1.16 Products

Theorem 11.1.17 (Cauchy product)**.** *Let $\sum_{n\geq 0} a_n = A$ and $\sum_{n\geq 0} b_n = B$ converge absolutely with real or complex terms and let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n$ converges absolutely to AB.*

Proof. Let A_N, B_N, C_N be the partial sums through index N. The product $A_N B_N$ is the sum of $a_k b_l$ over $0 \le k, l \le N$ and C_N is the sum over $k + l \le N$. Every pair in the first set and not in the second has $k > N/2$ or $l > N/2$. Hence

$$|A_N B_N - C_N| \le \sum_{k>N/2} |a_k| \sum_{l\ge 0} |b_l| + \sum_{k\ge 0} |a_k| \sum_{l>N/2} |b_l|, \tag{11.1.12}$$

with both tails tending to 0 (Problem 10.2.7). Since $A_N B_N \to AB$ and the right side of (11.1.12) tends to 0, also $C_N \to AB$. The same estimate with absolute values gives $\sum_{n\le N} |c_n| \le \sum |a_k| \sum |b_l|$ and hence absolute convergence. ∎

Since two power series about the same centre with radii R_1 and R_2 both converge absolutely for $|x - a| < \min(R_1, R_2)$, Theorem 11.1.17 gives

$$\Big(\sum_{n=0}^{\infty} a_n (x-a)^n\Big)\Big(\sum_{n=0}^{\infty} b_n (x-a)^n\Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)(x-a)^n \tag{11.1.13}$$

there. The coefficient of $(x - a)^n$ collects every way of splitting the power n between the two factors. The termwise product $\sum a_k b_k$ that Section 4.6 warns against omits all the cross terms.

Study guide

Compute the radius first, from (11.1.7) when the coefficient ratios have a limit and from (11.1.5) otherwise. Then test each endpoint as a numerical series and report the interval with the correct brackets. Inside the open interval the sum may be differentiated, integrated and multiplied freely. At an endpoint use Abel's theorem.

Marks are lost in two places. The first is an interval reported without endpoint work. The second is a differentiated series left starting at $n = 0$ that produces the undefined term $0 \cdot c_0 (x - a)^{-1}$.

Worked Example 11.1.1

Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\, 2^n}$.

Since $|c_n / c_{n+1}| = 2(n+1)/n \to 2$, $R = 2$ by (11.1.7) and the open interval is $(1, 5)$. At $x = 1$ the series $\sum (-1)^n / n$ converges by Theorem 10.6.2. At $x = 5$ it is the harmonic series and diverges. The interval of convergence is $[1, 5)$.

Worked Example 11.1.2

Find a power series for $\ln(1 + x)$ and determine where it represents the function.

The geometric series with ratio $-t$ gives

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n = 1 - t + t^2 - t^3 + \cdots, \qquad |t| < 1. \tag{11.1.14}$$

Since the 1-form $\mathrm{d}t/(1+t)$ is $\mathrm{d}\ln(1+t)$, for $|x|<1$ Theorem 11.1.15 integrates (11.1.14) from 0 to x:

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots,\qquad -1<x\le 1. \tag{11.1.15}$$

The radius is 1. At $x=-1$ the series is the negative of the harmonic series and diverges. At $x=1$ it is the alternating harmonic series and converges. Abel's theorem (Theorem 11.1.12) then gives $\sum(-1)^{n+1}/n=\lim_{x\to 1^-}\ln(1+x)=\ln 2$ by the continuity of the logarithm. Integration has added the endpoint $x=1$ that the series (11.1.14) lacks.

Problems

11.1.1 Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. ⋆ Hint · Answer · Solution

11.1.2 Let $c_n=2^{-n}$ for even n and $c_n=3^{-n}$ for odd n. Find the radius of convergence of $\sum c_n x^n$ and show that Corollary 11.1.7 does not apply. ⋆⋆ Hint · Answer · Solution

11.1.3 Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\sqrt{n}}$. ⋆ Hint · Answer · Solution

11.1.4 Differentiate $\displaystyle\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ term by term and use the result to evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$. ⋆⋆ Hint · Answer · Solution

11.1.5 Express $\displaystyle\int_0^{1/2}\frac{\mathrm{d}x}{1+x^4}$ as a series and evaluate it to five decimal places with an error bound. ⋆⋆ Hint · Answer · Solution

11.1.6 Let $\sum c_n x^n$ have radius R. Prove that $\sum c_n x^{2n}$ has radius $\sqrt{R}$, with $\sqrt{\infty}=\infty$. ⋆⋆ Hint · Answer · Solution

11.1.7 Let $\sum c_n x^n$ have radius $R>0$ and let its sum f be an even function on $(-R,R)$. Prove that $c_n=0$ for every odd n. ⋆⋆ Hint · Answer · Solution

11.1.8 Let $f(x)=\sum_{n\ge 1}\sin(nx)/n^3$. Prove that f is differentiable on $\mathbb{R}$ with $f'(x)=\sum_{n\ge 1}\cos(nx)/n^2$ and that f' is continuous. ⋆⋆ Hint · Answer · Solution

11.1.9 Show that $\sum_{n\ge 1}x^n/n^2$ converges uniformly on $[-1,1]$ and that its derived series converges at $x=-1$ and diverges at $x=1$. What does this show about Theorem 11.1.15 at the endpoints? ⋆⋆ Hint · Answer · Solution

11.1.10 Let $f_n(x)=n^2xe^{-nx}$ on $[0,1]$. Show that $f_n\to 0$ pointwise, compute $\int_0^1 f_n$, and explain which hypothesis of Theorem 11.1.2 fails for the series $\sum u_n$ with $u_1=f_1$ and $u_n=f_n-f_{n-1}$ for $n\ge 2$. Its partial sums are the f_n. ⋆⋆ Hint · Answer · Solution

11.2 Taylor and Maclaurin Series

Overview

Objectives: Prove that a power series is the Taylor series of its sum (Theorem 11.2.1); define the Taylor series generated by a smooth function (Definition 11.2.2); derive the standard Maclaurin series, the logarithm and arctangent ones with their endpoints by Theorem 11.1.12; obtain new series from known ones by substitution, multiplication by a power, products and termwise operations.
Prerequisites: Taylor polynomials (Definition 7.4.2 and Eq. (7.4.3)); power series, termwise differentiation and integration, Abel's theorem and the Cauchy product (Theorems 11.1.5, 11.1.12, 11.1.15 and 11.1.17); the geometric series in function form (Eq. (11.1.14)) and the logarithm series (Eq. (11.1.15)); the half-angle identity (Eq. (9.3.2)).
Key results: The coefficient formula (Theorem 11.2.1 and Eq. (11.2.1)); the Taylor series (Definition 11.2.2 and Eq. (11.2.3)); the standard Maclaurin series of Table 11.2.1.

Notation

$f^{(k)}(a)$	the kth derivative of f at a, with $f^{(0)} = f$
$P_n(x)$	the Taylor polynomial of order n

Section 11.1 started from a series and studied its sum. This section starts from a function and asks which power series represents it. A function has at most one power series about a given centre and its coefficients are the Taylor coefficients of Section 7.4.

Theorem 11.2.1 (Power series are Taylor series)**.** *Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $(a-R, a+R)$ with $R > 0$. Then f has derivatives of every order there and*

$$c_k = \frac{f^{(k)}(a)}{k!} \qquad (k \geq 0). \tag{11.2.1}$$

In particular two power series about a with the same sum near a have the same coefficients.

Proof. Applying Theorem 11.1.15 k times gives, on $(a-R, a+R)$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (x-a)^{n-k}. \tag{11.2.2}$$

At $x = a$ every term of (11.2.2) with $n > k$ vanishes and the term with $n = k$ is $k!\, c_k$. If two series have the same sum f near a, their coefficients are both $f^{(k)}(a)/k!$. ∎

Theorem 11.2.1 extends Eq. (7.4.3) from polynomials to power series. It makes the following definition the only candidate for a series representing f.

Definition 11.2.2 (Taylor and Maclaurin series)**.** Let f have derivatives of every order at a. The *Taylor series generated by f at a* is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \tag{11.2.3}$$

the series whose partial sums are the Taylor polynomials P_n of Definition 7.4.2. The case $a = 0$ is the *Maclaurin series* of f.

Example 11.2.3. The geometric series $\sum x^n$ has sum $1/(1-x)$ on $(-1,1)$. By Theorem 11.2.1 it is therefore the Maclaurin series of $1/(1-x)$ and $f^{(k)}(0) = k!$.

Example 11.2.4 (A Taylor series that misses its function (non-example))**.** Definition 11.2.2 builds a series from f with no guarantee that it converges anywhere except a, or that its sum is f where it does converge. Section 11.3 gives a smooth function whose Maclaurin series converges everywhere to the zero function and so has a sum differing from the function at every $x \neq 0$.

Taylor polynomials generalise the tangent line, the case of order 1 (Fig. 11.2.1).

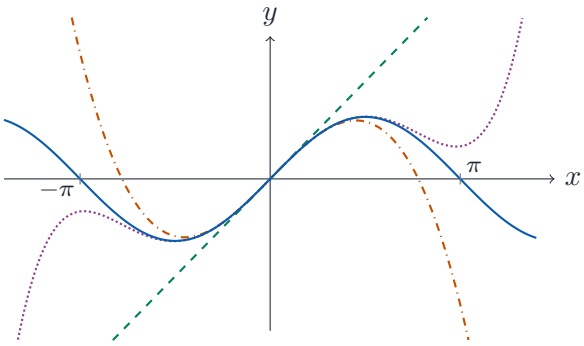


Figure 11.2.1: Taylor polynomials of $\sin x$ at $a = 0$: $y = \sin x$ (solid), the tangent $P_1(x) = x$ (dashed), $P_3(x) = x - x^3/6$ (dash-dotted) and $P_5(x) = x - x^3/6 + x^5/120$ (dotted). Each polynomial fits better than the one before and every fit is best near the origin. Because $\sin x$ has only odd-power terms, $P_2 = P_1$ and $P_4 = P_3$.

11.2.5 Standard series

Exponential, sine and cosine. Every derivative of e^x is e^x, with value 1 at 0. The Maclaurin series generated by e^x is therefore

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{11.2.4}$$

The derivatives of $\sin x$ cycle through $\sin, \cos, -\sin, -\cos$ with values $0, 1, 0, -1$ at 0 and those of $\cos x$ through $\cos, -\sin, -\cos, \sin$. Hence

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \tag{11.2.5}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \tag{11.2.6}$$

The exponential series has $R = \infty$ by (11.1.7). For the sine and cosine series the ratios of consecutive nonzero terms are $x^2/((2n+2)(2n+3))$ and $x^2/((2n+1)(2n+2))$. Both tend to 0 for every x and the ratio test (Theorem 10.5.2) gives absolute convergence. At this stage the equality signs in Eqs. (11.2.4) to (11.2.6) record generated series only: Theorem 11.3.2 proves that each sum equals its function for every real x. Problem 11.2.7 gives a second proof for the exponential.

Logarithm and arctangent. These two series come from the geometric series (11.1.14). They are equalities from the start inside $(-1,1)$ because Theorem 11.1.15 proves them there. Worked Example 11.1.2 integrated (11.1.14) to obtain the logarithm series (11.1.15). Substi-

tuting t^2 for t in (11.1.14) gives $1/(1+t^2) = \sum(-1)^n t^{2n}$ for $|t| < 1$. Integrating the 1-form $\mathrm{d}t/(1+t^2) = \mathrm{d}\arctan t$ from 0 to x gives

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1. \tag{11.2.7}$$

At $x = \pm 1$ the series converges by Theorem 10.6.2. Abel's theorem (Theorem 11.1.12) with the continuity of arctan extends the equality to both endpoints. At $x = 1$ it gives *Leibniz's series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{11.2.8}$$

Table 11.2.1: Six standard Maclaurin series. The first row is Theorem 10.2.7. The fifth and sixth rows are proved by Theorems 11.1.12 and 11.1.15 and the other three by Theorem 11.3.2.

Function	**Series**	**Valid for**
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	all x
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \le 1$
$\arctan x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$-1 \le x \le 1$

New series come from Table 11.2.1 without computing derivatives. Substituting an expression for x transfers both the series and its interval. Multiplying by a power of x shifts every exponent. Theorem 11.1.15 differentiates and integrates a series and (11.1.13) multiplies two series. By Theorem 11.2.1 a series obtained in any such way is the Taylor series of its sum.

Study guide

Compute derivatives only when no row of Table 11.2.1 adapts to the function. Substitution needs a matching change of interval: the geometric row gives $1/(1-2x)$ only for $|x| < \frac{1}{2}$ and the cosine row gives $\cos(x^3)$ for every x.
Two errors are common. The first is evaluating a series outside its interval of convergence. The second is dropping the factorial from an odd-power series: the sine has $(2n+1)!$ where the arctangent has $2n+1$.

Worked Example 11.2.1

Find the Maclaurin series for $x^2 e^{-3x}$ and say where it is valid.

Substituting $-3x$ into the exponential row and multiplying by x^2,

$$x^2 e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{n+2}}{n!} = x^2 - 3x^3 + \frac{9x^4}{2} - \frac{9x^5}{2} + \cdots,$$

valid for every real x. With $m = n+2$ the coefficient of x^m is $(-3)^{m-2}/(m-2)!$ for $m \geq 2$. By Theorem 11.2.1 the fifth derivative at 0 is $5! \cdot (-\frac{9}{2}) = -540$, found without differentiating.

Worked Example 11.2.2

Find the Taylor polynomial of order 3 for $f(x) = \ln x$ at $a = 1$ and the interval on which the Taylor series represents $\ln x$.
The derivatives $1/x$, $-1/x^2$, $2/x^3$ take the values $1, -1, 2$ at 1 and $\ln 1 = 0$. Hence

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

Writing $\ln x = \ln(1 + (x-1))$ and substituting $x-1$ into (11.1.15) gives the whole Taylor series, valid for $-1 < x - 1 \leq 1$, that is $0 < x \leq 2$.

Problems

11.2.1 Find the Maclaurin series for e^{-2x} and state where it is valid. ⋆ Hint · Answer · Solution

11.2.2 Find the Maclaurin series for $x\cos(x^2)$. ⋆ Hint · Answer · Solution

11.2.3 Use (11.2.7) to compute $\arctan \frac{1}{2}$ to four decimal places, with an error bound. ⋆⋆ Hint · Answer · Solution

11.2.4 Find the Maclaurin series for $e^x \sin x$ as far as the term in x^4. ⋆⋆ Hint · Answer · Solution

11.2.5 Find the Maclaurin series for $\sin^2 x$ in closed form. ⋆⋆ Hint · Answer · Solution

11.2.6 Let f be odd with derivatives of every order on $\mathbb{R}$. Prove that its Maclaurin series has only odd powers. ⋆ Hint · Answer · Solution

11.2.7 Let $E(x) = \sum_{n \geq 0} x^n/n!$. Prove that $E' = E$ and $E(0) = 1$ and deduce $E(x) = e^x$ for every real x without using a remainder estimate. ⋆⋆ Hint · Answer · Solution

11.2.8 With E as in Problem 11.2.7, prove $E(x)E(y) = E(x+y)$ for all real x and y from Theorem 11.1.17 and the binomial theorem. ⋆⋆ Hint · Answer · Solution

11.2.9 Find the Maclaurin series of $1/(1+x^2)$ and its radius of convergence. Explain why the radius is finite although the function has derivatives of every order on all of $\mathbb{R}$. ⋆⋆ Hint · Answer · Solution

11.2.10 A student puts $x = 2$ into (11.1.15) and writes $\ln 3 = 2 - 2 + \frac{8}{3} - 4 + \cdots$. Explain the error. ⋆ Hint · Answer · Solution

11.3 Convergence of Taylor Series

Overview

Objectives: Read Taylor's theorem (Theorem 7.4.8) as an exact formula for the remainder and prove from it that a Taylor series converges to its function when the remainder tends to 0 (Theorem 11.3.1); prove that the exponential, sine and cosine series converge to their functions everywhere (Theorem 11.3.2); distinguish analytic functions from smooth ones (Definition 11.3.4 and Proposition 11.3.5); define the complex exponential by its series (Definition 11.3.7), prove its addition law (Proposition 11.3.8) and prove Euler's formula (Theorem 11.3.9).
Prerequisites: Taylor's theorem and the remainder bound (Theorem 7.4.8, Eq. (7.4.6), and Corollary 7.4.9); Taylor series and the standard Maclaurin series (Definition 11.2.2 and Table 11.2.1); the limit $r^n/n! \to 0$ (Theorem 10.1.4); the growth ranking (Theorem 5.2.26); convergence in $\mathbb{C}$ and absolute convergence (Theorems 5.1.31 and 10.2.15 and Example 5.1.32); the binomial theorem and the Cauchy product (Theorems 4.6.14 and 11.1.17).
Key results: The convergence criterion (Theorem 11.3.1 and Eq. (11.3.2)); the exponential, sine and cosine series (Theorem 11.3.2); analytic functions and the smooth non-analytic function e^{-1/x^2} (Definition 11.3.4 and Proposition 11.3.5); the addition law (Proposition 11.3.8 and Eq. (11.3.6)); Euler's formula (Theorem 11.3.9 and Eq. (11.3.8)).

Notation

$R_n(x)$	the remainder $f(x) - P_n(x)$
c	the intermediate point supplied by Theorem 7.4.8
M	a bound on $\lvert f^{(n+1)}\rvert$ between a and x
$\exp z$	the complex exponential $\sum z^n/n!$ (Definition 11.3.7)

The partial sums of the Taylor series generated by f are the Taylor polynomials P_n. The series converges to $f(x)$ exactly when the remainder $R_n(x) = f(x) - P_n(x)$ tends to 0. Taylor's theorem (Theorem 7.4.8) gives that remainder exactly: if f is $n+1$ times differentiable on an open interval containing a and x, then for some c strictly between a and x

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}. \tag{11.3.1}$$

Applications replace $f^{(n+1)}(c)$ by a bound valid between a and x because the location of c is unknown.

Theorem 11.3.1 (Convergence criterion for Taylor series)**.** *Let f have derivatives of every order on an open interval containing a and x and suppose $\lvert f^{(n+1)}(t)\rvert \le M_n$ for every t between a and x inclusive. Then*

$$\lvert R_n(x)\rvert \le M_n\,\frac{\lvert x-a\rvert^{n+1}}{(n+1)!}. \tag{11.3.2}$$

If the right-hand side tends to 0 as $n \to \infty$, the Taylor series of f at a converges to $f(x)$.

Proof. The inequality is Corollary 7.4.9. If its right-hand side tends to 0, then $P_n(x) = f(x) - R_n(x) \to f(x)$ by Theorem 5.1.11 and the partial sums of the series converge to $f(x)$ (Definition 10.2.1). ∎

Theorem 11.3.2 (Exponential, sine and cosine series)**.** *For every real x,*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \tag{11.3.3}$$

Proof. Since every derivative of sin or cos is one of $\pm\sin$, $\pm\cos$, $M_n = 1$ serves in (11.3.2) for every n and every t. Theorem 11.3.1 then gives the two series because $|R_n(x)| \le |x|^{n+1}/(n+1)! \to 0$ by Theorem 10.1.4. Every derivative of e^t is e^t and is at most $e^{|x|}$ between 0 and x. With $M_n = e^{|x|}$, $|R_n(x)| \le e^{|x|}|x|^{n+1}/(n+1)! \to 0$. ∎

In each case the numerator of (11.3.2) grows geometrically in n and the factorial outgrows it for every fixed x. The logarithm and the arctangent have factorially growing derivatives and series of radius 1. For $f(x) = \ln(1+x)$ the derivatives are $f^{(n+1)}(x) = (-1)^n n!/(1+x)^{n+1}$ (Problem 11.3.4). For $f = \arctan$ the series (11.2.7) and Theorem 11.2.1 give $f^{(2n+1)}(0) = (-1)^n(2n)!$.

11.3.3 Analytic and smooth functions

Definition 11.3.4 (Analytic function)**.** A function f is *analytic at a* if it has derivatives of every order at a and its Taylor series at a converges to $f(x)$ for every x in some interval $(a-r, a+r)$ with $r > 0$. It is *analytic on an open interval* if it is analytic at every point of the interval.

By Theorem 11.2.1, f is analytic at a exactly when it equals the sum of some power series of positive radius about a near a. The exponential is analytic on $\mathbb{R}$ (Problem 11.3.7), the sine and cosine are analytic by the same estimate at any centre, and the sum of every power series is analytic on its open interval of convergence (Theorem 11.5.5). A function with derivatives of every order on an open interval is called *smooth* there. Every analytic function is smooth by Theorem 11.1.15. The converse fails.

Proposition 11.3.5 (A smooth function that is not analytic)**.** *Let $\phi(x) = e^{-1/x^2}$ for $x \neq 0$ and $\phi(0) = 0$. Then ϕ is smooth on $\mathbb{R}$ with $\phi^{(k)}(0) = 0$ for every k. Its Maclaurin series is the zero series and converges everywhere to 0 whereas $\phi(x) > 0$ for every $x \neq 0$. The function ϕ is not analytic at 0.*

Proof. For $x \neq 0$ induction on k gives $\phi^{(k)}(x) = q_k(1/x)e^{-1/x^2}$ with q_k a polynomial: $q_0 = 1$ and differentiating $q_k(1/x)e^{-1/x^2}$ by the chain and product rules gives $q_{k+1}(u) = -u^2 q_k'(u) + 2u^3 q_k(u)$. For every integer $m \ge 0$,

$$|x|^{-m} e^{-1/x^2} = t^{m/2} e^{-t} \to 0 \quad \text{as } x \to 0, \qquad t = 1/x^2 \to \infty, \tag{11.3.4}$$

by Theorem 5.2.26. Hence (11.3.4) applied to each monomial gives $q(1/x)e^{-1/x^2} \to 0$ for every polynomial q. Suppose $\phi^{(k)}(0) = 0$ (true for $k = 0$). Since the difference quotient $\phi^{(k)}(h)/h = h^{-1} q_k(1/h) e^{-1/h^2}$ has the same form and tends to 0, $\phi^{(k+1)}(0) = 0$ and $\phi^{(k+1)}$ is continuous at 0 by the same limit. Induction gives smoothness and $\phi^{(k)}(0) = 0$ for every k. If ϕ were analytic at 0, it would equal its Maclaurin series 0 on an interval around 0, contrary to $\phi(x) > 0$ for $x \neq 0$. ∎

Proposition 11.3.5 is the reason Theorem 11.3.1 needs its hypothesis. For ϕ every Taylor polynomial at 0 is 0. Hence $R_n(x) = \phi(x)$ for every n and never tends to 0. The operator chapter reaches the same function as Problem 7.4.11 with the sum of the Taylor series at 0 differing from $\phi(h)$ at every $h \neq 0$.

Common misconception

A Taylor series that converges need not converge to the function that generated it. Convergence of $\sum f^{(k)}(a)(x-a)^k/k!$ is a property of the numbers $f^{(k)}(a)$ alone. Agreement with $f(x)$ requires a separate argument such as Theorem 11.3.1. Proposition 11.3.5 shows that it can fail at every $x \neq a$.

11.3.6 Euler's formula

The exponential series converges absolutely for every complex argument (Problem 10.2.8). It therefore defines a function on $\mathbb{C}$ that agrees with e^x on the real line by Theorem 11.3.2.

Definition 11.3.7 (Complex exponential)**.** For a complex number z,

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{11.3.5}$$

the limit in $\mathbb{C}$ of the partial sums (Definition 10.2.1).

Proposition 11.3.8 (Addition law for the complex exponential)**.** *For all complex z and w,*

$$\exp(z+w) = \exp z \, \exp w. \tag{11.3.6}$$

Proof. The series of $\exp z$ and $\exp w$ converge absolutely by Problem 10.2.8. The induction proving Theorem 4.6.14 uses only the commutative, associative and distributive laws, and these hold in $\mathbb{C}$. Hence the nth Cauchy coefficient is

$$\sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \frac{(z+w)^n}{n!}. \tag{11.3.7}$$

By Theorem 11.1.17 the series with the terms (11.3.7) converges to $\exp z \, \exp w$. That series is the series of $\exp(z+w)$. ∎

For real z and w Proposition 11.3.8 is Problem 11.2.8.

Theorem 11.3.9 (Euler's formula)**.** *For every real θ,*

$$\exp(i\theta) = \cos\theta + i \sin\theta. \tag{11.3.8}$$

Proof. The powers of i satisfy $i^{2m} = (-1)^m$ and $i^{2m+1} = i(-1)^m$. The partial sum of (11.3.5) through index $2N+1$ therefore splits into its even and odd terms:

$$\sum_{n=0}^{2N+1} \frac{(i\theta)^n}{n!} = \sum_{m=0}^{N} \frac{(-1)^m \theta^{2m}}{(2m)!} + i \sum_{m=0}^{N} \frac{(-1)^m \theta^{2m+1}}{(2m+1)!}. \tag{11.3.9}$$

The left side of (11.3.9) is a subsequence of the partial sums of a convergent series and tends to $\exp(i\theta)$ by Theorem 5.1.23, whose proof applies in $\mathbb{C}$ with $|\cdot|$ the modulus. The two sums on the right tend to $\cos\theta$ and $\sin\theta$ by Theorem 11.3.2. A complex sequence converges when its real and imaginary parts do (Example 5.1.32). Uniqueness of limits (Theorem 5.1.31) gives (11.3.8). ∎

The proof rearranges nothing: it groups each partial sum of odd index into two finite sums. Setting $\theta = \pi$ gives $\exp(i\pi) = -1$. Since $\exp(z+w) = \exp z \exp w$ for complex z and w (Proposition 11.3.8), Euler's formula turns multiplication of points of the unit circle into addition of angles and yields the addition formulas for sine and cosine (Problem 11.3.8). The trigonometric functions of Section 4.5 are therefore the real and imaginary parts of one power series. In particular $|\exp(i\theta)|^2 = \cos^2\theta + \sin^2\theta = 1$.

Study guide

Applying (11.3.2) takes three steps. Name the derivative needed, the $(n+1)$st. Bound it over the whole interval between a and x. Substitute. For sin and cos the bound is 1 and for e^t it is the value at the right end of the interval.
Two errors are common. The first is bounding the nth derivative where the $(n+1)$st is needed. The second is bounding $f^{(n+1)}$ only at a or only at x although c may lie anywhere between them.

Worked Example 11.3.1

Estimate $e^{0.2}$ with the Taylor polynomial of order 3 at $a = 0$ and bound the error.
$P_3(0.2) = 1 + 0.2 + 0.02 + 0.00133\overline{3} = 1.22133\overline{3}$. The fourth derivative of e^t is at most $e^{0.2} < 2$ on $[0, 0.2]$, and (11.3.2) gives

$$|R_3(0.2)| \le 2 \cdot \frac{0.2^4}{4!} = \frac{0.0032}{24} \approx 0.000133.$$

The value is $e^{0.2} = 1.2214028$, with actual error 0.0000694.

Worked Example 11.3.2

How many terms of the exponential series compute e to within 10^{-4}?
With $x = 1$ and $M_n = 3 > e$, (11.3.2) gives $|R_n(1)| \le 3/(n+1)!$. Since $7! = 5040$ and $8! = 40320$, the bound falls below 10^{-4} first at $n + 1 = 8$. The eight terms of degree 0 to 7 sum to about 2.7182540, against $e \approx 2.7182818$.

Problems

11.3.1 Estimate $\sin(0.3)$ with the Taylor polynomial of order 3 at $a = 0$ and bound the error with Theorem 11.3.1. ⋆ Hint · Answer · Solution

11.3.2 Show that $\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| \le \frac{|x|^3}{6}$ for every real x. ⋆ Hint · Answer · Solution

11.3.3 How many terms of (11.2.6) compute $\cos 1$ to within 10^{-6}?
⋆⋆ Hint · Answer · Solution

11.3.4 Estimate $\ln(1.2)$ from (11.1.15) to within 10^{-4} using Theorem 10.6.3. Compare the bound with the one that Theorem 11.3.1 gives at $x = 0.2$ and at negative x.
⋆⋆ Hint · Answer · Solution

11.3.5 How many terms of the sine series (11.2.5) are needed to compute $\sin 2$ to within 10^{-6}? Answer with Theorem 11.3.1 and again with Theorem 10.6.3 and reconcile the two.
⋆⋆ Hint · Answer · Solution

11.3.6 Use Theorem 11.3.9 and Proposition 11.3.8 to show that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{n!} = e^{\cos\theta}\cos(\sin\theta)$.

⋆⋆ Hint · Answer · Solution

11.3.7 Prove that the Taylor series of e^x at any centre a converges to e^x for every real x. Deduce $e^{a+h} = e^a \sum_{k\geq 0} h^k/k!$. ⋆⋆ Hint · Answer · Solution

11.3.8 Derive $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and the formula for $\sin(\alpha+\beta)$ from Theorem 11.3.9 and $\exp(z+w) = \exp z \exp w$ (Proposition 11.3.8).

⋆⋆ Hint · Answer · Solution

11.3.9 Find two different smooth functions on $\mathbb{R}$ with the same Maclaurin series and say which of them is analytic at 0. ⋆ Hint · Answer · Solution

11.3.10 Let $g(x) = e^{-1/x}$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$. Prove that g is smooth on $\mathbb{R}$. Deduce that $\psi(x) = g(x)g(1-x)$ is smooth, positive on $(0,1)$ and zero outside it. Explain why no nonzero analytic function on $\mathbb{R}$ vanishes on an interval.

⋆⋆⋆ Hint · Answer · Solution

11.4 Binomial Series and Applications of Taylor Series

Overview

Objectives: Prove the binomial series for every real exponent (Theorem 11.4.2); integrate a series term by term to evaluate an integral with no elementary antiderivative, with an error bound from Theorem 10.6.3; evaluate an indeterminate limit by comparing leading coefficients of power series (Proposition 11.4.5).
Prerequisites: The generalised binomial coefficient (Eq. (4.6.18)); the radius from ratios (Corollary 11.1.7); termwise differentiation and integration and continuity of a power series (Theorems 11.1.11 and 11.1.15); Abel's theorem (Theorem 11.1.12); zero derivative implies constant (Corollary 7.2.2); the alternating series estimate (Theorem 10.6.3); numerical integration (Definition 9.7.6).
Key results: The binomial series (Theorem 11.4.2 and Eq. (11.4.1)); the series for $\int(\sin x)/x\,dx$ and $\int e^{-x^2}\,dx$ (Eqs. (11.4.6) and (11.4.7)); limits by leading coefficients (Proposition 11.4.5).

Notation

m	the exponent of the binomial, any real number
$\binom{m}{k}$	the generalised binomial coefficient of Eq. (4.6.18)
u_n	the size of the nth term of an alternating series

The binomial theorem expands $(1+x)^m$ for a nonnegative integer m as a finite sum. For every other real m the same coefficients give an infinite series with sum $(1+x)^m$ on $(-1,1)$. The second half applies termwise integration and the continuity of power series to two problems the earlier chapters cannot finish.

11.4.1 Binomial series

Theorem 11.4.2 (Binomial series)**.** *For any real number m and $|x|<1$,*

$$(1+x)^m=\sum_{k=0}^{\infty}\binom{m}{k}x^k=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\cdots, \tag{11.4.1}$$

with $\binom{m}{k}$ the generalised binomial coefficient of Eq. (4.6.18). When m is a nonnegative integer the series stops after the term in x^m and holds for every x.

Proof. If m is a nonnegative integer, $\binom{m}{k}=0$ for $k>m$ and the statement is the binomial theorem. Otherwise every coefficient is nonzero and

$$\left|\binom{m}{k+1}\Big/\binom{m}{k}\right|=\left|\frac{m-k}{k+1}\right|\xrightarrow[k\to\infty]{}1. \tag{11.4.2}$$

By (11.4.2) and Corollary 11.1.7 the radius is 1. Let $g(x)$ be the sum on $(-1,1)$. By (11.1.9) and the identity $(k+1)\binom{m}{k+1}+k\binom{m}{k}=m\binom{m}{k}$ (Problem 11.4.6),

$$(1+x)g'(x)=\sum_{k\ge0}\left((k+1)\binom{m}{k+1}+k\binom{m}{k}\right)x^k=m\,g(x). \tag{11.4.3}$$

By (11.4.3)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{g(x)}{(1+x)^m}\right] = \frac{(1+x)g'(x) - m\,g(x)}{(1+x)^{m+1}} = 0. \tag{11.4.4}$$

By (11.4.4) and Corollary 7.2.2, $g(x)/(1+x)^m$ is constant on $(-1,1)$. At $x=0$ the constant is 1. Hence $g(x) = (1+x)^m$. ∎

By Theorem 11.2.1 the binomial series is the Maclaurin series of $(1+x)^m$. Its coefficients $\binom{m}{k} = f^{(k)}(0)/k!$ agree with the derivatives $m(m-1)\cdots(m-k+1)(1+x)^{m-k}$ at 0. The fractional exponents $m = \pm\frac{1}{2}$ give $\sqrt{1+x}$ and $1/\sqrt{1+x}$ with first coefficients computed in Example 4.6.15. Neither function has a finite expansion.

For m not a nonnegative integer the endpoints $x = \pm 1$ depend on m (Problem 11.4.9). For $m > 0$ the series converges absolutely at both endpoints. For $-1 < m < 0$ it converges at $x = 1$ and diverges at $x = -1$. For $m \le -1$ it diverges at both. Where it converges, Theorem 11.1.12 gives the sum $\lim(1+x)^m$, equal to 2^m at $x = 1$ and to 0 at $x = -1$.

11.4.3 Nonelementary integrals

The integrands $(\sin x)/x$ and e^{-x^2} have no antiderivative that is a finite combination of the elementary functions of Chapter 4, by a theorem of Liouville that this book does not prove. The methods of Chapter 9 produce elementary antiderivatives and therefore fail on them. Power series supply antiderivatives of a different kind.

The sine row divided by x is the power series

$$S_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots, \tag{11.4.5}$$

with radius ∞. It equals $(\sin x)/x$ for $x \neq 0$ by Theorem 10.2.10, takes the value 1 at 0 and is continuous by Theorem 11.1.11. Integrating (11.4.5) term by term (Theorem 11.1.15),

$$\int \frac{\sin x}{x}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} = C + x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots \tag{11.4.6}$$

for every real x. Substituting $-x^2$ into the exponential row and integrating gives

$$\int e^{-x^2}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} = C + x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots, \tag{11.4.7}$$

again for every real x. Each series alternates for $x > 0$ and becomes a numerical alternating series between limits, with the error bound of Theorem 10.6.3. The error bounds of the Trapezoidal and Simpson rules (Theorems 9.7.3 and 9.7.7) need a bound on f'' or $f^{(4)}$ over the interval. For these integrands such a bound is laborious to obtain. The alternating bound needs only the size of the first omitted term.

11.4.4 Indeterminate limits

Proposition 11.4.5 (Limits by leading coefficients)**.** *Let* $f(x) = \sum_{n\ge k} c_n x^n$ *and* $g(x) = \sum_{n\ge k} d_n x^n$ *have positive radii, with* $d_k \neq 0$*. Then* $\lim_{x\to 0} f(x)/g(x) = c_k/d_k$*.*

Proof. For $x \neq 0$ inside both intervals, $f(x)/g(x) = F(x)/G(x)$ with $F(x) = \sum_{n\ge k} c_n x^{n-k}$ and $G(x) = \sum_{n\ge k} d_n x^{n-k}$. These power series converge wherever f and g do because $F(x) = f(x)/x^k$ and $G(x) = g(x)/x^k$ for $x \neq 0$. They are continuous at 0 by Theorem 11.1.11, with $F(0) = c_k$ and $G(0) = d_k \neq 0$. The quotient F/G is continuous at 0 (Theorem 5.3.6). ∎

For example, since $\sin x - x = \sum_{n\geq 1}(-1)^n x^{2n+1}/(2n+1)!$ begins at x^3 with coefficient $-\frac{1}{6}$, $\lim_{x\to 0}(\sin x - x)/x^3 = -\frac{1}{6}$. The series also gives the next correction: $(\sin x - x)/x^3 = -\frac{1}{6} + x^2/120 - \cdots$.

Study guide

For an integral, bring the integrand into a row of Table 11.2.1, integrate term by term inside the interval of convergence, substitute the limits and check that the sizes decrease before quoting Theorem 10.6.3. For a limit, subtract and divide series until the leading coefficients are visible and apply Proposition 11.4.5.
Two errors are common. The first is integrating a binomial series of radius 1 over an interval reaching past $x = 1$. The second is dropping the constant C from an indefinite integral of a series.

Worked Example 11.4.1

Find the first four terms of the binomial series for $\sqrt{1+x}$ and use them to estimate $\sqrt{1.1}$.
At $m = \frac{1}{2}$ the coefficients 1, $\frac{1}{2}$, $-\frac{1}{8}$ and $\frac{1}{16}$ (Example 4.6.15) give

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots, \qquad |x| < 1.$$

At $x = 0.1$ the four terms give 1.0488125, against $\sqrt{1.1} = 1.0488088$. Since the series alternates with decreasing sizes from the second term on, Theorem 10.6.3 bounds the error by $\left|\binom{1/2}{4}\right|(0.1)^4 = \frac{5}{128}\cdot 10^{-4} \approx 3.9 \times 10^{-6}$.

Worked Example 11.4.2

Evaluate $\displaystyle\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x$ to five decimal places, with an error bound.
Since the integrand is the continuous function S_0, (11.1.10) gives

$$\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} = 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} + \cdots.$$

The sizes 1, 0.0555556, 0.0016667, 0.0000283 and $u_4 = 1/3265920 \approx 3 \times 10^{-7}$ decrease to 0. Four terms leave an error of at most u_4 and sum to 0.9460828. The integral is therefore 0.94608 to five places. The value is 0.9460831.

Problems

11.4.1 Find the first three terms of the binomial series for $(1+x)^{1/3}$ and state where it is valid. ⋆ Hint · Answer · Solution

11.4.2 Evaluate $\displaystyle\int_0^{1/2} e^{-x^2}\,\mathrm{d}x$ to five decimal places, with an error bound.
⋆ Hint · Answer · Solution

11.4.3 Evaluate $\displaystyle\int_0^1 e^{-x^2}\,\mathrm{d}x$ to four decimal places with an error bound.
⋆⋆ Hint · Answer · Solution

11.4.4 Use the binomial series with $m = -\frac{1}{2}$ and termwise integration to find the Maclaurin series of $\arcsin x$ through the term in x^5. ⋆⋆ Hint · Answer · Solution

11.4.5 Prove that $\binom{-1/2}{k}(-4)^k = \binom{2k}{k}$ and deduce $\displaystyle\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$ for $|x| < \frac{1}{4}$.
⋆⋆ Hint · Answer · Solution

11.4.6 Prove that $(k+1)\binom{m}{k+1} + k\binom{m}{k} = m\binom{m}{k}$ for every real m and integer $k \geq 0$.
⋆ Hint · Answer · Solution

11.4.7 Use Proposition 11.4.5 to evaluate $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4}$ and show by an example that the proposition fails when $d_k = 0$ and $c_k \neq 0$. ⋆⋆ Hint · Answer · Solution

11.4.8 Show that the binomial series for $\sqrt{1+x}$ converges at $x = 1$ and that its sum there is $\sqrt{2}$, although Theorem 11.4.2 assumes $|x| < 1$. ⋆⋆ Hint · Answer · Solution

11.4.9 Let m be real and not a nonnegative integer and let $u_k = \left|\binom{m}{k}\right|$. Prove that the binomial series converges absolutely at $x = \pm 1$ when $m > 0$, converges at $x = 1$ and diverges at $x = -1$ when $-1 < m < 0$, and diverges at $x = \pm 1$ when $m \leq -1$.
⋆⋆⋆ Hint · Answer · Solution

11.4.10 Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ with Proposition 11.4.5 and say how many applications of L'Hôpital's rule the same limit needs. ⋆⋆ Hint · Answer · Solution

11.5 Analytic Functions and the Shift Operator

Overview

Objectives: Interchange the order of summation in an absolutely convergent double series (Theorem 11.5.2) and show by example that absolute convergence is needed; prove that the sum of a power series is analytic at every point of its open interval of convergence (Theorem 11.5.5); extend the shift theorem $E_h = T_h$ from polynomials to sums of power series (Theorem 11.5.7) and read off the limits of that extension.
Prerequisites: Absolute convergence and the Cauchy product (Theorems 10.2.15 and 11.1.17); termwise differentiation and the coefficient formula (Theorems 11.1.15 and 11.2.1); analytic functions and e^{-1/x^2} (Definition 11.3.4 and Proposition 11.3.5); the Taylor shift on polynomials and the shift theorem (Definition 7.4.11 and Theorem 7.4.12).
Key results: Interchange of summation (Theorem 11.5.2); re-expansion of a power series about a new centre (Theorem 11.5.5); the shift theorem for power series (Theorem 11.5.7 and Eq. (11.5.7)).

Notation

a_{mn}	a double sequence, indexed by $m, n \geq 0$
T_h	the shift $(T_h f)(x) = f(x+h)$
$E_h f$	the Taylor shift series $\sum_{k\geq 0} h^k f^{(k)}/k!$

Theorem 11.2.1 shows that a power series about a is the Taylor series of its sum at the centre a. It says nothing about other points of the interval. At a point $b \neq a$ the sum f has derivatives of every order and the question is whether its Taylor series at b converges to f near b. The answer needs a rearrangement of a sum indexed by two integers. Section 7.4 proved $\sum_{k\geq 0} h^k p^{(k)}(x)/k! = p(x+h)$ for polynomials (Definition 7.4.11 and Theorem 7.4.12), where the series stops after finitely many terms. For the sum of a power series the series is infinite and the same re-expansion settles when it converges to $f(x+h)$.

11.5.1 Double series

Theorem 11.5.2 (Interchange of summation in a double series)**.** *Let a_{mn} with $m, n \geq 0$ be real or complex numbers with*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{mn}| = B < \infty. \tag{11.5.1}$$

Then every row $\sum_n a_{mn}$ and every column $\sum_m a_{mn}$ converges absolutely and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}, \tag{11.5.2}$$

with both iterated series absolutely convergent.

Proof. Let $b_m = \sum_n |a_{mn}|$ (finite by hypothesis) with $\sum b_m = B$. Each column has $|a_{mn}| \leq b_m$ and converges absolutely by Theorem 10.4.1.
1. Nonnegative terms. For each N, linearity applied to the finitely many columns $n \leq N$ gives $\sum_{n\leq N}\sum_m |a_{mn}| = \sum_m \sum_{n\leq N} |a_{mn}| \leq B$. By Theorem 10.3.1 the series $\sum_n \sum_m |a_{mn}|$ converges with sum at most B. The same argument with rows and columns exchanged gives equality. The iterated sums of $|a_{mn}|$ agree.

2. General terms. Let $\varepsilon > 0$ and choose M with $\sum_{m>M} b_m < \varepsilon$. The left side L of (11.5.2) differs from $\sum_{m\le M}\sum_n a_{mn}$ by at most $\sum_{m>M} b_m < \varepsilon$. By linearity the finite sum over $m \le M$ equals $\sum_n \sum_{m\le M} a_{mn}$. The right side differs from this by

$$\Big|\sum_{n}\sum_{m>M} a_{mn}\Big| \le \sum_{n}\sum_{m>M} |a_{mn}| = \sum_{m>M} b_m < \varepsilon, \tag{11.5.3}$$

the equality being the nonnegative case applied to the rows $m > M$. The two sides of (11.5.2) differ by less than 2ε for every $\varepsilon > 0$ by (11.5.3) and the estimate on the left, and are equal. The bounds $|\sum_n a_{mn}| \le b_m$ and the nonnegative case give absolute convergence of both iterated series. ∎

Example 11.5.3 (Iterated sums that disagree (non-example))**.** Let $a_{mm} = 1$, $a_{m+1,m} = -1$ for every $m \ge 0$, and $a_{mn} = 0$ otherwise. Since each row $m \ge 1$ contains 1 and -1 and sums to 0 while row 0 sums to 1, the iterated sum by rows is 1. Since each column contains 1 and -1 and sums to 0, the iterated sum by columns is 0. Here $\sum_n |a_{mn}| = 2$ for $m \ge 1$ and the hypothesis of Theorem 11.5.2 fails.

11.5.4 Re-expansion about a new centre

Theorem 11.5.5 (Sums of power series are analytic)**.** *Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ have radius $R > 0$ and let $|b-a| < R$. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!}(x-b)^k \qquad \text{whenever } |x-b| < R - |b-a|. \tag{11.5.4}$$

In particular f is analytic on $(a-R, a+R)$.

Proof. Write $\beta = b-a$ and $t = x-b$. Then $x-a = \beta + t$ and $|\beta| + |t| < R$. By the binomial theorem

$$c_n (x-a)^n = \sum_{k=0}^{n} c_n \binom{n}{k} \beta^{n-k} t^k. \tag{11.5.5}$$

Let a_{nk} be the kth term of (11.5.5) for $k \le n$ and 0 for $k > n$. Then $\sum_k |a_{nk}| = |c_n|(|\beta| + |t|)^n$. The series $\sum_n |c_n|(|\beta|+|t|)^n$ converges by Theorem 11.1.5 because $|\beta|+|t| < R$. Theorem 11.5.2 gives

$$f(x) = \sum_n \sum_k a_{nk} = \sum_{k=0}^{\infty} \Big(\sum_{n=k}^{\infty} c_n \binom{n}{k} \beta^{n-k} \Big) t^k. \tag{11.5.6}$$

Differentiating the series of f k times by Theorem 11.1.15 and setting $x = b$ gives $f^{(k)}(b) = \sum_{n\ge k} \dfrac{n!}{(n-k)!} c_n \beta^{n-k}$. Hence the bracket in (11.5.6) is $f^{(k)}(b)/k!$. ∎

The guaranteed radius $R - |b-a|$ of (11.5.4) is the distance from b to the nearer endpoint of the original interval. The Taylor series at b may converge on a larger interval (Problem 11.5.1). By Proposition 11.3.5 the theorem has no converse: a smooth function need not be the sum of any power series near a point.

Worked Example 11.5.1

Re-expand $f(x) = 1/(1-x) = \sum x^n$ about $b = \frac{1}{2}$ and compare the radius with Theorem 11.5.5.

By Theorem 11.1.15, $f^{(k)}(x) = k!/(1-x)^{k+1}$. Hence $f^{(k)}(\tfrac{1}{2})/k! = 2^{k+1}$ and

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} 2^{k+1}\Big(x - \frac{1}{2}\Big)^k \qquad \text{for } \Big|x - \frac{1}{2}\Big| < \frac{1}{2}.$$

The radius $\frac{1}{2}$ equals $R - |b-a| = 1 - \frac{1}{2}$. The geometric series in $2(x-\frac{1}{2})$ diverges at $x = 1$ where f is undefined. The bound of the theorem is therefore attained here.

11.5.6 Shift operator on power series

For a function f with derivatives of every order at x, write

$$(E_h f)(x) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x) \tag{11.5.7}$$

whenever the series converges. On a polynomial the series stops and E_h is the operator of Definition 7.4.11. The series in (11.5.7) is the Taylor series of f at x evaluated at the point $x+h$. Theorem 11.5.5 decides its sum.

Theorem 11.5.7 (Shift theorem for power series)**.** *Let f be the sum of a power series about a with radius $R > 0$. If $|x-a| + |h| < R$, then the series (11.5.7) converges absolutely and*

$$(E_h f)(x) = f(x+h) = (T_h f)(x). \tag{11.5.8}$$

If also $|x-a| + |h| + |s| < R$, then $(E_h E_s f)(x) = (E_{h+s} f)(x)$.

Proof. Take $b = x$ in Theorem 11.5.5. Since $|h| < R - |x-a|$, the Taylor series of f at x converges at $x+h$ to $f(x+h)$. Its absolute convergence follows from the proof of Theorem 11.5.5 in which every iterated series converges absolutely. For the last statement, $E_s f = T_s f$ on the interval $|y-a| < R - |s|$ and $T_s f$ is the sum of a power series about $a - s$ with radius R. Applying the first statement to it at x gives $(E_h T_s f)(x) = f(x+h+s) = (E_{h+s} f)(x)$. ∎

Theorem 11.5.7 identifies the Taylor shift E_h with translation on sums of power series, within distance $R - |x-a|$ of the point. Outside the analytic functions the identity fails. For the function ϕ of Proposition 11.3.5 every derivative vanishes at 0 and hence $(E_h\phi)(0) = 0$ while $(T_h\phi)(0) = \phi(h) > 0$ for every $h \neq 0$.

Example 11.5.8. For the analytic function $f = \exp$ with $R = \infty$, (11.5.7) reads $\sum h^k e^x/k! = e^x e^h$ and Theorem 11.5.7 gives $e^{x+h} = e^x e^h$, the law of exponents. The function e^x equals its own derivative (Theorem 7.3.18), and E_h acts on it as multiplication by the number e^h.

Worked Example 11.5.2

Apply Theorem 11.5.7 to sin and derive the addition formula $\sin(x+h) = \sin x \cos h + \cos x \sin h$.

Since the sine is the sum of its Maclaurin series with $R = \infty$ (Theorem 11.3.2), (11.5.7) converges to $\sin(x+h)$ for all real x and h. Its derivatives cycle through $\sin, \cos, -\sin, -\cos$.

Since the series converges absolutely, Theorem 11.5.2 permits grouping the terms of even and odd order and gives

$$\sin(x+h) = \sin x \sum_{m\geq 0} \frac{(-1)^m h^{2m}}{(2m)!} + \cos x \sum_{m\geq 0} \frac{(-1)^m h^{2m+1}}{(2m+1)!} = \sin x \cos h + \cos x \sin h.$$

Study guide

Before reordering a sum indexed by two integers, check absolute convergence of the double series. The non-example above shows that the two orders of summation can disagree without it. To re-expand a power series about a new centre b, compute $f^{(k)}(b)$ by termwise differentiation and divide by $k!$. The guaranteed radius is the distance from b to the edge of the original interval.

Two errors are common. The first is applying $E_h = T_h$ to a smooth function that is not analytic. The second is taking $|h|$ past the distance from x to the edge of the interval of convergence where the series (11.5.7) may diverge.

Problems

11.5.1 Re-expand $1/(1-x)$ about $b = -\frac{1}{2}$. Find the radius of the new series and compare it with the radius $R - |b-a|$ of Theorem 11.5.5. ⋆⋆ Hint · Answer · Solution

11.5.2 Compute $(E_h f)(0)$ for $f(x) = \ln(1+x)$ and $|h| < 1$ and check Theorem 11.5.7. ⋆ Hint · Answer · Solution

11.5.3 Prove that $\displaystyle\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{1}{m^n} = 1$. ⋆⋆ Hint · Answer · Solution

11.5.4 Use Theorem 11.5.7 to prove $e^x e^y = e^{x+y}$ for all real x and y and compare with Problem 11.2.8. ⋆ Hint · Answer · Solution

11.5.5 Let $f = \sum c_n x^n$ have radius $R > 0$ and vanish on some interval $(\alpha, \beta) \subseteq (-R, R)$. Prove that $f = 0$ on $(-R, R)$. ⋆⋆ Hint · Answer · Solution

11.5.6 For the double sequence of the non-example after Theorem 11.5.2, write out rows 0 to 3 and columns 0 to 2 and show that the partial sums $\sum_{m\leq N}\sum_{n\leq N} a_{mn}$ over squares converge to 1 while the iterated sum by columns is 0. Which hypothesis of the theorem fails and where does its proof break? ⋆⋆ Hint · Answer · Solution

11.5.7 Let $g(x) = 1/(1+x^2)$ and $b \in \mathbb{R}$. Prove that the Taylor series of g at b has radius $\sqrt{1+b^2}$. ⋆⋆⋆ Hint · Answer · Solution

Appendix A

Proofs

This appendix gives complete proofs of the results that the main text states with a proof sketch or an informal argument. Each section names the result it proves and the places where the main text uses it before proving it from the definitions and theorems of the main text.

A.1 Length of a Curve and its Invariance

This section defines the length of a parametrised curve as the supremum of the lengths of its inscribed polygons and proves the facts about it that the main text uses. For the graph of a function whose derivative is continuous on a closed interval the length equals the integral Eq. (8.6.1) of Definition 8.6.2 (Theorem A.1.6). The derivation after Definition 8.6.2 is therefore a theorem. Rotations about the origin and reflections in lines through the origin preserve the length of every curve (Theorem A.1.9). A rotation preserves the direction of travel round the unit circle and a reflection reverses it (Corollary A.1.11). The quarter circle has length $\pi/2$ equal to the improper integral $\int_0^1 \mathrm{d}y/\sqrt{1-y^2}$ (Proposition A.1.12). From these facts Theorem A.1.14 constructs the point $P(\theta)$ of Definition 4.5.2 for every real θ. Every arc of the circle of given length and direction from $(1,0)$ ends at that point, whatever its parametrisation (Corollary A.1.22). Radian measure (Definition 4.5.2) and the remark after it, the proofs of Theorems 4.5.7 and 4.5.9 and the discussion of Theorem 8.6.6 use these facts. Points and displacements of the plane are written in bold, $\mathbf{v} = (v_1, v_2)$, with the length $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$ of Definition 3.5.2. The linear algebra of the plane used here (rotations, reflections, lengths, isometries and determinants) is proved in Sections 3.3 and 3.5.

Definition A.1.1 (Length of a curve)**.** A *curve* is a map $\gamma : [\alpha, \beta] \to \mathbb{R}^2$ with $\alpha < \beta$ and continuous coordinates x and y, written $\gamma(t) = (x(t), y(t))$. For a partition $P : \alpha = t_0 < t_1 < \cdots < t_n = \beta$ the *inscribed polygon* has length

$$\Lambda(\gamma, P) = \sum_{k=1}^{n} \bigl|\gamma(t_k) - \gamma(t_{k-1})\bigr|. \tag{A.1.1}$$

The *length* $\ell(\gamma)$ of γ is the supremum of $\Lambda(\gamma, P)$ over all partitions P when these numbers are bounded above (Definition 2.4.3). Otherwise $\ell(\gamma) = \infty$.

Lemma A.1.2 (Triangle inequality in the plane)**.** *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$ *and* $|\mathbf{u}| \le |u_1| + |u_2|$.

Proof. The first inequality is Theorem 3.5.5 with $n = 2$. The second is the inequality $\|\mathbf{u}\|_2 \le \|\mathbf{u}\|_1$ of Proposition 3.6.7. ∎

Lemma A.1.3 (Properties of length)**.** *Let* $\gamma : [\alpha, \beta] \to \mathbb{R}^2$ *be a curve.*

1. If the partition P' *contains every point of* P*, then* $\Lambda(\gamma, P) \le \Lambda(\gamma, P')$.

2. *For $\alpha < c < \beta$, $\ell(\gamma) = \ell(\gamma|_{[\alpha,c]}) + \ell(\gamma|_{[c,\beta]})$, where a sum with an infinite term is infinite.*
3. *If ϕ is a continuous bijection of an interval $[\alpha', \beta']$ onto $[\alpha, \beta]$ that is strictly increasing or strictly decreasing, then $\gamma \circ \phi$ is a curve with $\ell(\gamma \circ \phi) = \ell(\gamma)$.*
4. $|\gamma(\beta) - \gamma(\alpha)| \le \ell(\gamma)$.

Proof. (1) Adding one point s with $t_{k-1} < s < t_k$ replaces the term $|\gamma(t_k) - \gamma(t_{k-1})|$ by $|\gamma(s) - \gamma(t_{k-1})| + |\gamma(t_k) - \gamma(s)|$. By Lemma A.1.2 the new sum is at least the old term. Adding the points of P' one at a time proves (1).

(2) Write γ_1 and γ_2 for the two restrictions. Given a partition P of $[\alpha, \beta]$, the partition $P \cup \{c\}$ consists of a partition P_1 of $[\alpha, c]$ followed by a partition P_2 of $[c, \beta]$. By (1),

$$\Lambda(\gamma, P) \le \Lambda(\gamma, P \cup \{c\}) = \Lambda(\gamma_1, P_1) + \Lambda(\gamma_2, P_2) \le \ell(\gamma_1) + \ell(\gamma_2). \tag{A.1.2}$$

Hence (A.1.2) gives $\ell(\gamma) \le \ell(\gamma_1) + \ell(\gamma_2)$. Conversely any partitions P_1 of $[\alpha, c]$ and P_2 of $[c, \beta]$ together form a partition P of $[\alpha, \beta]$ with $\Lambda(\gamma_1, P_1) + \Lambda(\gamma_2, P_2) = \Lambda(\gamma, P) \le \ell(\gamma)$. Taking the supremum over P_1 and then over P_2 gives $\ell(\gamma_1) + \ell(\gamma_2) \le \ell(\gamma)$.

(3) The coordinates of $\gamma \circ \phi$ are continuous by Theorem 5.3.7. Because ϕ and its inverse are strictly monotone, ϕ sends the partitions of $[\alpha', \beta']$ one to one onto the partitions of $[\alpha, \beta]$ and reverses their order when ϕ decreases. Corresponding inscribed polygons have the same segments and therefore the same length.

(4) This is $\Lambda(\gamma, P)$ for the partition $P : \alpha < \beta$. ∎

Lemma A.1.4 (Scaling of length)**.** *Let γ be a curve and $\rho > 0$. For every partition P, $\Lambda(\rho\gamma, P) = \rho\,\Lambda(\gamma, P)$ and $\ell(\rho\gamma) = \rho\,\ell(\gamma)$. In particular an angle of $\theta > 0$ radians cuts off an arc of length $\rho\theta$ on the circle of radius ρ about the origin: for every real α the curve $t \mapsto \rho P(t)$, $\alpha \le t \le \alpha + \theta$, has length $\rho\theta$.*

Proof. The Euclidean length is a norm (Proposition 3.6.7). Hence $|\rho\gamma(t_k) - \rho\gamma(t_{k-1})| = \rho|\gamma(t_k) - \gamma(t_{k-1})|$ for each segment, and summing gives the first identity. The set of the numbers $\Lambda(\rho\gamma, P)$ is ρ times the set of the numbers $\Lambda(\gamma, P)$. One is bounded above exactly when the other is, and then Proposition 2.4.14 gives $\ell(\rho\gamma) = \rho\,\ell(\gamma)$. By Lemma A.1.17, $\ell(P|_{[\alpha,\alpha+\theta]}) = \theta$. With the first part this gives the last statement. The proof of Lemma A.1.17 does not use this lemma. ∎

Lemma A.1.5 (Length of curves with monotone coordinates)**.** *If each coordinate of the curve $\gamma = (x, y)$ is monotone on $[\alpha, \beta]$, then $\ell(\gamma) \le |x(\beta) - x(\alpha)| + |y(\beta) - y(\alpha)|$.*

Proof. Let P be a partition and write $\Delta x_k = x(t_k) - x(t_{k-1})$ and $\Delta y_k = y(t_k) - y(t_{k-1})$. By Lemma A.1.2 each segment has length at most $|\Delta x_k| + |\Delta y_k|$. Because x is monotone, the numbers Δx_k all have one sign and $\sum_k |\Delta x_k| = \left|\sum_k \Delta x_k\right| = |x(\beta) - x(\alpha)|$. The same is true of y. Hence every $\Lambda(\gamma, P)$ is at most the stated bound. ∎

Theorem A.1.6 (Length of a graph)**.** *Let f be differentiable on $[a, b]$ (Definition 6.1.13) with f' continuous on $[a, b]$ and let $\gamma(t) = (t, f(t))$ for $a \le t \le b$. Then*

$$\ell(\gamma) = L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,\mathrm{d}x. \tag{A.1.3}$$

For every $\varepsilon > 0$ there is also a $\delta > 0$ such that $|\Lambda(\gamma, P) - L| < \varepsilon$ for every partition P of mesh less than δ.

Proof. The function f is continuous on $[a,b]$ by Theorem 6.1.11, applied with one-sided limits at a and b. The integrand $g=\sqrt{1+(f')^2}$ is continuous on $[a,b]$ by Theorem 5.3.6 and the continuity of the square root (Corollary 5.4.13). The integral L therefore exists by Theorem 8.2.13. Let $\varepsilon>0$ and choose δ as in Definition 8.2.4 for g and ε.

Let P have mesh less than δ. On each $[t_{k-1},t_k]$ the Mean Value Theorem (Theorem 7.1.9) gives $c_k\in(t_{k-1},t_k)$ with $f(t_k)-f(t_{k-1})=f'(c_k)\,\Delta t_k$. The segment over $[t_{k-1},t_k]$ therefore has length

$$\sqrt{\Delta t_k^2+f'(c_k)^2\Delta t_k^2}=g(c_k)\,\Delta t_k.$$

The number $\Lambda(\gamma,P)$ is the Riemann sum of g with these tags and lies within ε of L. This proves the second statement.

Let P be any partition. Adding the points $a+j(b-a)/N$, $0<j<N$, with $N>(b-a)/\delta$ gives a partition P' of mesh less than δ that contains P. By Lemma A.1.3(1) and the second statement, $\Lambda(\gamma,P)\le\Lambda(\gamma,P')<L+\varepsilon$. The polygon lengths are therefore bounded above with supremum $\ell(\gamma)\le L+\varepsilon$. The partition P' has mesh less than δ and gives $\ell(\gamma)\ge\Lambda(\gamma,P')>L-\varepsilon$. Since $\varepsilon>0$ was arbitrary, $\ell(\gamma)=L$. ∎

The theorem fails to apply wherever f' is unbounded, at an end of the interval or inside it. For $f(x)=(\sqrt[3]{x})^2$ on $[-1,1]$ the derivative is unbounded near the interior point 0 (Example 8.6.3). The theorem applies on every closed interval that avoids 0. For $g(y)=\sqrt{1-y^2}$ on $[0,1]$ the derivative is unbounded near the end 1. Proposition A.1.12 treats that curve through a limit.

Definition A.1.7 (Rotations and reflections)**.** Let c and s be real numbers with $c^2+s^2=1$ and write $\mathbf{u}=(c,s)$. The *rotation* $R_{c,s}$ and the *reflection* $F_{c,s}$ in the line through the origin and $\mathbf{u}$ are the linear maps of Definitions 3.3.21 and 3.3.22. By Theorem 3.5.10 they are

$$R_{c,s}(x,y)=(cx-sy,\ sx+cy), \tag{A.1.4}$$

$$F_{c,s}(\mathbf{v})=2(\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}-\mathbf{v}=((c^2-s^2)x+2csy,\ 2csx+(s^2-c^2)y), \tag{A.1.5}$$

where $\mathbf{v}=(x,y)$ and $\mathbf{v}\cdot\mathbf{u}=cx+sy$.

The definition uses only the numbers c and s and no trigonometric function. The reflections in the x-axis, in the y-axis and in the line $y=x$ are $F_{1,0}(x,y)=(x,-y)$, $F_{0,1}(x,y)=(-x,y)$ and $F_{c,c}(x,y)=(y,x)$ with $c=1/\sqrt{2}$ (Section 3.3). The rotation $R_{c,-s}$ sends the point (c,s) of the unit circle to $(c^2+s^2,-sc+cs)=(1,0)$. Once Definition 4.5.5 is available, $R_{\cos\theta,\sin\theta}$ is the rotation through the angle θ. For $\mathbf{v}$ and $\mathbf{w}$ in the plane $\det(\mathbf{v},\mathbf{w})=v_1w_2-v_2w_1$ is the determinant of Definition 3.5.13.

Lemma A.1.8 (Rotations and reflections preserve distance)**.** *Let T be $R_{c,s}$ or $F_{c,s}$. Then for all $\mathbf{v}$, $\mathbf{w}$ in the plane:*

1. *$T(\mathbf{v}+\mathbf{w})=T\mathbf{v}+T\mathbf{w}$, $|T\mathbf{v}|=|\mathbf{v}|$ and $|T\mathbf{v}-T\mathbf{w}|=|\mathbf{v}-\mathbf{w}|$;*
2. *$\det(T\mathbf{v},T\mathbf{w})=\det(\mathbf{v},\mathbf{w})$ for $T=R_{c,s}$ and $\det(T\mathbf{v},T\mathbf{w})=-\det(\mathbf{v},\mathbf{w})$ for $T=F_{c,s}$;*
3. *T maps the unit circle onto itself.*

Proof. (1) The maps are linear (Definitions 3.3.21 and 3.3.22) and preserve lengths and distances by Theorem 3.5.10. (2) is Corollary 3.5.16. (3) is the last statement of Theorem 3.5.10. ∎

Theorem A.1.9 (Invariance of length)**.** *Let T be a rotation $R_{c,s}$ or a reflection $F_{c,s}$ and let γ be a curve. Then $T\circ\gamma$ is a curve and $\ell(T\circ\gamma)=\ell(\gamma)$.*

Proof. The coordinates of $T \circ \gamma$ are linear combinations of those of γ and are continuous by Theorem 5.3.6. For every partition P, Lemma A.1.8(1) gives

$$\Lambda(T \circ \gamma, P) = \sum_{k=1}^{n} \big|T\gamma(t_k) - T\gamma(t_{k-1})\big| = \sum_{k=1}^{n} \big|\gamma(t_k) - \gamma(t_{k-1})\big| = \Lambda(\gamma, P). \tag{A.1.6}$$

By (A.1.6) the two curves have the same set of polygon lengths and therefore the same supremum. ∎

Definition A.1.10 (Direction of travel)**.** A curve γ with values on the unit circle *travels anticlockwise* if there is a $\delta > 0$ such that $\det(\gamma(t), \gamma(u)) > 0$ whenever $\alpha \le t < u \le \beta$ and $u - t < \delta$. It *travels clockwise* if instead $\det(\gamma(t), \gamma(u)) < 0$ under the same conditions.

Corollary A.1.11 (Direction under rotations and reflections)**.** *Let γ be a curve on the unit circle that travels anticlockwise. Then $R_{c,s} \circ \gamma$ travels anticlockwise and $F_{c,s} \circ \gamma$ travels clockwise. Both have length $\ell(\gamma)$. The same statement with the two directions exchanged is true for a curve that travels clockwise.*

Proof. Both image curves lie on the unit circle by Lemma A.1.8(3). With the same δ as for γ, Lemma A.1.8(2) gives $\det(R\gamma(t), R\gamma(u)) = \det(\gamma(t), \gamma(u)) > 0$ and $\det(F\gamma(t), F\gamma(u)) = -\det(\gamma(t), \gamma(u)) < 0$. The lengths agree by Theorem A.1.9. ∎

Proposition A.1.12 (Arc length on the quarter circle)**.** *Let $\mathbf{q}(y) = (\sqrt{1-y^2},\, y)$ for $0 \le y \le 1$. Then $\mathbf{q}$ is a curve on the unit circle from $(1,0)$ to $(0,1)$ that travels anticlockwise. For $0 < s < 1$ the length of $\mathbf{q}|_{[0,s]}$ is $\Theta(s) = \int_0^s \mathrm{d}y/\sqrt{1-y^2}$, as in Theorem 8.6.6. The length of $\mathbf{q}$ is finite and*

$$\ell(\mathbf{q}) = \lim_{s \to 1^-} \Theta(s) = \sup_{0 \le s < 1} \Theta(s) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{\pi}{2}, \tag{A.1.7}$$

an improper integral (Section 9.8). The upper semicircle $\mathbf{u}$, the curve on $[0,2]$ equal to $\mathbf{q}(y)$ for $0 \le y \le 1$ and to $(-\sqrt{1-(2-y)^2},\, 2-y)$ for $1 \le y \le 2$, has length π. The curve $\mathbf{C}$ on $[0,4]$ equal to $\mathbf{u}(y)$ for $0 \le y \le 2$ and to $F_{1,0}(\mathbf{u}(4-y))$ for $2 \le y \le 4$ runs once round the unit circle from $(1,0)$ and has length 2π.

The length $\ell(\mathbf{C})$ is the circumference of the unit circle. Definition 4.5.2 defines π as half this circumference. The proof below finds $\ell(\mathbf{C}) = 4\,\ell(\mathbf{q})$ and $\ell(\mathbf{q}) = \sup\Theta$ without using the number π. The value $\pi/2$ in (A.1.7) then follows from the definition. Theorem 8.6.6(2) cites this proposition for $\sup\Theta = \pi/2$. The proof uses only part (1) of that theorem.

Proof. The coordinates of $\mathbf{q}$ are continuous (Corollary 5.4.13) and $\mathbf{q}(y)$ lies on the unit circle. For $0 \le t < u \le 1$,

$$\det(\mathbf{q}(t), \mathbf{q}(u)) = u\sqrt{1-t^2} - t\sqrt{1-u^2}.$$

If $t = 0$ this is $u > 0$. If $t > 0$ both products are nonnegative and the difference of their squares is $u^2(1-t^2) - t^2(1-u^2) = u^2 - t^2 > 0$. The first product is therefore the larger. Hence $\mathbf{q}$ travels anticlockwise with any δ.

1. Length of $\mathbf{q}$. Fix s with $0 < s < 1$. The reflection $F_{c,c}(x,y) = (y,x)$ with $c = 1/\sqrt{2}$ maps $\mathbf{q}|_{[0,s]}$ to the graph $y \mapsto (y, g(y))$ of $g(y) = \sqrt{1-y^2}$ over $[0,s]$. On $[0,s]$ the function g is differentiable with continuous derivative $g'(y) = -y/\sqrt{1-y^2}$ by Worked Example 6.1.1 and the

chain rule (Theorem 6.3.4). This derivative satisfies $1+g'(y)^2 = 1/(1-y^2)$. By Theorems A.1.6 and A.1.9,

$$\ell(\mathbf{q}|_{[0,s]}) = \int_0^s \sqrt{1+g'(y)^2}\,\mathrm{d}y = \Theta(s). \tag{A.1.8}$$

Because g is not differentiable at 1, Theorem A.1.6 does not apply on $[0,1]$. On $[s,1]$ the first coordinate of $\mathbf{q}$ decreases and the second increases. By Lemma A.1.3(2) and Lemma A.1.5,

$$\ell(\mathbf{q}) = \Theta(s) + \ell(\mathbf{q}|_{[s,1]}), \qquad 0 \le \ell(\mathbf{q}|_{[s,1]}) \le \sqrt{1-s^2} + (1-s). \tag{A.1.9}$$

In particular $\ell(\mathbf{q})$ is finite. The bound $\sqrt{1-s^2}+(1-s)$ tends to 0 as $s \to 1^-$ by the continuity of the square root at 0 (Corollary 5.4.13). Hence $\Theta(s) \to \ell(\mathbf{q})$. Because Θ is increasing and bounded (Theorem 8.6.6(1)), Proposition 5.2.23 identifies this limit with the supremum of Θ on $[0,1)$. The limit of $\Theta(s)$ is by definition the improper integral.

2. Semicircle and circle. On $[1,2]$ the second piece of $\mathbf{u}$ is $y \mapsto F_{0,1}(\mathbf{q}(2-y))$. Its length is $\ell(\mathbf{q})$ by Lemma A.1.3(3) and Theorem A.1.9. Both pieces equal $(0,1)$ at $y=1$. The curve $\mathbf{u}$ therefore has length $2\,\ell(\mathbf{q})$ by Lemma A.1.3(2). It ends at $\mathbf{u}(2) = (-1,0)$ and $F_{1,0}$ fixes this point. The two pieces of $\mathbf{C}$ therefore agree at $y=2$. The second piece is $F_{1,0}\circ\mathbf{u}$ reparametrised by $y \mapsto 4-y$ and has length $2\,\ell(\mathbf{q})$ by the same two results. Hence $\ell(\mathbf{C}) = 4\,\ell(\mathbf{q})$. The pieces $\mathbf{q}$, $F_{0,1}\circ\mathbf{q}$ and their images under $F_{1,0}$ lie in the four closed quadrants and cover the circle. By Definition 4.5.2, $\pi = \frac{1}{2}\ell(\mathbf{C}) = 2\,\ell(\mathbf{q})$. This proves $\ell(\mathbf{q}) = \pi/2$ and $\ell(\mathbf{u}) = \pi$. ∎

A.1.13 Arc-length parametrisation of the unit circle

Definition 4.5.2 names the point $P(\theta)$ reached by travelling a distance θ round the unit circle. The construction below defines $P(\theta)$ for every real θ from the quarter circle $\mathbf{q}$ and two reflections and proves that it has the properties the definition asks for. Extend Θ to $[0,1]$ by $\Theta(1) = \pi/2$. By Proposition A.1.12, $\Theta(b) = \ell(\mathbf{q}|_{[0,b]})$ for $0 < b \le 1$. The extended function is continuous on $[0,1)$ by Theorem 8.6.6(1) and at 1 because $\Theta(s) \to \pi/2$ as $s \to 1^-$. It is strictly increasing because Θ is strictly increasing on $[0,1)$ with supremum $\pi/2$ and therefore stays below $\pi/2$ there. Since $\Theta(0) = 0$, the Intermediate Value Theorem (Theorem 5.4.2) makes it a bijection from $[0,1]$ onto $[0,\pi/2]$. Let $\sigma : [0,\pi/2] \to [0,1]$ be its inverse, continuous and strictly increasing by Theorem 5.4.9, with $\sigma(0) = 0$ and $\sigma(\pi/2) = 1$. For $0 \le a < b \le 1$, Lemma A.1.3(2) gives

$$\ell(\mathbf{q}|_{[a,b]}) = \Theta(b) - \Theta(a), \tag{A.1.10}$$

directly when $a = 0$ and from $\Theta(b) = \Theta(a) + \ell(\mathbf{q}|_{[a,b]})$ when $a > 0$.

Theorem A.1.14 (Arc-length parametrisation of the unit circle)**.** *Define P on $[0,2\pi]$ by*

$$P(\theta) = \begin{cases} (\sqrt{1-\sigma(\theta)^2},\ \sigma(\theta)) = \mathbf{q}(\sigma(\theta)), & 0 \le \theta \le \pi/2, \\ F_{0,1}(P(\pi-\theta)), & \pi/2 \le \theta \le \pi, \\ F_{1,0}(P(2\pi-\theta)), & \pi \le \theta \le 2\pi, \end{cases} \tag{A.1.11}$$

and for every real θ let $P(\theta) = P(\theta - 2\pi k)$, where k is the integer with $2\pi k \le \theta < 2\pi(k+1)$ (Lemma 2.4.11). Then:

1. *P is well defined: the formulas agree at $\pi/2$ and at π with $P(2\pi) = P(0) = (1,0)$. Hence $P(\theta+2\pi) = P(\theta)$ for every real θ;*
2. *P is continuous on $\mathbb{R}$;*

3. *the restriction of P to $[0,2\pi)$ is a bijection onto the unit circle, and P maps $(0,\pi)$ into the open upper half circle $\{y>0\}$;*

4. *for $0\le\alpha<\beta\le 2\pi$ the curve $P|_{[\alpha,\beta]}$ has length $\beta-\alpha$. The curve $P|_{[0,2\pi]}$ travels anticlockwise. In particular, for $0<\theta<2\pi$ the arc $P|_{[0,\theta]}$ from $P(0)=(1,0)$ to $P(\theta)$ travels anticlockwise and has length θ;*

5. *for $0<\theta<2\pi$ the curve $t\mapsto P(-t)$, $0\le t\le\theta$, from $(1,0)$ to $P(-\theta)$ travels clockwise and has length θ;*

6. *for $0\le\theta<\pi/2$, $P(\theta)=(\sqrt{1-s^2},s)$ with s the unique number in $[0,1)$ satisfying $\Theta(s)=\theta$.*

Proof. The second formula of (A.1.11) uses P on $[0,\pi/2]$ and the third uses P on $[0,\pi]$. The cases therefore define P in turn. Write $P=(x,y)$.

(1) At $\theta=\pi/2$ the first formula of (A.1.11) gives $\mathbf{q}(1)=(0,1)$ and the second gives $F_{0,1}(0,1)=(0,1)$. At $\theta=\pi$ the second formula gives $F_{0,1}(P(0))=(-1,0)$ and the third gives $F_{1,0}(-1,0)=(-1,0)$. At $\theta=2\pi$ the third formula gives $F_{1,0}(P(0))=(1,0)=P(0)$. The rule for real θ therefore agrees with the formulas on all of $[0,2\pi]$. Replacing θ by $\theta+2\pi$ replaces k by $k+1$ and leaves $\theta-2\pi k$ unchanged.

(2) On $[0,\pi/2]$ the coordinates $\sqrt{1-\sigma^2}$ and σ are continuous (Corollary 5.4.13 and Theorem 5.3.6). On $[\pi/2,\pi]$ and on $[\pi,2\pi]$ the coordinates of P are continuous because they are those of P on an earlier interval composed with $\theta\mapsto\pi-\theta$ or $\theta\mapsto 2\pi-\theta$ and with a sign change. At $\pi/2$ and π the one-sided limits both equal the common value of (1). By Theorem 5.2.11, P is continuous on $[0,2\pi]$. On $(2\pi k,2\pi(k+1))$, $P(\theta)=P(\theta-2\pi k)$ is continuous. At $\theta=2\pi k$ the limit from the right is $P(0)$ and the limit from the left is $P(2\pi)=P(0)$. Hence P is continuous at every real number.

(3) Every point of the unit circle lies in exactly one of the sets

$$A_1=\{x>0,\ y\ge 0\},\quad A_2=\{x\le 0,\ y>0\},\quad A_3=\{x<0,\ y\le 0\},\quad A_4=\{x\ge 0,\ y<0\},$$

as the cases $y>0$, $y=0$ and $y<0$ show. The points of A_1 are $(\sqrt{1-v^2},v)$ with $0\le v<1$, each for exactly one v. Since σ maps $[0,\pi/2)$ bijectively onto $[0,1)$, P maps $[0,\pi/2)$ bijectively onto A_1. For $\pi/2\le\theta<\pi$ the number $v=\sigma(\pi-\theta)$ runs bijectively through $(0,1]$ and $P(\theta)=(-\sqrt{1-v^2},v)$. These are the points of A_2, each once. The same argument with $v\in[0,1)$ shows that P maps $(\pi/2,\pi]$ bijectively onto $\{x<0,\ y\ge 0\}$. Likewise P maps $(0,\pi/2]$ bijectively onto $\{x\ge 0,\ y>0\}$. Applying $F_{1,0}$ and $\theta\mapsto 2\pi-\theta$, P maps $[\pi,3\pi/2)$ bijectively onto A_3 and $[3\pi/2,2\pi)$ bijectively onto A_4. The four intervals partition $[0,2\pi)$ and the four sets partition the circle. For $0<\theta<\pi/2$ the second coordinate of $P(\theta)$ is $\sigma(\theta)>0$. For $\pi/2\le\theta<\pi$ the point $P(\theta)$ lies in A_2, where $y>0$. Hence P maps $(0,\pi)$ into $\{y>0\}$.

(4) The length and the direction are shown in turn.

1. Length. If $[\alpha,\beta]\subseteq[0,\pi/2]$, then $P|_{[\alpha,\beta]}=\mathbf{q}\circ\sigma$ with σ a continuous increasing bijection of $[\alpha,\beta]$ onto $[\sigma(\alpha),\sigma(\beta)]$. By Lemma A.1.3(3) and Eq. (A.1.10) the length is $\Theta(\sigma(\beta))-\Theta(\sigma(\alpha))=\beta-\alpha$. If $[\alpha,\beta]\subseteq[\pi/2,\pi]$, then $P|_{[\alpha,\beta]}$ is $F_{0,1}$ applied to $P|_{[\pi-\beta,\pi-\alpha]}$ reparametrised by $\theta\mapsto\pi-\theta$. By Theorem A.1.9 and Lemma A.1.3(3) its length is $(\pi-\alpha)-(\pi-\beta)=\beta-\alpha$. Splitting at $\pi/2$ with Lemma A.1.3(2) gives the length $\beta-\alpha$ for every $[\alpha,\beta]\subseteq[0,\pi]$. The same reflection argument with $F_{1,0}$ and $\theta\mapsto 2\pi-\theta$ extends it to $[\alpha,\beta]\subseteq[\pi,2\pi]$. Splitting at π extends it to every $[\alpha,\beta]\subseteq[0,2\pi]$.

2. Direction. We show that $\det(P(t), P(u)) > 0$ whenever $0 \le t < u \le 2\pi$ and $u - t < \pi/2$. On $[0, \pi]$ the coordinate y is nonnegative and x is strictly decreasing from 1 to -1: $\sqrt{1-\sigma(\theta)^2}$ decreases on $[0, \pi/2]$ and $-\sqrt{1-\sigma(\pi-\theta)^2}$ decreases on $[\pi/2, \pi]$. Hence $y = \sqrt{1-x^2}$ there.

- If $t, u \in [0, \pi]$, put $x_1 = x(t) > x_2 = x(u)$. Then $\det = x_1\sqrt{1-x_2^2} - x_2\sqrt{1-x_1^2}$. If $x_2 \ge 0$ both products are nonnegative and the difference of their squares is $x_1^2 - x_2^2 > 0$. If $x_1 \le 0$ the same computation with $|x_2| > |x_1|$ applies to the two terms $|x_2|\sqrt{1-x_1^2}$ and $|x_1|\sqrt{1-x_2^2}$. If $x_1 > 0 > x_2$ both terms of $x_1\sqrt{1-x_2^2} + |x_2|\sqrt{1-x_1^2}$ are nonnegative and both vanish only for $x_1 = 1$ and $x_2 = -1$. That case is $t = 0$ and $u = \pi$, excluded by $u - t < \pi/2$.
- If $t, u \in [\pi, 2\pi]$, then by Lemma A.1.8(2) $\det(P(t), P(u)) = -\det(P(2\pi - t), P(2\pi - u)) = \det(P(2\pi - u), P(2\pi - t)) > 0$ by the first case.
- If $t < \pi < u$, then $\pi/2 < t < \pi < u < 3\pi/2$. With $v = \sigma(\pi - t) \in (0, 1)$ the second formula gives $P(t) = (-\sqrt{1-v^2}, v)$ with $x(t) < 0 < y(t)$. The point $P(2\pi - u)$ has the same signs because $\pi/2 < 2\pi - u < \pi$. Its reflection $P(u)$ has $x(u) < 0$ and $y(u) < 0$. Both terms $x(t)y(u)$ and $-y(t)x(u)$ of the determinant are positive.

(5) For $0 \le t \le \theta$ the definition gives $P(-t) = P(2\pi - t)$. The third formula gives $P(2\pi - t) = F_{1,0}(P(t))$ for $t \le \pi$. For $t \ge \pi$ it gives $P(t) = F_{1,0}(P(2\pi - t))$. Since $F_{1,0}$ undoes itself, the same identity follows on applying it. The curve is therefore $F_{1,0} \circ P|_{[0,\theta]}$. By Corollary A.1.11 and (4) it travels clockwise with length θ.

(6) Here $\sigma(\theta)$ is the $s \in [0, 1)$ with $\Theta(s) = \theta$. ∎

Lemma A.1.15 (Symmetries of the parametrisation)**.** *For every real θ,*

$$P(-\theta) = F_{1,0}(P(\theta)), \qquad P(\pi - \theta) = F_{0,1}(P(\theta)), \qquad P(\tfrac{\pi}{2} - \theta) = F_{c,c}(P(\theta)), \tag{A.1.12}$$

where $c = 1/\sqrt{2}$ and $F_{c,c}(x, y) = (y, x)$.

Proof. By periodicity it suffices to take $\theta \in [0, 2\pi)$. The three reflections $F_{1,0}(x, y) = (x, -y)$, $F_{0,1}(x, y) = (-x, y)$ and $F_{c,c}(x, y) = (y, x)$ each undo themselves. Direct substitution gives $F_{1,0}F_{0,1} = F_{0,1}F_{1,0}$, $F_{c,c}F_{0,1} = F_{1,0}F_{c,c}$ and $F_{c,c}F_{1,0} = F_{0,1}F_{c,c}$.

(i) First identity. $P(-\theta) = P(2\pi - \theta)$. For $\theta \in [0, \pi]$ the third formula of (A.1.11) gives $P(2\pi - \theta) = F_{1,0}(P(\theta))$. For $\theta \in [\pi, 2\pi)$ the third formula gives $P(\theta) = F_{1,0}(P(2\pi - \theta))$. Applying $F_{1,0}$ to both sides gives the claim.

(ii) Second identity. For $\theta \in [0, \pi]$ it is the second formula of (A.1.11), read in either direction. For $\theta \in [\pi, 2\pi)$, the first identity and the case just proved give

$$F_{0,1}P(\theta) = F_{0,1}F_{1,0}P(2\pi - \theta) = F_{1,0}F_{0,1}P(2\pi - \theta) = F_{1,0}P(\theta - \pi) = P(\pi - \theta).$$

(iii) Third identity. Take first $\theta \in [0, \pi/2]$ and put $s = \sigma(\theta)$ and $c_0 = \sqrt{1-s^2}$. Then $P(\theta) = (c_0, s)$ and $F_{c,c}P(\theta) = (s, c_0) = \mathbf{q}(c_0)$. The claim is $\sigma(\pi/2 - \theta) = c_0$, that is, $\Theta(c_0) = \pi/2 - \theta$. If $s = 1$ then $c_0 = 0$ and both sides are 0. Otherwise, by Eq. (A.1.10), $\ell(\mathbf{q}|_{[s,1]}) = \pi/2 - \theta$. The function $\psi(v) = \sqrt{1-v^2}$ is a continuous decreasing bijection of $[0, c_0]$ onto $[s, 1]$ that satisfies

$$F_{c,c}(\mathbf{q}(\psi(v))) = (\psi(v), \sqrt{1-\psi(v)^2}) = (\sqrt{1-v^2}, v) = \mathbf{q}(v).$$

By Theorem A.1.9 and Lemma A.1.3(3), $\Theta(c_0) = \ell(\mathbf{q}|_{[0,c_0]}) = \ell(\mathbf{q}|_{[s,1]}) = \pi/2 - \theta$.

For $\theta \in [\pi/2, \pi]$, with $\pi - \theta \in [0, \pi/2]$,

$$F_{c,c}P(\theta) = F_{c,c}F_{0,1}P(\pi - \theta) = F_{1,0}F_{c,c}P(\pi - \theta) = F_{1,0}P(\theta - \tfrac{\pi}{2}) = P(\tfrac{\pi}{2} - \theta).$$

For $\theta \in [\pi, 2\pi)$, with $2\pi - \theta \in (0, \pi]$,

$$F_{c,c}P(\theta) = F_{c,c}F_{1,0}P(2\pi - \theta) = F_{0,1}F_{c,c}P(2\pi - \theta) = F_{0,1}P(\theta - \tfrac{3\pi}{2}) = P(\tfrac{5\pi}{2} - \theta) = P(\tfrac{\pi}{2} - \theta),$$

using the second identity and periodicity. ∎

A.1.16 Rotations and the parametrisation

The quarter turn $J = R_{0,1}$, $J(x, y) = (-y, x)$, equals $F_{0,1}F_{c,c}$ with $c = 1/\sqrt{2}$ by direct substitution. By Lemma A.1.15, $JP(\theta) = F_{0,1}P(\frac{\pi}{2} - \theta) = P(\theta + \frac{\pi}{2})$ for every real θ. Since J^4 is the identity, induction gives

$$P(\theta + \tfrac{m\pi}{2}) = J^m P(\theta) \qquad \text{for all real } \theta \text{ and all integers } m, \tag{A.1.13}$$

where $J^{-1} = J^3$. By Theorem A.1.14(1) and (3), $P(u) = P(v)$ exactly when $u - v$ is an integer multiple of 2π: reduce u and v to $[0, 2\pi)$ by subtracting multiples of 2π and use injectivity there.

Lemma A.1.17 (Parametrisation on the whole line)**.** *For all real $\alpha < \beta$ the curve $P|_{[\alpha,\beta]}$ has length $\beta - \alpha$. If $t < u$ and $u - t < \pi/2$, then $\det(P(t), P(u)) > 0$. In particular every restriction of P to a closed interval travels anticlockwise.*

Proof. Choose an integer k with $\alpha - 2\pi k \in [0, 2\pi)$ (Lemma 2.4.11). Since $P(t + 2\pi k) = P(t)$, Lemma A.1.3(3) with $\phi(t) = t + 2\pi k$ shows that $P|_{[\alpha,\beta]}$ and $P|_{[\alpha-2\pi k,\beta-2\pi k]}$ have the same length. We may therefore take $0 \le \alpha < 2\pi$. Split $[\alpha, \beta]$ at the multiples of 2π inside it. Each piece is $[\alpha, \beta]$ itself or a translate by a multiple of 2π of a subinterval of $[0, 2\pi]$. Its length is its width by Theorem A.1.14(4) and the same shift. Adding the pieces by Lemma A.1.3(2) gives $\beta - \alpha$.

For the second statement choose an integer m with $t' = t - m\pi/2 \in [0, \pi/2)$ and put $u' = u - m\pi/2$. Then $0 \le t' < u' < \pi$ and $u' - t' < \pi/2$. By Eq. (A.1.13) and Lemma A.1.8(2), $\det(P(t), P(u)) = \det(J^m P(t'), J^m P(u')) = \det(P(t'), P(u')) > 0$ by Theorem A.1.14(4). ∎

On $[0, \pi]$ the first coordinate of P is continuous and strictly decreasing from 1 to -1 (proof of Theorem A.1.14(4)). Let $\xi : [-1, 1] \to [0, \pi]$ be its inverse, continuous by Theorem 5.4.9. Write H for the open upper half circle, the points (x, y) of the unit circle with $y > 0$.

Lemma A.1.18 (Chart on the upper half circle)**.** *For $X = (x, y)$ in H put $h(X) = \xi(x)$. Then $h(X) \in (0, \pi)$ and $P(h(X)) = X$. For X_1 and X_2 in H with $\det(X_1, X_2) > 0$, $h(X_1) < h(X_2)$.*

Proof. Since $y > 0$, $x \in (-1, 1)$ and $h(X) \in (0, \pi)$. The point $P(\xi(x))$ has first coordinate x. Its second coordinate is nonnegative because P maps $[0, \pi]$ into $y \ge 0$. On the unit circle that second coordinate is $\sqrt{1 - x^2} = y$. Hence $P(h(X)) = X$.

Write $X_i = (x_i, y_i)$. If $x_1 = x_2$ then $X_1 = X_2$ and the determinant is 0. If $x_1 > x_2$ the first case in the proof of Theorem A.1.14(4) gives $\det(X_1, X_2) > 0$ because two points of H are never $(1, 0)$ and $(-1, 0)$. If $x_1 < x_2$ the same case with the points exchanged gives $\det(X_1, X_2) = -\det(X_2, X_1) < 0$. A positive determinant therefore forces $x_1 > x_2$. Since ξ is decreasing, $h(X_1) < h(X_2)$. ∎

Theorem A.1.19 (Uniqueness of arcs of given length)**.** *Let $L > 0$, let a be real and let $Q : [0, L] \to \mathbb{R}^2$ be a curve on the unit circle that travels anticlockwise, with $Q(0) = P(a)$ and $\ell(Q|_{[0,t]}) = t$ for every $t \in (0, L]$. Then $Q(t) = P(a + t)$ for every $t \in [0, L]$. In particular the arc ends at $P(a + L)$, a point determined by its starting point and its length.*

Proof. Let E be the set of $t \in [0, L]$ with $Q(u) = P(a+u)$ for every $u \in [0, t]$ and let $T = \sup E$. The set contains 0. Every $u < T$ lies below some element of E, and continuity of Q and P at T gives $Q(T) = P(a+T)$. Hence $T \in E$. Suppose $T < L$.

Let $B = Q(T) = P(b)$ with $b = a + T$. One of B, JB, J^2B, J^3B lies in H because their second coordinates are y, x, $-y$ and $-x$ for $B = (x, y) \neq (0, 0)$. Fix such an $m \in \{0, 1, 2, 3\}$. By Eq. (A.1.13), $P(b + m\pi/2) = J^m B = P(h(J^m B))$. Hence $b + m\pi/2 = h(J^m B) + 2\pi k$ for an integer k. The second coordinate of $J^m Q(t)$ is continuous in t and positive at T. By Lemma 5.3.8 there is an $\eta > 0$ with $T + \eta \leq L$ and η less than the δ of Definition A.1.10 for Q, such that $J^m Q(t) \in H$ for $T \leq t \leq T + \eta$. For these t put

$$\tau(t) = h(J^m Q(t)) + 2\pi k - \tfrac{m\pi}{2}.$$

The function τ is continuous with $\tau(T) = b$. By Eq. (A.1.13), periodicity and Lemma A.1.18, $P(\tau(t)) = J^{-m} P(h(J^m Q(t))) = Q(t)$.

For $T \leq t < u \leq T + \eta$, Lemma A.1.8(2) gives $\det(J^m Q(t), J^m Q(u)) = \det(Q(t), Q(u)) > 0$. By Lemma A.1.18, $\tau(t) < \tau(u)$. The function τ is therefore a continuous increasing bijection of $[T, t]$ onto $[b, \tau(t)]$ for each $t \in (T, T + \eta]$ with $Q|_{[T,t]} = P \circ \tau$. By Lemma A.1.3(3) and Lemma A.1.17, $\ell(Q|_{[T,t]}) = \tau(t) - b$. By Lemma A.1.3(2) and the hypothesis, the same length is $t - T$, directly when $T = 0$ and as $\ell(Q|_{[0,t]}) - \ell(Q|_{[0,T]})$ when $T > 0$. Hence $\tau(t) = a + t$ and $Q(t) = P(a + t)$ for $T \leq t \leq T + \eta$. Then $T + \eta \in E$, contradicting $T = \sup E$. Therefore $T = L$. ∎

Corollary A.1.20 (Rotations shift the parameter)**.** *Let R be a rotation $R_{c,s}$ and let φ be a real number with $R(1, 0) = P(\varphi)$. Then $R(P(\theta)) = P(\theta + \varphi)$ for every real θ.*

Proof. Let $L > 0$ and $Q(t) = R(P(t))$ for $0 \leq t \leq L$. By Lemma A.1.17 and Corollary A.1.11, Q is a curve on the unit circle that travels anticlockwise with $\ell(Q|_{[0,t]}) = \ell(P|_{[0,t]}) = t$. Also $Q(0) = R(1, 0) = P(\varphi)$. By Theorem A.1.19, $R(P(t)) = P(\varphi + t)$ for $0 \leq t \leq L$. Since L is arbitrary the identity holds for every $\theta \geq 0$. For $\theta < 0$ choose an integer k with $\theta + 2\pi k \geq 0$ and use periodicity on both sides. ∎

Lemma A.1.21 (Arc-length reparametrisation)**.** *Let $Q : [\alpha, \beta] \to \mathbb{R}^2$ be a curve on the unit circle that travels anticlockwise and has finite length $L = \ell(Q)$. Put $s(\alpha) = 0$ and $s(t) = \ell(Q|_{[\alpha,t]})$ for $\alpha < t \leq \beta$. Then $L > 0$ and s is a continuous strictly increasing bijection of $[\alpha, \beta]$ onto $[0, L]$. The curve $R = Q \circ s^{-1}$ on $[0, L]$ lies on the unit circle and travels anticlockwise, with $R(0) = Q(\alpha)$, $R(L) = Q(\beta)$ and $\ell(R|_{[0,u]}) = u$ for every $u \in (0, L]$.*

Proof. Let δ be as in Definition A.1.10 for Q. By Lemma A.1.3(2), $s(u) - s(t) = \ell(Q|_{[t,u]})$ for $\alpha \leq t < u \leq \beta$, and every such length is at most L.

1. Strict increase. Let $\alpha \leq t < u \leq \beta$ and choose u' with $t < u' \leq u$ and $u' - t < \delta$. Then $\det(Q(t), Q(u')) > 0$. Hence $Q(u') \neq Q(t)$ because the determinant of a vector with itself is 0. By Lemma A.1.3(2) and (4),

$$s(u) - s(t) = \ell(Q|_{[t,u]}) \geq \ell(Q|_{[t,u']}) \geq \big|Q(u') - Q(t)\big| > 0.$$

In particular $L = s(\beta) > s(\alpha) = 0$.

2. Continuity from the right. Let $\alpha \leq t_0 < \beta$ and $\varepsilon > 0$. By Lemma 2.4.4 there is a partition P of $[t_0, \beta]$ with $\Lambda(Q|_{[t_0,\beta]}, P) > \ell(Q|_{[t_0,\beta]}) - \varepsilon$. Let p_1 be the point of P after t_0. The coordinates of Q are continuous at t_0. By Lemma A.1.2 there is a t_1 with $t_0 < t_1 < p_1$ and $|Q(t_1) - Q(t_0)| < \varepsilon$. Adding t_1 to P gives a partition P' with $\Lambda(Q|_{[t_0,\beta]}, P') \geq \Lambda(Q|_{[t_0,\beta]}, P)$ by Lemma A.1.3(1). The

first segment of P' has length less than ε and the others form an inscribed polygon of $Q|_{[t_1,\beta]}$. Hence

$$\ell(Q|_{[t_0,\beta]}) - \varepsilon < \varepsilon + \ell(Q|_{[t_1,\beta]}).$$

Since $\ell(Q|_{[t_0,t_1]}) = \ell(Q|_{[t_0,\beta]}) - \ell(Q|_{[t_1,\beta]})$ by Lemma A.1.3(2), this gives $\ell(Q|_{[t_0,t_1]}) < 2\varepsilon$. For $t_0 < t \le t_1$ the first paragraph and Lemma A.1.3(2) give $0 < s(t) - s(t_0) \le \ell(Q|_{[t_0,t_1]}) < 2\varepsilon$.
3. Continuity from the left. Let $\alpha < t_0 \le \beta$. The same argument on $[\alpha, t_0]$, with the added point t_1 below t_0 and above the last point of the partition before t_0, gives $0 < s(t_0) - s(t) < 2\varepsilon$ for $t_1 \le t < t_0$. By Theorem 5.2.11, s is continuous on $[\alpha, \beta]$.
4. Reparametrisation. Since $s(\alpha) = 0$ and $s(\beta) = L$, the Intermediate Value Theorem (Theorem 5.4.2) and strict increase make s a bijection of $[\alpha, \beta]$ onto $[0, L]$. Its inverse is continuous and strictly increasing by Theorem 5.4.9. Hence R is a curve on the unit circle with $R(0) = Q(\alpha)$ and $R(L) = Q(\beta)$. For $0 < u \le L$ the curve $R|_{[0,u]}$ is $Q|_{[\alpha,s^{-1}(u)]}$ composed with s^{-1}. By Lemma A.1.3(3) its length is $s(s^{-1}(u)) = u$. By the Heine–Cantor theorem (Theorem 5.5.10) there is a $\delta' > 0$ with $s^{-1}(w) - s^{-1}(v) < \delta$ whenever $0 \le v < w \le L$ and $w - v < \delta'$. For such v and w, $\det(R(v), R(w)) = \det(Q(s^{-1}(v)), Q(s^{-1}(w))) > 0$. Hence R travels anticlockwise. ∎

Corollary A.1.22 (Endpoint of an arc of given length)**.** *Let $Q : [\alpha, \beta] \to \mathbb{R}^2$ be a curve on the unit circle with finite length L and $Q(\alpha) = P(a)$. If Q travels anticlockwise, then $Q(\beta) = P(a+L)$. If Q travels clockwise, then $Q(\beta) = P(a-L)$. In particular every arc of length $\theta > 0$ from $(1,0)$ ends at $P(\theta)$ when travelled anticlockwise and at $P(-\theta)$ when travelled clockwise, whatever its parametrisation.*

Proof. Let Q travel anticlockwise. Lemma A.1.21 gives $L > 0$ and a curve R on $[0, L]$ that satisfies the hypotheses of Theorem A.1.19 with $R(0) = P(a)$. That theorem gives $Q(\beta) = R(L) = P(a + L)$. Let Q travel clockwise. By Corollary A.1.11 the curve $F_{1,0} \circ Q$ travels anticlockwise and has length L. It starts at $F_{1,0}(P(a)) = P(-a)$ by Lemma A.1.15. By the first case it ends at $P(L-a)$. Applying $F_{1,0}$ and Lemma A.1.15 again gives $Q(\beta) = F_{1,0}(P(L-a)) = P(a - L)$. The last statement is the case $a = 0$ and $L = \theta$. ∎

By Theorem A.1.14(4) and (5), $P(\theta)$ is the point that Definition 4.5.2 describes: the end of the arc of length $|\theta|$ from $(1,0)$ travelled anticlockwise for $\theta > 0$ and clockwise for $\theta < 0$. By Corollary A.1.22 every arc of the same length and direction from $(1,0)$ ends at $P(\theta)$, whatever its parametrisation. With $\cos\theta$ and $\sin\theta$ the coordinates of $P(\theta)$ (Definition 4.5.5), Theorem A.1.14(1) gives part (2) of Theorem 4.5.7 and Lemma A.1.15 gives parts (3) and (4). Corollary A.1.20 gives the rotation step in the proof of Theorem 4.5.9.

A.2 Partial Fraction Decomposition

Section 9.5 integrates rational functions with Theorem 9.5.2 and Section 10.2 sums telescoping series with it. This section proves the theorem. The theorem assumes that the denominator is given in factored form

$$q(x) = a \prod_i (x - r_i)^{m_i} \prod_j (x^2 + b_j x + c_j)^{n_j}, \tag{A.2.1}$$

with $a \neq 0$, distinct real numbers r_i, distinct pairs (b_j, c_j) and $b_j^2 - 4c_j < 0$ for every j. That every real polynomial has such a factorisation follows from the fundamental theorem of algebra. This book does not prove it. We prove that every proper fraction p/q is a sum of the terms Eqs. (9.5.2) and (9.5.3) and that the coefficients A_k, B_k, C_k are unique. The proof follows the

integer arithmetic of Theorems 1.4.14 and 1.4.15, with the division algorithm for polynomials (Theorem 4.2.4) in place of division with remainder.

A polynomial φ *divides* a polynomial g, written $\varphi \mid g$, if $g = \varphi h$ for a polynomial h. Because the degree of a product of nonzero polynomials is the sum of the degrees, a nonzero polynomial divides no nonzero polynomial of smaller degree. Two polynomials are *coprime* if they are not both zero and have no common divisor of positive degree.

Lemma A.2.1 (Bézout's identity for polynomials)**.** *Polynomials f and g are coprime if and only if $uf + vg = 1$ for some polynomials u and v.*

Proof. If $uf + vg = 1$ then f and g are not both zero. A common divisor δ of f and g divides $uf + vg = 1$ and therefore has degree 0.

Conversely, let f and g be coprime. The set S of nonzero polynomials of the form $uf + vg$ contains f or g. By Assumption 1.4.1, applied to the numbers $1 + \deg d$ for $d \in S$, it contains an element $d = u_0 f + v_0 g$ of least degree. Divide f by d (Theorem 4.2.4): $f = Qd + r$ with $r = 0$ or $\deg r < \deg d$. Since $r = (1 - Qu_0)f + (-Qv_0)g$, a nonzero r would be an element of S of smaller degree than d. Hence $r = 0$ and $d \mid f$. In the same way $d \mid g$. Because f and g are coprime, d has degree 0 and is a nonzero constant κ. Then $(u_0/\kappa)f + (v_0/\kappa)g = 1$. ∎

Lemma A.2.2 (Coprime polynomials)**.** *Let φ, ψ and χ be polynomials.*

1. *If φ and ψ are coprime and $\varphi \mid \psi\chi$, then $\varphi \mid \chi$.*
2. *If φ is coprime to ψ and to χ, then φ is coprime to $\psi\chi$.*
3. *If $\varphi_1, \dots, \varphi_N$ are pairwise coprime and $e_1, \dots, e_N \in \mathbb{N}$, then $\varphi_1^{e_1}$ is coprime to $\varphi_2^{e_2} \cdots \varphi_N^{e_N}$.*

Proof. (1) By Lemma A.2.1, $u\varphi + v\psi = 1$. Then $\chi = (u\chi)\varphi + v(\psi\chi)$ with φ dividing both terms.

(2) Multiplying $u\varphi + v\psi = 1$ and $u'\varphi + v'\chi = 1$ gives the identity

$$1 = \varphi(uu'\varphi + uv'\chi + u'v\psi) + (vv')\,\psi\chi \tag{A.2.2}$$

that Lemma A.2.1 requires.

(3) By (2) and induction on e, a polynomial coprime to ψ is coprime to ψ^e. Hence φ_1 is coprime to each $\varphi_i^{e_i}$ with $i \geq 2$. By (2) again it is coprime to their product Π. Applying the same argument to Π and φ_1 shows that Π is coprime to $\varphi_1^{e_1}$. ∎

Lemma A.2.3 (Distinct factors are coprime)**.** *Any two of the factors $x - r_i$ and $x^2 + b_j x + c_j$ of q are coprime.*

Proof. Suppose δ of positive degree divides two of the factors. A divisor δ of $x - r$ of positive degree has degree 1. Because $x - r = \delta h$ forces $\deg h = 0$, it is $\kappa(x - r)$ for a constant $\kappa \neq 0$. If δ divides $x - r$ and $x - r'$, then $\delta = \kappa(x - r) = \kappa'(x - r')$ vanishes at r and at r'. Hence $r = r'$. If δ divides $x - r$ and an irreducible quadratic Q, then $Q(r) = 0$ in contradiction to Theorem 4.2.15. If δ divides two irreducible quadratics $Q \neq Q'$, its degree is 1 or 2. Degree 1 would give Q a real root, again excluded. Degree 2 gives $Q = \kappa\delta$ and $Q' = \kappa'\delta$ for constants κ and κ'. Both have leading coefficient 1. Hence $\kappa = \kappa'$ and $Q = Q'$. This contradicts $Q \neq Q'$. ∎

Lemma A.2.4 (Partial fractions over coprime factors)**.** *Let q_1 and q_2 be coprime polynomials of positive degree and let p be zero or of degree less than $\deg(q_1 q_2)$. There are polynomials p_1 and p_2, each zero or of degree less than $\deg q_1$ and $\deg q_2$ respectively, with $p = p_1 q_2 + p_2 q_1$. Hence*

$$\frac{p(x)}{q_1(x)q_2(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} \tag{A.2.3}$$

wherever $q_1(x)q_2(x) \neq 0$.

Proof. By Lemma A.2.1, $uq_1 + vq_2 = 1$. Divide pv by q_1: $pv = sq_1 + p_1$ with $p_1 = 0$ or $\deg p_1 < \deg q_1$. Then

$$p = p(uq_1 + vq_2) = puq_1 + (sq_1 + p_1)q_2 = p_1q_2 + p_2q_1, \qquad p_2 = pu + sq_2.$$

The polynomial $p_2q_1 = p - p_1q_2$ is zero or of degree less than $\deg q_1 + \deg q_2$ because both p and p_1q_2 are. Hence $p_2 = 0$ or $\deg p_2 < \deg q_2$. Dividing $p = p_1q_2 + p_2q_1$ by q_1q_2 gives (A.2.3). ∎

Lemma A.2.5 (Partial fractions over a power of one factor)**.** *Let φ have degree $d \geq 1$, let $m \in \mathbb{N}$ and let p be zero or of degree less than md. There are polynomials $N_1, \dots, N_m$, each zero or of degree less than d, with*

$$p = \sum_{k=1}^{m} N_k \varphi^{m-k}, \qquad \frac{p}{\varphi^m} = \sum_{k=1}^{m} \frac{N_k}{\varphi^k}. \tag{A.2.4}$$

Proof. By induction on m. For $m = 1$ take $N_1 = p$. Let $m \geq 2$ and divide p by φ: $p = p'\varphi + N_m$ with $N_m = 0$ or $\deg N_m < d$. Then $p'\varphi = p - N_m$ is zero or of degree less than md. Hence p' is zero or of degree less than $(m-1)d$. The inductive hypothesis gives $p' = \sum_{k=1}^{m-1} N_k \varphi^{m-1-k}$. Therefore $p = \sum_{k=1}^{m-1} N_k \varphi^{m-k} + N_m$. Dividing by φ^m gives (A.2.4). ∎

Proof of Theorem 9.5.2. Let $\varphi_1, \dots, \varphi_N$ be the distinct factors of q and $e_1, \dots, e_N$ their multiplicities with $q = a\varphi_1^{e_1} \cdots \varphi_N^{e_N}$. They are pairwise coprime by Lemma A.2.3.

1. Existence. We prove by induction on N that a fraction p/q with p zero or of degree less than $\deg q$ is a sum

$$\frac{p}{q} = \sum_{i=1}^{N} \sum_{k=1}^{e_i} \frac{N_{i,k}}{\varphi_i^k}, \tag{A.2.5}$$

where each $N_{i,k}$ is zero or of degree less than $\deg \varphi_i$. For $N = 1$ apply Lemma A.2.5 to p/a and $\varphi_1^{e_1}$. For $N \geq 2$ the polynomials $q_1 = \varphi_1^{e_1}$ and $q_2 = \varphi_2^{e_2} \cdots \varphi_N^{e_N}$ are coprime by Lemma A.2.2(3). Lemma A.2.4 applied to p/a gives $p/q = p_1/q_1 + p_2/q_2$. Lemma A.2.5 expands p_1/q_1 and the inductive hypothesis expands p_2/q_2. For a linear factor $\varphi_i = x - r$ the numerators $N_{i,k}$ are constants A_k as in Eq. (9.5.2). For a quadratic factor they are $B_kx + C_k$ as in Eq. (9.5.3).

2. Uniqueness. The difference of two decompositions Eq. (A.2.5) of the same fraction is an expression of the same form with numerators $D_{i,k}$. Its value is 0 wherever $q(x) \neq 0$. Suppose some $D_{i,k} \neq 0$. Fix such an i and let k be the largest index with $D_{i,k} \neq 0$. Let $\Pi = \prod_{l \neq i} \varphi_l^{e_l}$ and multiply the expression by $\varphi_i^k \Pi$. Every term with $l \neq i$ becomes a polynomial multiple of φ_i^k. Every term $D_{i,j}/\varphi_i^j$ with $j < k$ becomes a polynomial multiple of φ_i. Hence

$$D_{i,k}\Pi + \varphi_i H = 0$$

for a polynomial H, at every x with $q(x) \neq 0$. The polynomial q has finitely many roots (Corollary 4.2.7). The left side therefore vanishes at infinitely many points and is the zero polynomial by Corollary 4.2.8. Since $\varphi_i \mid D_{i,k}\Pi$ and φ_i is coprime to Π (Lemma A.2.2(3)), Lemma A.2.2(1) gives $\varphi_i \mid D_{i,k}$. This contradicts $0 \leq \deg D_{i,k} < \deg \varphi_i$. Every $D_{i,k}$ is therefore zero and the two decompositions coincide. ∎

A.3 Trapezoidal Rule Error

This section proves Lemma A.3.1, the error of the Trapezoidal Rule of Definition 9.7.2 on a single subinterval. The proof of Theorem 9.7.3 in Section 9.7 sums the formula of the lemma over the n subintervals. Section 9.7 applies the resulting bound in Worked Example 9.7.1 and to choose the number of subintervals.

Lemma A.3.1 (Trapezoidal error on one subinterval)**.** *Let f'' be continuous on $[\alpha, \beta]$ and $h = \beta - \alpha > 0$. There is a $c \in (\alpha, \beta)$ with*

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x = \frac{h}{2}(f(\alpha) + f(\beta)) - \frac{h^3}{12} f''(c). \tag{A.3.1}$$

Proof. For $0 \le t \le h$ let

$$F(t) = \frac{t}{2}(f(\alpha) + f(\alpha + t)) - \int_{\alpha}^{\alpha+t} f(x)\,\mathrm{d}x. \tag{A.3.2}$$

By Theorem 8.3.4, the chain rule and the product rule (Theorems 6.3.1 and 6.3.4), F is differentiable on $[0, h]$ with one-sided derivatives at the ends. Its first two derivatives are

$$F'(t) = \tfrac{1}{2}(f(\alpha) + f(\alpha + t)) + \tfrac{t}{2} f'(\alpha + t) - f(\alpha + t) = \tfrac{1}{2}(f(\alpha) - f(\alpha + t)) + \tfrac{t}{2} f'(\alpha + t),$$
$$F''(t) = -\tfrac{1}{2} f'(\alpha + t) + \tfrac{1}{2} f'(\alpha + t) + \tfrac{t}{2} f''(\alpha + t) = \tfrac{t}{2} f''(\alpha + t).$$

In particular $F(0) = F'(0) = 0$. Let $K = F(h)/h^3$ and $G(t) = F(t) - Kt^3$. Since $G(0) = G(h) = 0$, Rolle's theorem (Theorem 7.1.7) gives $t_1 \in (0, h)$ with $G'(t_1) = 0$. The function $G'(t) = F'(t) - 3Kt^2$ is differentiable on $[0, t_1]$ with $G'(0) = 0 = G'(t_1)$. Rolle's theorem again gives $t_2 \in (0, t_1)$ with

$$0 = G''(t_2) = \tfrac{t_2}{2} f''(\alpha + t_2) - 6Kt_2.$$

Since $t_2 > 0$, $K = f''(c)/12$ with $c = \alpha + t_2 \in (\alpha, \beta)$. Substituting this value of K into $F(h) = Kh^3$ gives (A.3.1). ∎

A.4 Simpson's Rule Error

This section proves Lemma A.4.1, the error of Simpson's Rule of Definition 9.7.6 on a single panel. The proof of Theorem 9.7.7 in Section 9.7 sums the formula of the lemma over the $n/2$ panels. Section 9.7 applies the resulting bound in Worked Example 9.7.2 and to choose the number of subintervals. The method is that of Appendix A.3 with one more application of Rolle's theorem.

Lemma A.4.1 (Simpson error on one panel)**.** *Let $f^{(4)}$ be continuous on $[\mu - h, \mu + h]$ with $h > 0$. There is a $c \in (\mu - h, \mu + h)$ with*

$$\int_{\mu-h}^{\mu+h} f(x)\,\mathrm{d}x = \frac{h}{3}(f(\mu - h) + 4f(\mu) + f(\mu + h)) - \frac{h^5}{90} f^{(4)}(c). \tag{A.4.1}$$

Proof. For $0 \le t \le h$ let

$$F(t) = \frac{t}{3}(f(\mu - t) + 4f(\mu) + f(\mu + t)) - \int_{\mu-t}^{\mu+t} f(x)\,\mathrm{d}x. \tag{A.4.2}$$

With $\Phi(x) = \int_{\mu-h}^{x} f$, the integral is $\Phi(\mu+t) - \Phi(\mu-t)$ with derivative $f(\mu+t) + f(\mu-t)$ by Theorems 6.3.4 and 8.3.4. Differentiating three times,

$$\begin{aligned} F'(t) &= -\tfrac{2}{3}(f(\mu+t) + f(\mu-t)) + \tfrac{4}{3}f(\mu) + \tfrac{t}{3}(f'(\mu+t) - f'(\mu-t)), \\ F''(t) &= -\tfrac{1}{3}(f'(\mu+t) - f'(\mu-t)) + \tfrac{t}{3}(f''(\mu+t) + f''(\mu-t)), \\ F'''(t) &= \tfrac{t}{3}(f'''(\mu+t) - f'''(\mu-t)), \end{aligned}$$

on $[0,h]$, with one-sided derivatives at the ends. In particular $F(0) = F'(0) = F''(0) = 0$. Let $K = F(h)/h^5$ and $G(t) = F(t) - Kt^5$. Rolle's theorem (Theorem 7.1.7) applied to G on $[0,h]$ gives $t_1 \in (0,h)$ with $G'(t_1) = 0$. Since $G'(0) = 0 = G'(t_1)$, Rolle's theorem on $[0,t_1]$ gives $t_2 \in (0,t_1)$ with $G''(t_2) = 0$. Since $G''(0) = 0 = G''(t_2)$, Rolle's theorem on $[0,t_2]$ gives $t_3 \in (0,t_2)$ with

$$0 = G'''(t_3) = \tfrac{t_3}{3}(f'''(\mu+t_3) - f'''(\mu-t_3)) - 60Kt_3^2.$$

The Mean Value Theorem (Theorem 7.1.9) for f''' on $[\mu-t_3, \mu+t_3]$ gives $c \in (\mu-t_3, \mu+t_3)$ with $f'''(\mu+t_3) - f'''(\mu-t_3) = 2t_3 f^{(4)}(c)$. Hence $\frac{2}{3}t_3^2 f^{(4)}(c) = 60Kt_3^2$ and $K = f^{(4)}(c)/90$. Substituting this value of K into $F(h) = Kh^5$ gives (A.4.1). ∎

A.5 Rearrangement of Absolutely Convergent Series

This section proves Theorem 10.6.7: a rearrangement of an absolutely convergent series converges absolutely to the same sum. Section 10.6 states it beside the Riemann rearrangement theorem as the case in which reordering leaves the sum unchanged.

Proof of Theorem 10.6.7. Let $S = \sum a_n$, $A = \sum|a_n|$, and let s_n and t_m be the partial sums of $\sum a_n$ and of $\sum a_{\sigma(m)}$.

For each m let $K = \max\{\sigma(1), \dots, \sigma(m)\}$. The indices $\sigma(1), \dots, \sigma(m)$ are distinct and at most K. Hence $\sum_{j=1}^{m}|a_{\sigma(j)}| \le \sum_{k=1}^{K}|a_k| \le A$. Because the partial sums of $\sum|a_{\sigma(m)}|$ are bounded, that series converges by Theorem 10.3.1.

Let $\varepsilon > 0$. The tail $\sum_{k>N}|a_k|$ equals $A - \sum_{k\le N}|a_k|$ and tends to 0. Choose N with $\sum_{k>N}|a_k| < \varepsilon$. By Corollary 10.2.16 applied to the tail, $|S - s_N| = \left|\sum_{k>N} a_k\right| < \varepsilon$. Because σ is a bijection, each of $1, \dots, N$ equals $\sigma(j)$ for some j. Let M be the largest of these N values of j. For $m \ge M$ the sum t_m contains every term $a_1, \dots, a_N$ once. The remaining terms of t_m are a_k for distinct indices $k > N$. Hence

$$|t_m - s_N| \le \sum_{k>N}|a_k| < \varepsilon \qquad \text{and} \qquad |t_m - S| \le |t_m - s_N| + |s_N - S| < 2\varepsilon. \tag{A.5.1}$$

Therefore $t_m \to S$. ∎

A.6 Riemann Rearrangement Theorem

This section proves Theorem 10.6.8: the terms of a conditionally convergent series can be reordered to converge to any real number, to diverge to $+\infty$ or to $-\infty$, or to oscillate with no limit. Section 10.6 states the theorem and outlines the greedy construction. Section 4.6 cites it in its warning against reordering infinite sums. The proof below makes the construction precise with a sequence of targets.

Lemma A.6.1 (Positive and negative terms of a conditionally convergent series)**.** *Let $\sum a_n$ converge conditionally. Let $\pi_1 < \pi_2 < \cdots$ be the indices n with $a_n \ge 0$ and $\kappa_1 < \kappa_2 < \cdots$ those*

with $a_n < 0$. Both lists are infinite with $\sum_k a_{\pi_k} = +\infty$ and $\sum_k a_{\kappa_k} = -\infty$. The same is true after any finite number of terms is removed from either list.

Proof. Let $p_n = \max(a_n, 0)$ and $q_n = \max(-a_n, 0)$. Then $a_n = p_n - q_n$ and $|a_n| = p_n + q_n$. If $\sum p_n$ and $\sum q_n$ both converged, then $\sum |a_n|$ would converge by Theorem 10.2.10. If exactly one converged, say $\sum q_n$, then $\sum p_n = \sum (a_n + q_n)$ would converge as well. Hence both diverge to $+\infty$ by Theorem 10.3.1. If either list were finite, the series $\sum p_n$ or $\sum q_n$ would have finitely many nonzero terms and would converge. Both lists are therefore infinite. The partial sums of $\sum_k a_{\pi_k}$ are the partial sums of $\sum p_n$ taken at $n = \pi_K$ because $p_n = 0$ for every other index. They therefore tend to $+\infty$. In the same way $\sum_k a_{\kappa_k} = -\sum_k q_{\kappa_k}$ tends to $-\infty$. Remove finitely many terms from either list. Once the number of terms taken passes the position of the last removed term, each partial sum of the shortened list differs from a partial sum of the full list by the fixed sum of the removed terms. The shortened sums therefore tend to the same infinite limit. ∎

Lemma A.6.2 (Rearrangement towards prescribed targets)**.** *Let $\sum a_n$ converge conditionally and let $\{\alpha_j\}$ and $\{\beta_j\}$ be real sequences with $\alpha_j \le \beta_j$ and $\alpha_j \le \beta_{j+1}$ for every j. There is a rearrangement $\sum a_{\sigma(m)}$ with partial sums T_m and indices $m_1 < m_1' < m_2 < m_2' < \cdots$ that have the following properties for every j:*

1. *$T_{m_j} > \beta_j$ and $T_{m_j'} < \alpha_j$;*
2. *$T_m \ge \alpha_j - d_j'$ for $m_j \le m \le m_{j+1}$;*
3. *$T_m \le \beta_{j+1} + d_{j+1}$ for $m_j' \le m \le m_{j+1}'$;*

where $d_j = |a_{\sigma(m_j)}|$ and $d_j' = |a_{\sigma(m_j')}|$. Both $d_j \to 0$ and $d_j' \to 0$.

Proof. Use the lists of Lemma A.6.1 in order and never reuse a term. At stage j first take unused terms of the list $\{a_{\pi_k}\}$, in order and at least one, until the running sum first exceeds β_j. Call the step at which this happens m_j. Then take unused terms of the list $\{a_{\kappa_k}\}$, in order and at least one, until the running sum first falls below α_j. Call this step m_j'. Each stage ends after finitely many steps because the unused terms of each list form a series diverging to $+\infty$ or to $-\infty$ (Lemma A.6.1). Every stage uses at least one term of each list. Each index therefore occurs exactly once and σ is a bijection. The construction gives property (1).

The running sum does not decrease from step m_j' to step m_{j+1} and does not increase from step m_j to step m_j'. The sum $T_{m_j'-1}$ is either $T_{m_j} > \beta_j \ge \alpha_j$ or a sum that has not yet fallen below α_j. In both cases $T_{m_j'-1} \ge \alpha_j$ and $T_{m_j'} \ge \alpha_j - d_j'$. Every sum from step m_j to step m_{j+1} is at least $T_{m_j'}$. Property (2) follows. Likewise $T_{m_{j+1}-1}$ is either $T_{m_j'} < \alpha_j \le \beta_{j+1}$ or a sum that has not yet exceeded β_{j+1}. Hence $T_{m_{j+1}} \le \beta_{j+1} + d_{j+1}$. Every sum from step m_j' to step m_{j+1}' is at most $T_{m_{j+1}}$. Property (3) follows. The indices $\sigma(m_j)$ and $\sigma(m_j')$ are distinct and only finitely many of them lie below any bound. Since $a_n \to 0$ (Theorem 10.2.4), both $d_j \to 0$ and $d_j' \to 0$. ∎

Proof of Theorem 10.6.8. Apply Lemma A.6.2 with the targets below and write

$$e_j = \sup_{i \ge j} \max(d_i, d_i'),$$

so that $e_j \to 0$.

For a real L take $\alpha_j = \beta_j = L$. Every $m \ge m_j$ with $j \ge 2$ lies in some range of (2) with index $i \ge j$ and in some range of (3) with index $i + 1 \ge j$. Hence $L - e_j \le T_m \le L + e_j$ and $T_m \to L$.

For $+\infty$ take $\alpha_j = j-1$ and $\beta_j = j$. By (2), $T_m \geq j-1-e_j$ for every $m \geq m_j$. Hence $T_m \to +\infty$. Applying this case to the conditionally convergent series $\sum(-a_n)$ gives a rearrangement of $\sum a_n$ diverging to $-\infty$.

For oscillation take $\alpha_j = -1$ and $\beta_j = 1$. By (1), $T_{m_j} > 1$ and $T_{m'_j} < -1$ for every j. The partial sums are not eventually within 1 of any number and do not tend to $+\infty$ or to $-\infty$. The rearranged series therefore has no limit, finite or infinite. ∎

A.7 Abel's Theorem

This section proves Theorem 11.1.12: a power series that converges at an endpoint of its interval of convergence converges uniformly up to that endpoint and has a sum continuous there. Sections 11.1 and 11.2 use it to extend the logarithm and arctangent series to $x = 1$. The proof uses summation by parts, the device of Problem 10.6.11, and the completeness of $C([a,b])$ (Theorem 10.1.17).

Proof of Theorem 11.1.12. The substitution $x = a + Rt$ turns the series into $\sum c_n R^n t^n$, a series converging at $t = 1$. Write $b_n = c_n R^n$. It suffices to prove that $\sum b_n t^n$ converges uniformly on $[0,1]$.

Let $B = \sum_{k\geq 0} b_k$ and $B_n = \sum_{k=0}^{n} b_k$. For fixed n the partial sums $B_m - B_n$, $m > n$, of the tail $\sum_{k>n} b_k$ tend to $B - B_n$ by Theorem 5.1.9. The tail therefore converges with sum $r_n = B - B_n$. Since $B_n \to B$, $r_n \to 0$. Also $b_k = r_{k-1} - r_k$ for $k \geq 1$. The quantity $\varepsilon_n = \sup_{k\geq n}|r_k|$ tends to 0. For $m > n \geq 0$ and $t \in [0,1]$, collecting the coefficient of each r_k gives

$$\sum_{k=n+1}^{m} b_k t^k = \sum_{k=n+1}^{m} (r_{k-1} - r_k)t^k = r_n t^{n+1} - r_m t^m - \sum_{k=n+1}^{m-1} r_k (t^k - t^{k+1}). \tag{A.7.1}$$

In (A.7.1) each $t^k - t^{k+1}$ is nonnegative and these differences telescope to $t^{n+1} - t^m$. Hence

$$\Big|\sum_{k=n+1}^{m} b_k t^k\Big| \leq \varepsilon_n (t^{n+1} + t^m + t^{n+1} - t^m) \leq 2\varepsilon_n \tag{A.7.2}$$

for every $t \in [0,1]$. The partial sums of $\sum b_n t^n$ are polynomials. The estimate (A.7.2) states that they form a Cauchy sequence in $C([0,1])$ with the sup norm. By Theorem 10.1.17 they converge uniformly on $[0,1]$ to a continuous function that equals the sum of the series at every $t \in [0,1]$.

Returning to x the series converges uniformly on $[a, a+R]$ to a function continuous there. With Theorem 11.1.11 on $(a-R, a+R)$ the sum is continuous on $(a-R, a+R]$ and its value at $a+R$ is the limit from the left. For the endpoint $a-R$, apply the result to the series $\sum c_n(-1)^n(y-a)^n$ with value at $y = a+R$ equal to the value of the original series at $a-R$. ∎

The converse of Theorem 11.1.12 fails. The geometric series $\sum(-1)^n x^n$ has sum $1/(1+x)$ tending to $\frac{1}{2}$ as $x \to 1^-$ whereas the series $\sum(-1)^n$ at $x = 1$ diverges. A limit of the sum at an endpoint does not make the series converge there.

A.8 Termwise Differentiation

Part (3) of Theorem 11.1.2 assumes that every term has a continuous derivative and integrates the derivatives in its proof. This section removes the continuity assumption with the Mean Value Theorem in place of the integral.

Theorem A.8.1 (Termwise differentiation)**.** *Let each u_n be differentiable on $[a,b]$, one-sided at the endpoints. Suppose that $\sum u_n(x_0)$ converges for some $x_0 \in [a,b]$ and that $\sum u_n'$ converges uniformly on $[a,b]$ to g. Then $\sum u_n$ converges uniformly on $[a,b]$ to a function f with $f' = g$ there.*

Proof. Write $s_N = \sum_{n \le N} u_n$. Since $s_K - s_N$ is differentiable for $K > N$, the Mean Value Theorem (Theorem 7.1.9) gives a point ξ between x_0 and x for each $x \neq x_0$ with

$$(s_K - s_N)(x) - (s_K - s_N)(x_0) = (x - x_0)(s_K' - s_N')(\xi). \tag{A.8.1}$$

Hence $\|s_K - s_N\|_\infty \le |s_K(x_0) - s_N(x_0)| + (b-a)\|s_K' - s_N'\|_\infty$ with the second norm a supremum over $[a,b]$. Since both terms tend to 0, $\{s_N\}$ is a Cauchy sequence of continuous functions in the sup norm and converges uniformly to a continuous f (Theorem 10.1.17).

Fix $x_1 \in [a,b]$ and for $x \neq x_1$ put

$$\phi_N(x) = \frac{s_N(x) - s_N(x_1)}{x - x_1}, \qquad \phi(x) = \frac{f(x) - f(x_1)}{x - x_1}.$$

Let $\varepsilon > 0$ and choose N_0 with $\|s_N' - g\|_\infty < \varepsilon/2$ for $N > N_0$. The Mean Value Theorem applied to $s_K - s_N$ between x_1 and x gives $|\phi_K(x) - \phi_N(x)| = |(s_K' - s_N')(\xi)| < \varepsilon$ for $K > N > N_0$. Letting $K \to \infty$, Theorem 5.1.10 gives $|\phi(x) - \phi_N(x)| \le \varepsilon$ for every $x \neq x_1$. Fix one $N > N_0$ and choose $\delta > 0$ with $|\phi_N(x) - s_N'(x_1)| < \varepsilon$ for $0 < |x - x_1| < \delta$. For such x,

$$|\phi(x) - g(x_1)| \le |\phi(x) - \phi_N(x)| + |\phi_N(x) - s_N'(x_1)| + |s_N'(x_1) - g(x_1)| < 3\varepsilon. \tag{A.8.2}$$

Hence $\phi(x) \to g(x_1)$ as $x \to x_1$ and $f'(x_1) = g(x_1)$. ∎

Appendix B

Hints

1.1 Statements and Connectives

1.1.1 Use the implication and biconditional columns of Table 1.1.1. p. 13

1.1.2 The negation of $P \implies Q$ is $P \wedge \neg Q$. p. 13

1.1.3 Compute the column $P \implies Q$ first. p. 13

1.1.4 An implication is false in only one situation. For the second part try $P = \text{F}$ and $R = \text{F}$. p. 13

1.1.5 "P is necessary for Q" means $Q \implies P$. p. 13

1.1.6 Use the second part of Theorem 1.1.10 with $\neg P$ in place of P. p. 13

1.2 Quantifiers

1.2.1 For an existential claim name the element. For a universal claim look for a counterexample. p. 17

1.2.2 Flip each quantifier and negate the final inequality. p. 18

1.2.3 The negation of an implication is a conjunction. p. 18

1.2.4 Solve $1/n < \varepsilon$ for the integer n. p. 18

1.2.5 Take $\varepsilon = \frac{1}{4}$ and use indices $n \geq 4$. p. 18

1.2.6 Apply the negation procedure, then De Morgan's law to the conjunction. p. 18

1.3 Methods of Proof

1.3.1 Write the integers as $2k + 1$ and $2l + 1$. p. 22

1.3.2 Prove the contrapositive. p. 22

1.3.3 One checked case refutes each statement. p. 22

1.3.4 Expand $(a - b)^2$. For the second part apply the first to $\sqrt{a}$ and $\sqrt{b}$. p. 22

1.3.5 Use Lemma 1.3.3 and Problem 1.3.1. p. 23

1.3.6 Factor $x^3 - 8$ and complete the square in the quadratic factor. p. 23

1.4 Mathematical Induction

1.4.1 In the inductive step, show that $\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$. p. 29

1.4.2 Factor $(k+1)^2/4$ from $\frac{k^2(k+1)^2}{4} + (k+1)^3$. p. 29

1.4.3 Write $4^{k+1} - 1 = 4(4^k - 1) + 3$. p. 29

1.4.4 Induct from the base $n_0 = 0$ using Corollary 1.4.3. p. 29

1.4.5 Check the base case. p. 29

1.4.6 Check $12, 13, 14, 15$ directly and, for $k + 1 \geq 16$, apply the inductive hypothesis to $k + 1 - 4$. p. 29

1.4.7 Divide 120 by 84 and 84 by the remainder. Then substitute back. p. 29

1.4.8 If $p \nmid a$, show that p and a are coprime and apply Theorem 1.4.15. p. 29

1.4.9 Write $c = ak$ and apply Theorem 1.4.15 to $b \mid ak$. p. 29

1.5 Sets and Functions

1.5.1 Use Definition 1.5.4. Each element of A pairs with each element of B. p. 36

1.5.2 For injectivity test $f(x_1) = f(x_2)$. For surjectivity try to solve $f(x) = y$ in the domain. p. 36

1.5.3 Complete the square: $f(x) = (x - 2)^2 - 1$. p. 36

1.5.4 Solve $y(x - 1) = x + 2$ for x. p. 36

1.5.5 The domain of $f \circ g$ is the set of x with $g(x) \geq 0$. p. 36

1.5.6 Apply g to $f(x_1) = f(x_2)$. For the example let A have one element. p. 36

1.5.7 For the example use x^2 with $S = [-1, 0]$ and $T = [0, 1]$. p. 37

2.1 Number Systems

2.1.1 Multiply by 100 and subtract. p. 43

2.1.2 Divide 5 by 12 and watch the remainders. p. 43

2.1.3 A set that contains two numbers but misses one between them is not an interval. p. 43

2.1.4 Show first that if 3 divides m^2 then 3 divides m. p. 43

2.1.5 Square $s + t$ and use Proposition 2.1.4 with $p = 6$. p. 43

2.1.6 Compare with Worked Example 2.1.2. p. 43

2.1.7 Write $n = 2^a 5^b$ and multiply numerator and denominator to reach a power of 10. p. 43

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2.2.2 Look at the inverses. p. 47

2.2.3 Apply (F3) twice. p. 47

2.2.4 Use Proposition 2.2.5(vi) and multiply once more. p. 47

2.2.5 Try negative numbers. For the proof compare ac with bc and bc with bd. p. 47

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2.3 Inequalities and Absolute Value

2.3.1 Factor the quadratic. p. 53

2.3.2 $|y| > 7$ means $y < -7$ or $y > 7$. p. 53

2.3.3 The factor $(x - 1)^2$ is never negative and does not change sign at 1. p. 53

2.3.4 Do not multiply by x. Write the inequality as $\dfrac{x^2 - 1}{x} > 0$. p. 53

2.3.5 $|2x - 6| = 2|x - 3|$. p. 53

2.3.6 Use Corollary 2.3.9 to show $|x| > \frac{1}{2}$. p. 53

2.3.7 In the inductive step multiply by $1 + x$ and discard a nonnegative x^3 term. p. 53

2.3.8 Apply Theorem 2.3.12 to the side lengths a and b. p. 53

2.4 Completeness

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2.4.2 For the infimum use Theorem 2.4.10. p. 61

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2.4.4 Show that $\sup T$ is an upper bound of S. p. 61

2.4.5 Each b is an upper bound of A. p. 61

2.4.6 Use the uniqueness in Theorem 2.4.8 and Proposition 2.2.5(vi). p. 61

2.4.7 Show $\frac{1}{3} - t_k = \dfrac{1}{3 \cdot 10^k}$. p. 61

3.1 Vectors in the Line and the Plane

3.1.1 Work component by component. For $\mathbf{w}$ add $-\mathbf{u}$ to both sides. p. 67

3.1.2 Move from P by one third and by two thirds of $\overrightarrow{PQ}$. p. 67

3.1.3 The midpoint of BC is $\frac{1}{2}(B + C)$. p. 67

3.1.4 Compare the first components and then the second. p. 67

3.1.5 Write $M = \frac{1}{2}(A + B)$ and $N = \frac{1}{2}(A + C)$. p. 67

3.1.6 Use Proposition 3.1.6. A vector does not change when the origin moves. p. 67

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3.3.4 Use Theorem 3.3.17. p. 82

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3.4.6 Use Theorem 3.4.5(3). p. 87

3.4.7 Generalise Worked Example 3.4.2: find p_i with $E_{a_j} p_i = \delta_{ij}$. p. 87

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3.5.5 Expand both squares with Proposition 3.5.3(5). p. 94

3.5.6 Apply Theorem 3.5.4 in $\mathbb{R}^3$ to (a, b, c) and $(1, 1, 1)$. p. 94

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3.6.3 Bound each component of $A\mathbf{x}$. Test $(1, 1)$ for the ∞-norm and $\mathbf{e}_1$ for the 1-norm. p. 103

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4.5.4 Use $\cos 2x = 2\cos^2 x - 1$ and solve a quadratic in $\cos x$. p. 136

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9.3.4 Use Theorem 9.3.15. p. 303

9.3.5 Write $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x$ and apply Eq. (9.3.2). p. 303

9.3.6 On $[0, \frac{\pi}{2}]$, $0 \le \sin x \le 1$. Divide the inequalities by W_{2m+1} and use $W_{2m-1}/W_{2m+1} = \frac{2m+1}{2m}$. p. 304

9.3.7 Apply Eq. (9.3.3). For an integer $k \neq 0$, $\int_{-\pi}^{\pi} \cos kx\,dx = 0$. p. 304

9.3.8 Write $\tan^n x = \tan^{n-2} x\,(\sec^2 x - 1)$. p. 304

9.3.9 F is undefined at π and is an antiderivative only on $(-\pi, \pi)$ and on $(\pi, 3\pi)$ separately. Split the integral at π and use one-sided limits of F. p. 304

9.3.10 With $\theta = \arctan t$, $\sin^2\theta = t^2/(1+t^2)$ and $d\theta = dt/(1+t^2)$. p. 304

9.4 Trigonometric Substitutions

9.4.1 Use $x = 2\sec\theta$ on $(0, \frac{\pi}{2})$ and $\tan^2\theta = \sec^2\theta - 1$. p. 309

9.4.2 By Eq. (9.4.4), $x^2 + 4x + 13 = (x+2)^2 + 9$. p. 309

9.4.3 Use $x = 2\sin\theta$ and Theorem 9.3.7 for $\sin^2\theta\cos^2\theta$. p. 309

9.4.4 Use the second interval of the secant row of Table 9.4.1, on which $\sqrt{x^2-25} = -5\tan\theta$. p. 309

9.4.5 Use the chain rule and Theorem 6.4.10, and $\sqrt{x^2} = |x|$. p. 309

9.4.6 On $[\frac{\pi}{2}, \pi]$ the cosine is negative or zero, and $\sqrt{1-\sin^2\theta} = -\cos\theta$. p. 309

9.4.7 With $x = a\tan\theta$ the pullback is $a^2\sec^3\theta\,d\theta$. Use Worked Example 9.3.4. p. 309

9.4.8 The radicand is $1 + (f')^2 = 1 + 4x^2$. Substitute $u = 2x$ and use Problem 9.4.7 with $a = 1$. p. 309

9.4.9 Follow Example 9.4.3. Compare the hypotheses of Theorem 8.5.2 with those of Theorem 9.1.4. p. 309

9.4.10 The graph of $y = \sqrt{4-x^2}$ is the upper half of a circle of radius 2. Use Example 9.4.3. p. 310

9.4.11 The average value is $\dfrac{1}{b-a}\int_a^b f$ (Definition 8.2.19). Reuse the area from Problem 9.4.10. p. 310

9.5 Integration of Rational Functions by Partial Fractions

9.5.1 Divide first by Eq. (9.5.1). p. 315

9.5.2 Use the form $\dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$. Cover-up gives A and C. p. 315

9.5.3 On this interval $x - 3 < 0 < x + 3$. p. 315

9.5.4 Compute the discriminant of $x^2 - x - 20$. p. 315

9.5.5 Use $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1}$. p. 315

9.5.6 Neither root of the denominator lies in $[0, 1]$. p. 315

9.5.7 As in the proof of Proposition 9.5.4, $g(x) = \sum_{k=1}^{m} A_k(x-r)^{m-k} + (x-r)^m R(x)$ with R differentiable near r. p. 315

9.5.8 Write the decomposition of $f(-x)$ and use uniqueness in Theorem 9.5.2. p. 315

9.5.9 By the product rule $q'(r_i) = \prod_{j \neq i}(r_i - r_j)$. For the sum, compare the coefficients of x^{n-1} in the cleared identity. p. 316

9.5.10 Use Eq. (9.5.6) with $k = 1$ and $n = 2$. p. 316

9.6 Integral Tables and Computer Algebra Systems

9.6.1 Use $n = 2$ and $a = -1$, then $n = 1$. p. 319

9.6.2 Subtract one answer from the other and use Eq. (9.3.1). p. 320

9.6.3 A real logarithm needs a positive argument. For $y < 0$ the complex logarithm is $\ln|y| + i\pi$. Compare with Problem 9.5.3. p. 320

9.6.4 Use the chain rule and $\sqrt{a^2} = |a|$. p. 320

9.6.5 The arctangent term of F has period 2π. p. 320

9.6.6 Evaluate the difference at $x = 0$ and at a point of $(1, \infty)$, or let $x \to \infty$. p. 320

9.6.7 Apply the Mean Value Theorem (Theorem 7.1.9) on the interval between c and $c + h$. p. 320

9.6.8 Take $n = 3$ and $a = 1$. p. 320

9.7 Numerical Integration

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9.7.2 Compute $f^{(4)}$ and use Eq. (9.7.7). p. 326

9.7.3 Use $h = \frac{1}{4}$. The integral is $e - 1$. p. 326

9.7.4 $f^{(4)}(x) = 24/x^5$ is decreasing on $[1, 2]$. p. 326

9.7.5 Follow Lemma A.3.1 with $F(t) = 2t\, f(\mu) - \int_{\mu-t}^{\mu+t} f$ and $G(t) = F(t) - Kt^3$, $K = F(h)/h^3$. p. 326

9.7.6 Write T_m and T_{2m} with the points of the finer partition and compare the weights. p. 326

9.7.7 $f''(x) = -\frac{1}{4}x^{-3/2}$ on $(0, 1]$. p. 326

9.7.8 T_n is the average of the left and right Riemann sums on the partition of mesh h. p. 326

9.8 Improper Integrals

9.8.1 The integrand is undefined at $x = 1$. Split there. p. 332

9.8.2 Split at 0 by Eq. (9.8.3). p. 332

9.8.3 Compare with $1/x^2$ by Theorem 9.8.10. p. 332

9.8.4 Theorem 9.8.9 gives convergence only from an upper bound. p. 332

9.8.5 Show $\ln(x + 1) < x$ for $x > 0$. p. 332

9.8.6 Integrate by parts on $[1, b]$ and use $\lim_{b\to\infty}(\ln b)/b^2 = 0$ (Theorem 5.2.26). p. 332

9.8.7 For $n \geq 1$ integrate by parts on $[0, b]$ with $u = x^n$ and use $b^n e^{-b} \to 0$ (Theorem 5.2.26). p. 332

9.8.8 For convergence follow the proof of Proposition 9.8.14. For $p \leq 1$ use $|\sin x|/x^p \geq |\sin x|/x$ on $[1, \infty)$. p. 332

9.8.9 Compare with Eq. (8.2.6). Its constant is the length of the interval. p. 332

9.8.10 For $\int g$ use Problem 9.8.8. For $\int f$ use the divergence of $\int_1^\infty \sin^2 x/x$ from the proof of Proposition 9.8.14. p. 332

9.8.11 Place at each $n \geq 2$ a triangle of height 1 and base $[n - 2^{-n}, n + 2^{-n}]$. Use Lemma 9.8.8 with the finite geometric sum Eq. (4.6.11). p. 333

10.1 Sequences and Complete Spaces

10.1.1 Use Eqs. (10.1.1) to (10.1.3). p. 341

10.1.2 Treat the even and the odd terms separately. p. 341

10.1.3 Use the fifth limit of Theorem 10.1.4 and the product law. p. 341

10.1.4 For $n \geq 4$, split off the first four factors of $2^n/n!$. p. 341

10.1.5 Show by induction that $0 < a_n < 2$ and that the sequence increases. Take limits in the recursion only after that. p. 341

10.1.6 Compare $\sup_{k\ge n}(a_k+b_k)$ with $\sup_{k\ge n}a_k+\sup_{k\ge n}b_k$. For strictness try $a_n=(-1)^n$. p. 342

10.1.7 If $\mathbf{v}_n\to\mathbf{w}$, compare consecutive terms with the triangle inequality. p. 342

10.1.8 $f_n'(x)=n(1-nx)e^{-nx}$ vanishes only at $x=1/n$. p. 342

10.1.9 A continuous $f\ge 0$ with $f(x_0)>0$ is at least $f(x_0)/2$ on an interval around x_0. p. 342

10.1.10 Copy the proofs of Theorem 5.1.7 and of the second half of Theorem 5.1.28. p. 342

10.1.11 Rationalise: $\sqrt{x^2+1/n}-|x|=\dfrac{1/n}{\sqrt{x^2+1/n}+|x|}$. p. 342

10.1.12 For $m>n$ the difference g_n-g_m vanishes outside $[\frac12,\frac12+\frac1n]$ and has size at most 1. For the second part, show that a limit g would have $\int_0^{1/2}|g|=0$ and $\int_{1/2+\delta}^{1}|g-1|=0$ for every $\delta>0$. p. 342

10.2 Infinite Series

10.2.1 The first term is the $n=0$ term. p. 347

10.2.2 Partial fractions, then (10.2.4). p. 347

10.2.3 Show that each piece converges before applying Theorem 10.2.10. p. 348

10.2.4 The ratio is $r=(x-2)/3$. p. 348

10.2.5 Take $M_n=2^{-n}$ in Theorem 10.2.18. The tail of the series is itself geometric. p. 348

10.2.6 Each of the n terms is at least $1/(2n)$. p. 348

10.2.7 The partial sums of the tail are $\mathbf{s}_m-\mathbf{s}_n$ for $m>n$. p. 348

10.2.8 Fix an integer $K>2|z|$ and use the bound in the proof of Theorem 10.1.4. p. 348

10.2.9 The sum $1/(1-x)$ is unbounded on $(-1,1)$ while each s_N is bounded there. p. 348

10.2.10 Write 1 once, then $-\frac12$ twice, then $\frac13$ three times, then $-\frac14$ four times, and so on. p. 348

10.2.11 Given a Cauchy sequence $\{\mathbf{x}_n\}$, choose $n_1<n_2<\cdots$ with $\|\mathbf{x}_m-\mathbf{x}_n\|<2^{-k}$ for $m,n\ge n_k$. Sum the differences $\mathbf{x}_{n_{k+1}}-\mathbf{x}_{n_k}$ and use Problem 10.1.10. p. 348

10.3 Integral Test

10.3.1 Write each general term as $1/n^p$. p. 352

10.3.2 $f(x)=1/(x^2+1)$ has an arctangent antiderivative (Proposition 9.1.7). Bound the sum with (10.3.3). p. 352

10.3.3 For $p>0$ substitute $u=\ln x$ as in Worked Example 10.3.2. For $p\le 0$ compare with $\sum 1/(n\ln n)$ or with $\sum 1/n$. p. 352

10.3.4 Apply Eq. (10.3.5) with $f(x)=x^{-3}$ and $n=4$. p. 352

10.3.5 By Eq. (10.3.5), $R_n\le\int_n^\infty x^{-2}\,dx$. p. 352

10.3.6 $d_n-d_{n+1}=\int_n^{n+1}dx/x-\frac{1}{n+1}$. For the bounds use (10.3.3) with $f(x)=1/x$. p. 352

10.3.7 If $f(x_0)<0$ for some x_0, then $f\le f(x_0)$ on $[x_0,\infty)$. p. 352

10.3.8 Add to e^{-x} a tent of height $1/n$ and half-width $1/n^2$ centred at each integer $n\ge 2$. p. 352

10.3.9 Let $n\to\infty$ in both inequalities. p. 352

10.3.10 Group the terms in blocks $2^k\le n<2^{k+1}$ and bound each block by its first and its last term. p. 352

10.4 Comparison Tests

10.4.1 Adding n to the denominator makes the fraction smaller than $1/n^2$. p. 356

10.4.2 Compare with the harmonic series from below. p. 356

10.4.3 Compare with $1/n$. p. 356

10.4.4 By the growth ranking, $\ln n\le n^{1/4}$ for large n. p. 356

10.4.5 Keep the dominant term of each line and limit-compare with what remains. p. 356

10.4.6 $2|ab|\le a^2+b^2$. p. 356

10.4.7 For the converse, convergence of the second series forces $a_n/(1+a_n) \to 0$ and hence $a_n \to 0$. p. 356

10.4.8 Try $a_n = 1/n^2$ and $a'_n = 1/(n \ln n)$ for $n \geq 2$. p. 356

10.4.9 Let $r_n = \sum_{k>n} b_k$ and $a_n = b_n/\sqrt{r_{n-1}}$. Show that $a_n \leq 2(\sqrt{r_{n-1}} - \sqrt{r_n})$. p. 356

10.5 Ratio and Root Tests

10.5.1 Cancel the factorials with (4.6.14). p. 359

10.5.2 The polynomial factor contributes a ratio tending to 1. p. 360

10.5.3 $3^{n^2} = (3^n)^n$. p. 360

10.5.4 Treat even and odd n separately. For the sum, split into the two geometric series of odd and even indices. p. 360

10.5.5 Apply Lemma 10.5.4 to $a_n = n!/n^n$ and use Theorem 5.1.20. p. 360

10.5.6 The ratio is $x(1+1/n)^{-n}$. At $x = e$ use the fact that $(1+1/n)^n$ increases to e (Theorem 5.1.20). p. 360

10.5.7 Compute ρ and then compare with $\sum 1/n^2$. p. 360

10.5.8 For the second part look at the harmonic series. p. 360

10.5.9 Bound s_n above and identify the limit along the even indices. p. 360

10.5.10 Rewrite the hypothesis as $na_n - (n+1)a_{n+1} \geq (p-1)a_{n+1}$ and sum from N to m. p. 360

10.6 Alternating Series and Rearrangement

10.6.1 Theorem 10.6.2 and then limit comparison of $\sum 1/(2n+1)$ with $\sum 1/n$. p. 364

10.6.2 Test $\sum |a_n|$ with the ratio test. p. 364

10.6.3 Treat $p > 1$, $0 < p \leq 1$ and $p \leq 0$ separately. p. 364

10.6.4 Solve $1/(n+1)! \leq 0.001$. p. 364

10.6.5 Take 1; then negative terms $-\frac{1}{2}, -\frac{1}{4}, \ldots$ until the running sum is negative; then the next positive term. p. 364

10.6.6 $0 \leq a_n^{\pm} \leq |a_n|$, $|a_n| = a_n^+ + a_n^-$ and $a_n^+ = a_n + a_n^-$. p. 364

10.6.7 If only finitely many terms were negative, compare the series with $\sum |a_n|$. p. 364

10.6.8 Use Problem 10.6.6 and Theorem 10.2.10. p. 365

10.6.9 Eventually $a_n \leq 1$. For the example use $a_n = (-1)^n/\sqrt{n}$ and Theorem 10.6.2. p. 365

10.6.10 Each block sums to 0. The cubes of a block sum to $6c_k^3$. p. 365

10.6.11 Summation by parts: $\sum_{k=n+1}^{m} a_k b_k = a_m B_m - a_{n+1} B_n - \sum_{k=n+1}^{m-1} (a_{k+1} - a_k) B_k$. Then apply Theorem 10.2.12. p. 365

11.1 Power Series

11.1.1 Use (11.1.7). p. 372

11.1.2 Compute $|c_n|^{1/n}$ along even and odd n. p. 372

11.1.3 $R = 1$. Test $x = 1$ and $x = -1$ separately. p. 372

11.1.4 Multiply the derived series by x before setting $x = \frac{1}{2}$. p. 372

11.1.5 Substitute $t = x^4$ in (11.1.14), integrate with (11.1.10), and bound the error with Theorem 10.6.3. p. 372

11.1.6 Apply Theorem 11.1.5 to the series in $t = x^2$ or compute the upper limit for the coefficients of x^m directly. p. 372

11.1.7 The series $\sum c_n x^n - \sum c_n (-x)^n$ represents $f(x) - f(-x) = 0$. Differentiate it term by term k times at 0. p. 372

11.1.8 Apply the M-test to both series on an interval $[-b, b]$ and then part (3) of Theorem 11.1.2. p. 372

11.1.9 M-test with $M_n = 1/n^2$. The derived series is $\sum x^{n-1}/n$. p. 372

11.1.10 Integrate by parts. The maximum of f_n is at $x = 1/n$. p. 372

11.2 Taylor and Maclaurin Series

11.2.1 Substitute $-2x$ into (11.2.4). p. 376

11.2.2 Substitute x^2 into the cosine row and multiply by x. p. 376

11.2.3 Add terms until the first omitted term is below 10^{-5}. p. 376

11.2.4 Use (11.1.13) with the exponential and sine rows. p. 376

11.2.5 Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ from Eq. (9.3.2) and the cosine row at $2x$. p. 376

11.2.6 Differentiate $f(-x) = -f(x)$ k times with the chain rule. p. 376

11.2.7 Use Theorem 11.1.15 and Theorem 7.3.18 with $\lambda = 1$. p. 376

11.2.8 The nth Cauchy coefficient is $\sum_{k=0}^{n} \frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}$. p. 376

11.2.9 Substitute $-x^2$ into the geometric row. At $x = 1$ look at the terms. p. 376

11.2.10 Compare $x = 2$ with the interval of (11.1.15). p. 376

11.3 Convergence of Taylor Series

11.3.1 $M_3 = 1$. p. 380

11.3.2 $1 - x^2/2$ is P_2 for cos at 0. p. 380

11.3.3 $|R_n(1)| \le 1/(n+1)!$. p. 380

11.3.4 The error after n terms is at most $0.2^{n+1}/(n+1)$. The $(n+1)$st derivative of $\ln(1+x)$ is $(-1)^n n!/(1+x)^{n+1}$. p. 380

11.3.5 The first bound is $2^{n+1}/(n+1)!$. The term sizes are $2^{2k+1}/(2k+1)!$. p. 380

11.3.6 The series is the real part of $\sum \exp(i\theta)^n/n!$, and $\exp(i\theta)^n = \exp(in\theta)$ by Proposition 11.3.8. p. 381

11.3.7 Between a and x, every derivative of e^t is at most $e^{|a|+|x-a|}$. p. 381

11.3.8 Expand $\exp(i(\alpha+\beta)) = \exp(i\alpha)\exp(i\beta)$ and compare real and imaginary parts. p. 381

11.3.9 Add the function ϕ of Proposition 11.3.5 to e^x. p. 381

11.3.10 Follow the proof of Proposition 11.3.5 with $g^{(k)}(x) = p_k(1/x)e^{-1/x}$ for $x > 0$. For the last part, an analytic function vanishing on an interval has a point at the end of its zero set where all derivatives vanish. p. 381

11.4 Binomial Series and Applications of Taylor Series

11.4.1 Equation (4.6.18) at $m = \frac{1}{3}$. p. 384

11.4.2 Equation (11.4.7) between 0 and $\frac{1}{2}$ and then Theorem 10.6.3. p. 384

11.4.3 The sizes are $1/(n!(2n+1))$. p. 384

11.4.4 $\mathrm{d}\arcsin t = (1-t^2)^{-1/2}\,\mathrm{d}t$. Substitute $-t^2$ into the binomial series. p. 385

11.4.5 Write the k factors of $\binom{-1/2}{k}$ as $-\frac{2j-1}{2}$ and multiply numerator and denominator by $2 \cdot 4 \cdots (2k) = 2^k k!$. p. 385

11.4.6 $(k+1)\binom{m}{k+1} = (m-k)\binom{m}{k}$ by Eq. (4.6.18). p. 385

11.4.7 The numerator begins at x^4. p. 385

11.4.8 From $k = 1$ on the terms alternate in sign and $\left|\binom{1/2}{k+1}\Big/\binom{1/2}{k}\right| = \frac{k-1/2}{k+1} < 1$. Show that the sizes tend to 0 and then use Theorem 11.1.12. p. 385

11.4.9 By Eq. (4.6.18), $(k+1)u_{k+1} = |m-k|u_k$. For $k > m$ this reads $ku_k - (k+1)u_{k+1} = mu_k$. p. 385

11.4.10 Substitute (11.2.4). p. 385

11.5 Analytic Functions and the Shift Operator

11.5.1 $1 - x = \frac{3}{2} - (x + \frac{1}{2})$. p. 389

11.5.2 $f^{(k)}(0) = (-1)^{k+1}(k-1)!$ for $k \ge 1$. p. 389

11.5.3 Sum over n first, after exchanging the order with Theorem 11.5.2. p. 389

11.5.4 Apply the theorem to exp at the point x with $h = y$. p. 389

11.5.5 By Theorem 11.5.5 the sum is analytic on $(-R, R)$. Adapt the last part of Problem 11.3.10. p. 389

11.5.6 The square $m, n \le N$ contains $N+1$ entries 1 and N entries -1. p. 389

11.5.7 Write $g(x) = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$ and expand each fraction as a complex geometric series in $x - b$. The coefficient of $(x-b)^k$ is $-\operatorname{Im}((i-b)^{-k-1})$, with absolute value

$(1+b^2)^{-(k+1)/2}|\sin((k+1)\theta)|$ for θ the argument of $i-b$. p. 389

Appendix C

Answers

1.1 Statements and Connectives

1.1.1 (a) true; (b) false; (c) true; (d) true. p. 13

1.1.2 Converse: if $x^2 > 9$ then $x > 3$. Contrapositive: if $x^2 \le 9$ then $x \le 3$. Negation: $x > 3$ and $x^2 \le 9$. The implication and its contrapositive are true for every x. The converse fails at $x = -4$ and the negation is false for every x. p. 13

1.1.3 The column of $(P \implies Q) \implies P$ is $\mathrm{T}, \mathrm{T}, \mathrm{F}, \mathrm{F}$, the same as the column of P. p. 13

1.1.4 Both sides of the equivalence are false exactly when $P = Q = \mathrm{T}$ and $R = \mathrm{F}$. With $P = Q = R = \mathrm{F}$ the statement $(P \implies Q) \implies R$ is false and $P \implies (Q \implies R)$ is true. p. 13

1.1.5 (a) If 4 divides n then n is even: true. (b) If n is even then 4 divides n: false at $n = 2$. (c) If n is even then n^2 is even: true. p. 13

1.1.6 $P \vee Q \equiv \neg P \implies Q$ and $P \wedge Q \equiv \neg(P \implies \neg Q)$. p. 13

1.2 Quantifiers

1.2.1 (a) true; (b) false; (c) false; (d) true. p. 17

1.2.2 (a) $\exists x \in \mathbb{R}\ \forall n \in \mathbb{N},\ n \le x$; the original is true. (b) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ y^2 < x$; the original is true. p. 18

1.2.3 $\exists x, y \in I\ (x < y \wedge f(x) \ge f(y))$. Take $x = -1$ and $y = 0$. p. 18

1.2.4 $N = 100$ and $N = 333$. p. 18

1.2.5 With $\varepsilon = \frac{1}{4}$, every $n \ge 4$ has $|1/n - \frac{1}{2}| \ge \frac{1}{4}$. p. 18

1.2.6 Negation: $\forall x \in S\ (\neg P(x) \vee \exists y \in S\ (P(y) \wedge y \ne x))$, that is, either no element satisfies P or at least two do. The first uniqueness statement is false and the second true. p. 18

1.3 Methods of Proof

1.3.1 $(2k+1)(2l+1) = 2(2kl + k + l) + 1$ and $(2k+1) + (2l+1) = 2(k + l + 1)$. p. 22

1.3.2 If n is even, $n^2 = 2(2k^2)$ is even. p. 22

1.3.3 (a) $x = -2$; (b) 9; (c) $a = b = 1$. p. 22

1.3.4 $a^2 + b^2 - 2ab = (a-b)^2 \ge 0$. Then $a + b = (\sqrt{a})^2 + (\sqrt{b})^2 \ge 2\sqrt{a}\sqrt{b} = 2\sqrt{ab}$. p. 22

1.3.5 For n even every term is even; for n odd the first two terms are odd and their sum is even. p. 23

1.3.6 $x^3 - 8 = (x-2)(x^2 + 2x + 4)$ and $x^2 + 2x + 4 = (x+1)^2 + 3 > 0$. p. 23

1.4 Mathematical Induction

1.4.1 Base $1 = 1 \cdot 2 \cdot 3/6$. Step: factor $(k+1)/6$ out of $k(k+1)(2k+1)/6 + (k+1)^2$ to get $(k+1)(2k^2 + 7k + 6)/6$. p. 29

1.4.2 Step: $\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k^2+4k+4)}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2$. p. 29

1.4.3 Base $4 - 1 = 3$. Step: $4^{k+1} - 1 = 4(4^k - 1) + 3$ and both terms are divisible by 3. p. 29

1.4.4 Step: $\frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{1-r^{k+2}}{1-r}$. The sum is 364. p. 29

1.4.5 The base case is missing and fails: $1^2 + 1 = 2$ is even. The claim is false for every n (Example 1.3.13). p. 29

1.4.6 $12 = 4 \cdot 3$, $13 = 4 \cdot 2 + 5$, $14 = 4 + 5 \cdot 2$, $15 = 5 \cdot 3$; for $n \geq 16$ add one 4 to the representation of $n - 4$. p. 29

1.4.7 $\gcd(84, 120) = 12 = 3 \cdot 84 - 2 \cdot 120$. p. 29

1.4.8 A proof. p. 29

1.4.9 Euclid's lemma gives $b \mid k$ where $c = ak$. With $a = b = 2$ and $c = 2$ the conclusion $4 \mid 2$ fails. p. 29

1.5 Sets and Functions

1.5.1 $\{1, 2, 3, 4, 5\}$, $\{3, 4\}$, $\{1, 2\}$, $\{1, 2, 6\}$; $A \times B$ has 12 elements. p. 36

1.5.2 (a) injective, not surjective; (b) surjective, not injective; (c) injective, not surjective; (d) bijective. p. 36

1.5.3 $f([0, 3]) = [-1, 3]$, $f^{-1}(\{0\}) = \{1, 3\}$ and $f^{-1}([-1, 0]) = [1, 3]$. p. 36

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7.3 Differentiation as a Linear Operator

7.3.1 $A=\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$, $A^2=\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix}$, $A^3=0$. p. 240

7.3.2 $f(x)=1/x+c_1$ for $x<0$ and $f(x)=1/x+c_2$ for $x>0$ with arbitrary constants c_1,c_2. p. 240

7.3.3 $f(x)=\cos x-2\sin x$. p. 240

7.3.4 Matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}$; $D^n(xe^x)=(x+n)e^x$. p. 240

7.3.5 $e^x(x^3+15x^2+60x+60)$. p. 240

7.3.6 $J(3x^2-4x+1) = x^3-2x^2+x$; $JDp = x^2 = p-5$. p. 240

7.3.7 $(x|x|)' = 2|x|$; $(|x|^3)' = 3x|x|$ and $(|x|^3)'' = 6|x|$ with no derivative at 0. p. 240

7.3.8 Applying $D-\lambda\,\mathrm{id}$ gives $b(\mu-\lambda)e^{\mu x}=0$. Hence $b=0$ and then $a=0$. p. 240

7.3.9 $D(f^{n+1}) = D(f^n)f + f^nDf = (n+1)f^nDf$. p. 240

7.3.10 $A = \frac{1}{2}(f(0)+f'(0))$ and $B = \frac{1}{2}(f(0)-f'(0))$. p. 240

7.3.11 $1/\sqrt{n}$, $\sqrt{n}$ and $1/\sqrt{n}+\sqrt{n}$. The ratio $\|g_n\|_{C^1}/\|g_n\|_\infty = 1+n$ is unbounded. p. 240

7.4 Taylor's Theorem

7.4.1 $x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$. p. 247

7.4.2 $x + \dfrac{x^3}{3}$. p. 247

7.4.3 $P_2(9.2) = 3 + \frac{0.2}{6} - \frac{0.04}{216} \approx 3.0331481$, with error at most $\frac{0.008}{3888} \approx 2.06\times10^{-6}$. p. 247

7.4.4 $n=5$. p. 247

7.4.5 $\sin x - x = R_2(x) = -\cos(c)\,x^3/6$. p. 247

7.4.6 $-\frac{1}{6}$. p. 247

7.4.7 Since $g'' = 1-\cos x \geq 0$, g lies above its tangent line $y=0$ at 0. p. 247

7.4.8 $\begin{pmatrix} 1 & h & h^2 \\ 0 & 1 & 2h \\ 0 & 0 & 1 \end{pmatrix}$ with third column $h^2+2hx+x^2 = (x+h)^2$. p. 247

7.4.9 The quotient equals $\frac{1}{2}(f''(c_1)+f''(c_2))$ with c_1, c_2 within $|h|$ of a. p. 247

7.4.10 $q!(e-P_q(1))$ would be an integer strictly between 0 and 1. p. 247

7.4.11 Every derivative at 0 is 0. Hence $P_n = 0$ for every n although $f(h) = e^{-1/h^2} > 0$. p. 247

8.1 Antiderivatives

8.1.1 $\dfrac{x^4}{2} - \dfrac{5x^2}{2} + 7x + C$. p. 253

8.1.2 $\dfrac{1}{2}\arctan(2x) + C$. p. 253

8.1.3 $F(x) = x^3 - 2$. p. 253

8.1.4 $G = F + C$ on I and the value at x_0 forces $C=0$. p. 253

8.1.5 $F(x) = 2\ln x$ for $x>0$ and $F(x) = 2\ln(-x)+1$ for $x<0$. p. 253

8.1.6 $F'(x) = |x|$ for $x\neq 0$ by the power rule, and $F'(0) = \lim_{h\to0}\frac{1}{2}|h| = 0$. p. 253

8.1.7 Any such F would equal $|x|+F(0)$ on $(-1,1)$. That function has no derivative at 0. p. 253

8.1.8 $G-F = -\dfrac{\pi}{2}$ on $(0,\infty)$ and $G-F = \dfrac{\pi}{2}$ on $(-\infty,0)$. p. 253

8.1.9 $f(x) = 2x\sin(1/x) - \cos(1/x)$ for $x\neq0$ and $f(0)=0$. Since $f(1/(2k\pi)) = -1$ for every k, f is not continuous at 0. p. 253

8.2 Definite Integral

8.2.1 -14. p. 262

8.2.2 The sums are $\dfrac{(n+1)^2}{4n^2} \to \dfrac{1}{4}$. p. 262

8.2.3 A point with $f(p)>0$ would force the integral to be at least the positive number $\frac{1}{2}f(p)(\beta-\alpha)$ for an interval $[\alpha,\beta]$ around p. p. 262

8.2.4 Every Riemann sum of f lies in $[0, 2\|P\|]$. Take $\delta = \varepsilon/2$. A function $\tilde{g}$ equal to g except at p has Riemann sums $S(\tilde{g}) = S(g) + (\tilde{g}(p)-g(p))S(f)$. p. 262

8.2.5 $f_n(0)=0$ and $f_n(x)=0$ once $n\geq 2/x$. The integral is the triangle area $\frac{1}{2}\cdot\frac{2}{n}\cdot n = 1$. Since $\|f_n-0\|_\infty = n \not\to 0$, the sequence does not converge to 0 in the sup norm and the theorem does not apply. p. 262

8.2.6 $U(f,P)-L(f,P) = \sum_k (f(x_k)-f(x_{k-1}))\Delta x_k \leq \|P\|(f(b)-f(a))$. The bound is less than ε when $\|P\| < \varepsilon/(f(b)-f(a)+1)$. p. 263

8.2.7 $|f|=1$ has integral 1. On every partition the Riemann sums of f take both values 1 and -1. The inequality presumes that f is integrable. Integrability of $|f|$ does not supply that hypothesis. p. 263

8.2.8 f has infinitely many jump discontinuities and is integrable by Problem 8.2.6. Piecewise continuity is sufficient for integrability and not

necessary. p. 263

8.3 Fundamental Theorem of Calculus

8.3.1 2. p. 269

8.3.2 $2x\cos(x^2)$. p. 269

8.3.3 $-3x\sin x$. p. 269

8.3.4 $F'(x) = 26x^2$. p. 270

8.3.5 $f(x) = x^{3/2}$ and $a = 9$. p. 270

8.3.6 $f(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = \dfrac{\mathrm{d}}{\mathrm{d}x}0 = 0.$ p. 270

8.3.7 $F(x) = |x| - 1$. Continuity of the integrand at 0 fails. F is still continuous because $|F(x) - F(y)| \le |x - y|$. p. 270

8.3.8 With $g = f$ the integral of the continuous function $f^2 \ge 0$ is 0. Hence $f^2 = 0$. p. 270

8.3.9 $\|g_n\|_\infty \le \frac{1}{n}$ and $\|g_n'\|_\infty = |g_n'(0)| = n$. Differentiation sends a sequence tending to 0 to one with norms tending to ∞. Integration from 0 recovers g_n exactly. p. 270

8.3.10 $G'(x) = f(x) - f(-x) = 0$ and $G(0) = 0$. Hence $G = 0$. p. 270

8.4 Indefinite Integrals and the Substitution Method

8.4.1 $\dfrac{(3x^2 + 4x)^5}{10} + C.$ p. 273

8.4.2 $-6\sqrt{1 - r^3} + C.$ p. 273

8.4.3 $\dfrac{1}{2}\sin^6\left(\dfrac{x}{3}\right) + C.$ p. 273

8.4.4 With $u = \alpha x + \beta$ the 1-form $\mathrm{d}x$ is $\frac{1}{\alpha}\,\mathrm{d}u$. p. 273

8.4.5 $\dfrac{(\ln x)^2}{2} + C.$ p. 273

8.4.6 $\ln(\ln x) + C.$ p. 273

8.4.7 The image $g(I)$ is an interval inside $(0, \infty)$ or inside $(-\infty, 0)$. On it $\ln|u|$ is an antiderivative of $1/u$. Theorem 8.4.2 then applies. p. 273

8.4.8 $\ln(-\ln x) + C_1$ on $(0, 1)$ and $\ln(\ln x) + C_2$ on $(1, \infty)$ with independent constants. p. 273

8.4.9 $\sin(x^2) + C$. The theorem asks only that g be differentiable on an interval with values in an interval on which f is continuous. Its proof differentiates $F \circ g$ by the chain rule and never inverts g. p. 274

8.5 Definite Integral Substitutions and the Area Between Curves

8.5.1 $\dfrac{15}{8}$. p. 278

8.5.2 $\dfrac{1}{3}$. p. 278

8.5.3 $\dfrac{9}{2}$. p. 278

8.5.4 $\dfrac{1}{2}$. p. 278

8.5.5 The substitution $g(t) = \frac{\pi}{2} - t$ exchanges the two integrals. p. 278

8.5.6 The substitution $x = t + p$ gives $\int_p^{a+p} f = \int_0^a f$. Additivity then gives the result. p. 278

8.5.7 $\dfrac{\pi^2}{4}$. p. 278

8.5.8 $e^4 - e$. The interval $[0, 1]$ is traversed twice in opposite directions and the two contributions cancel. p. 279

8.5.9 The value is $\dfrac{\pi}{2}$. The substitution $g(x) = 1/x$ is undefined at $0 \in [-1, 1]$. Hence Theorem 8.5.2 does not apply. p. 279

8.6 Arc Length

8.6.1 $\dfrac{5}{3}$. p. 284

8.6.2 $\dfrac{17}{12}$. p. 284

8.6.3 $\dfrac{3}{8} + \ln 2.$ p. 284

8.6.4 Polygon lengths do not decrease under refinement. The trivial partition gives the chord. The polygon lengths converge to L. p. 284

8.6.5 $g'(x) = f'(x - h)$ and the substitution $u = x - h$ maps $[a + h, b + h]$ onto $[a, b]$. p. 284

8.6.6 $g'(x) = f'(x/\lambda)$. The substitution $x = \lambda u$ turns the length integral of g into λ times that of f. p. 284

8.6.7 $L(f) = \sqrt{2}$ and $L(2f) = \sqrt{5} \ne 2\sqrt{2}$. The map is not linear. p. 284

8.6.8 $L(f_n) \ge n\int_0^\pi |\cos(n^2 x)|\,\mathrm{d}x = 2n$, while $f_n \to 0$ in the sup norm and the zero function

has length π. p. 284

9.1 Strategy for a General Integrand

9.1.1 $2\sqrt{x} - \dfrac{4}{\sqrt{x}} + C$. p. 290

9.1.2 $\dfrac{1}{4}\arctan\dfrac{x+1}{2} + C$. p. 290

9.1.3 $2\sqrt{x} - 2\arctan\sqrt{x} + C$. p. 290

9.1.4 $x^2 + bx + c = (x + \frac{b}{2})^2 + (\frac{d}{2})^2$. The template gives $\frac{2}{d}\arctan\frac{x+b/2}{d/2}$. p. 290

9.1.5 The pullback is $-\mathrm{d}t$ and the antiderivative is $-(\pi - \arcsin x) = \arcsin x - \pi$. It differs from $\arcsin x$ by the constant $-\pi$. p. 291

9.1.6 $\frac{1}{2}x^2 + 4x + C_1$ on $(-\infty, 4)$ and $\frac{1}{2}x^2 + 4x + C_2$ on $(4, \infty)$. It is not an antiderivative on $\mathbb{R}$ because the integrand has no value at 4. p. 291

9.1.7 $F(x) = \int_0^x e^{t^2}\,\mathrm{d}t$ with $F(0) = 0$ and $F'(0) = 1$. The integrand belongs to none of the classes. p. 291

9.1.8 $F \circ \varphi$ is an antiderivative of $-f(-t) = f(t)$. Hence $F(-t) = F(t) + C$. The value $t = 0$ gives $C = 0$. p. 291

9.2 Integration by Parts

9.2.1 $(x^2 - 2x + 2)e^x + C$. p. 296

9.2.2 $\dfrac{e^x}{2}(\sin x + \cos x) + C$. p. 296

9.2.3 $x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$. p. 296

9.2.4 $1 - \dfrac{2}{e}$. p. 296

9.2.5 With $v = -(x-t)^n/n$ the remainder of order $n-1$ becomes $\dfrac{f^{(n)}(a)}{n!}(x-a)^n$ plus the remainder of order n. The bound follows from Theorem 8.2.15(iv) and $\left|\int_a^x |x-t|^n\,\mathrm{d}t\right| = |x-a|^{n+1}/(n+1)$. p. 297

9.2.6 $x((\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6) + C$. p. 297

9.2.7 Both sides are the family $\ln x + C$. Adding the constant 1 to the family does not change it. Families cannot be subtracted as numbers. p. 297

9.2.8 $\dfrac{1}{3}(x^3 - 1)e^{x^3} + C$. p. 297

9.2.9 The boundary term is at most $(|f(a)| + |f(b)|)/\lambda$ and the remaining integral at most $(b-a)\|f'\|_\infty/\lambda$. p. 297

9.2.10 $\int e^x \sin x\,\mathrm{d}x = e^x\sin x - e^x\cos x - \int e^x\sin x\,\mathrm{d}x$, hence $\frac{1}{2}e^x(\sin x - \cos x) + C$. Because no derivative of $\sin x$ is 0, the remainder never vanishes. p. 297

9.3 Trigonometric Integrals

9.3.1 $-\dfrac{\cos 2x}{4} - \dfrac{\cos 6x}{12} + C$. p. 303

9.3.2 $\dfrac{\sec^5 x}{5} - \dfrac{\sec^3 x}{3} + C$. p. 303

9.3.3 $\dfrac{1}{4}\sec^3 x\tan x + \dfrac{3}{8}\sec x\tan x + \dfrac{3}{8}\ln|\sec x + \tan x| + C$. p. 303

9.3.4 $\dfrac{2}{\sqrt{3}}\arctan\Big(\dfrac{1}{\sqrt{3}}\tan\dfrac{x}{2}\Big) + C$. p. 303

9.3.5 $\dfrac{\pi}{16}$. p. 303

9.3.6 $1 \le W_{2m}/W_{2m+1} \le (2m+1)/(2m)$ and hence $W_{2m}/W_{2m+1} \to 1$. By Eq. (9.3.8) the ratio is $\frac{\pi}{2}$ divided by the product. p. 304

9.3.7 By Eq. (9.3.3) each integrand is a combination of $\cos(m \pm n)x$ or $\sin(m \pm n)x$. Only $\cos 0x = 1$ contributes, with $\frac{1}{2}\cdot 2\pi = \pi$ when $m = n$. p. 304

9.3.8 $\dfrac{\tan^3 x}{3} - \tan x + x + C$. p. 304

9.3.9 $\dfrac{2\pi}{\sqrt{3}}$. The function F jumps by $-2\pi/\sqrt{3}$ at π. Theorem 8.3.5 does not apply to it on $[0, 2\pi]$. p. 304

9.3.10 $\dfrac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x) + C$. The integrand depends only on $\sin^2 x$. As a rational function of $\tan x$ it has lower degree than as a function of $\tan(x/2)$. p. 304

9.4 Trigonometric Substitutions

9.4.1 $\sqrt{x^2 - 4} - 2\arccos\dfrac{2}{x} + C$. p. 309

9.4.2 $\ln\Big(x + 2 + \sqrt{x^2 + 4x + 13}\Big) + C$. p. 309

9.4.3 $2\arcsin\dfrac{x}{2} + \dfrac{x(x^2-2)\sqrt{4-x^2}}{4} + C$. p. 309

9.4.4 $-\ln\left(\sqrt{x^2-25}-x\right)+C$. p. 309

9.4.5 $\arccos\dfrac{1}{x}+C$ on $(1,\infty)$ and $-\arccos\dfrac{1}{x}+C$ on $(-\infty,-1)$. p. 309

9.4.6 $\dfrac{\pi}{4}$. Writing $\cos\theta$ for the radical gives the wrong value $-\dfrac{\pi}{4}$. p. 309

9.4.7 $a^2\int\sec^3\theta\,d\theta=\frac{a^2}{2}(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta))$, which in x is the stated formula up to the constant $-\frac{a^2}{2}\ln a$. p. 309

9.4.8 $\dfrac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5})\approx 1.4789$. p. 309

9.4.9 πab. Theorem 8.5.2 needs only a continuous derivative of $g(\theta)=a\sin\theta$ on the closed interval and a continuous integrand on its image $[-a,a]$. The open intervals of the table serve indefinite integrals whose integrands, such as $1/\sqrt{a^2-x^2}$, are undefined at $\pm a$. p. 309

9.4.10 2π. p. 310

9.4.11 $\dfrac{\pi}{2}$. p. 310

9.5 Integration of Rational Functions by Partial Fractions

9.5.1 $x+\dfrac{3}{2}\ln|x^2-1|+C$. p. 315

9.5.2 $\dfrac{2}{3}\ln|x-1|+\dfrac{1}{3}\ln|x+2|-\dfrac{1}{x+2}+C$. p. 315

9.5.3 $\dfrac{1}{6}\ln\dfrac{3-x}{3+x}+C$. Without the bars the term $\ln(x-3)$ is undefined at every point of the interval. p. 315

9.5.4 $\dfrac{16}{9}\ln|x-5|+\dfrac{11}{9}\ln|x+4|+C$. Completing the square gives the difference of squares $(x-\frac{1}{2})^2-\frac{81}{4}$. No arctangent template applies. p. 315

9.5.5 $\ln|x-1|-\dfrac{2}{x-1}+\dfrac{1}{2}\ln(x^2+1)-\arctan x+C$. p. 315

9.5.6 $9\ln 3-14\ln 2$. p. 315

9.5.7 g is a polynomial in $x-r$ of degree $m-1$ with coefficients $A_m, A_{m-1},\ldots$ plus the term $(x-r)^m R$. The first $m-1$ derivatives of that term vanish at r. In Worked Example 9.5.2, $g=2x-1$ gives $C=-3$, $B=2$ and $A=0$. p. 315

9.5.8 Replacing x by $-x$ gives the decomposition $\sum_i(\dfrac{-B_i}{x-r_i}+\dfrac{-A_i}{x+r_i})$ of the same function f. Uniqueness gives $A_i=-B_i$. p. 315

9.5.9 Cover-up gives the coefficient $1/q'(r_i)$. In $1=\sum_i q(x)/(q'(r_i)(x-r_i))$ the coefficient of x^{n-1} on the right is $\sum_i 1/q'(r_i)$ and on the left 0. For the example, $\frac{1}{3}-\frac{1}{2}+\frac{1}{6}=0$. p. 316

9.5.10 $\dfrac{1}{4}+\dfrac{\pi}{8}$. p. 316

9.6 Integral Tables and Computer Algebra Systems

9.6.1 $-(x^2+2x+2)e^{-x}+C$. p. 319

9.6.2 $\dfrac{1}{2}\sin^2 x+C$ and $-\dfrac{1}{2}\cos^2 x+C$. Their difference is the constant $\dfrac{1}{2}$. p. 320

9.6.3 With the real logarithm only on $(3,\infty)$. With the complex logarithm the expression equals $\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|$ on $(3,\infty)$ and on $(-\infty,-3)$, and that function plus the constant $\frac{i\pi}{6}$ on $(-3,3)$. The general real antiderivative is $\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|+C_i$ with an independent constant on each of $(-\infty,-3)$, $(-3,3)$ and $(3,\infty)$. p. 320

9.6.4 $\dfrac{d}{dx}\arcsin\dfrac{x}{a}=\dfrac{1}{a\sqrt{1-x^2/a^2}}=\dfrac{|a|}{a}\cdot\dfrac{1}{\sqrt{a^2-x^2}}$. The correct entry for every $a\neq 0$ is $\arcsin\dfrac{x}{|a|}+C$. p. 320

9.6.5 $\dfrac{4\pi}{\sqrt{3}}$. p. 320

9.6.6 $F_2-F_1=\dfrac{\pi}{4}$ on $(-\infty,1)$ and $-\dfrac{3\pi}{4}$ on $(1,\infty)$. p. 320

9.6.7 $\dfrac{F(c+h)-F(c)}{h}=f(\xi_h)$ with ξ_h between c and $c+h$. Continuity gives $f(\xi_h)\to f(c)$. No value at π makes F_0 continuous there. p. 320

9.6.8 $6-2e$. p. 320

9.7 Numerical Integration

9.7.1 Bound $\frac{\pi^3}{864} \approx 0.035887$; $T_6 = \frac{\pi(\sqrt{3}+2)}{12} \approx 0.977049$, integral 1, $|E_T| \approx 0.022951$. p. 325

9.7.2 $S_2 = 36$, the value of the integral. Simpson's Rule is exact for every polynomial of degree at most 3. p. 326

9.7.3 $T_4 \approx 1.727222$, $E_T \approx 0.008940$; $S_4 \approx 1.718319$, $E_S \approx 0.0000370$. The predictions are $\frac{e-1}{12\cdot 16} \approx 0.008949$ and $\frac{e-1}{180\cdot 256} \approx 0.0000373$. p. 326

9.7.4 Bound $\frac{1}{1920} \approx 0.000521$; $S_4 = \frac{1747}{2520} \approx 0.693254$, integral $\ln 2 \approx 0.693147$, $|E_S| \approx 0.000107$. p. 326

9.7.5 Rolle's theorem twice and the Mean Value Theorem give $K = -f''(c)/3$. With $\Delta = 2h$ the constant $\frac{h^3}{3}$ is $\frac{\Delta^3}{24}$. Summing as in Theorem 9.7.3 gives the composite formula with constant $\frac{1}{24}$. p. 326

9.7.6 Both sides have the weights $\frac{h}{3}(1, 4, 2, 4, \dots, 4, 1)$ with $h = (b-a)/(2m)$. The combination cancels the leading term $\frac{(b-a)^2}{12n^2}(f'(b) - f'(a))$ of the trapezoid error. p. 326

9.7.7 $T_1 = 0.5$, $T_2 \approx 0.603553$, $T_4 \approx 0.643283$; errors -0.166667, -0.063113, -0.023384; ratios about 2.64 and 2.70. The second derivative is unbounded on $(0, 1]$. The theorem does not apply. p. 326

9.7.8 $T_n = \frac{1}{2}(L_n + R_n)$ with the left and right sums L_n and R_n. Both tend to the integral by Theorem 8.2.13. p. 326

9.8 Improper Integrals

9.8.1 It diverges. The function $-1/(x-1)$ is undefined at 1 and so is no antiderivative on $[0, 2]$. A positive integrand cannot have a negative integral. p. 332

9.8.2 It diverges: $\int_{-\infty}^{0} e^x \, dx = 1$ but $\int_0^{\infty} e^x \, dx = \infty$. p. 332

9.8.3 It converges. p. 332

9.8.4 A lower bound with a convergent integral proves nothing. From $\frac{1}{\sqrt{x^4+x}} \le \frac{1}{x^2}$ the integral converges. p. 332

9.8.5 It diverges. p. 332

9.8.6 $\frac{1}{4}$. p. 332

9.8.7 The case $n = 0$ is Worked Example 9.8.1. Parts gives $\int_0^b x^n e^{-x} \, dx = -b^n e^{-b} + n \int_0^b x^{n-1} e^{-x} \, dx$, and the limit is $n \cdot (n-1)!$. p. 332

9.8.8 Parts gives $\int_1^b \frac{\sin x}{x^p} \, dx = \cos 1 - \frac{\cos b}{b^p} - p \int_1^b \frac{\cos x}{x^{p+1}} \, dx$ with an absolutely convergent last integral. For $p > 1$ compare with x^{-p}. For $p \le 1$ compare with $|\sin x|/x$. p. 332

9.8.9 The integral is $n \cdot \frac{1}{n} + \frac{1}{2n}$. Such a K would give $1 < 1 + \frac{1}{2n} \le \frac{K}{n}$ for every n. p. 332

9.8.10 $f/g = 1 + \sin x/\sqrt{x} \to 1$. If $\int f$ converged, then $\int \sin^2 x/x = \int f - \int g$ would converge. The functions change sign. p. 332

9.8.11 Let $f(x) = \max(0, 1 - 2^n|x-n|)$ for $|x-n| \le \frac{1}{2}$, $n \ge 2$, and $f = 0$ elsewhere. The partial integrals are at most $\sum_{n\ge 2} 2^{-n} \le \frac{1}{2}$. p. 333

10.1 Sequences and Complete Spaces

10.1.1 $a_{10} = 29$, $s_{10} = 155$ and $b_n \to 0$. p. 341

10.1.2 $\sup a_n = \frac{3}{2}$, $\inf a_n = -2$, $\limsup a_n = 1$, $\liminf a_n = -1$. p. 341

10.1.3 e^3 and e^{-2}. p. 341

10.1.4 0. p. 341

10.1.5 The sequence converges to 2. p. 341

10.1.6 With $a_n = (-1)^n$ and $b_n = -a_n$ the left side is 0 and the right side is 2. p. 342

10.1.7 Convergence of the norms does not imply convergence of the vectors. p. 342

10.1.8 On $[0, 1]$ the norm is e^{-1} for every n and the convergence is therefore not uniform. On $[\frac{1}{2}, 1]$ it is $\frac{n}{2}e^{-n/2}$ for $n \ge 2$ and tends to 0. p. 342

10.1.9 $\|x^n\|_1 = \dfrac{1}{n+1}$ and $\|x^n\|_\infty = 1$. A constant K would give $1 \le K/(n+1)$ for every n. p. 342

10.1.10 Both are the real proofs with the norm in place of the absolute value. p. 342

10.1.11 The difference is largest at $x = 0$ where it equals $1/\sqrt{n}$. p. 342

10.1.12 $\|g_n - g_m\|_1 \le 1/n$. A continuous limit would have to equal 0 on $[0, \frac{1}{2}]$ and 1 on $(\frac{1}{2}, 1]$. p. 342

10.2 Infinite Series

10.2.1 5. p. 347

10.2.2 1. p. 347

10.2.3 2. p. 348

10.2.4 It converges for $-1 < x < 5$, with sum $\dfrac{3}{5-x}$. p. 348

10.2.5 $f(x) = \dfrac{2}{2-x}$, with $\|f - s_N\|_\infty = 2^{-N}$ and $N = 10$. p. 348

10.2.6 The block of length n ending at $2n$ is at least $\frac{1}{2}$. p. 348

10.2.7 $\mathbf{r}_n = \lim_{m\to\infty}(\mathbf{s}_m - \mathbf{s}_n) = \mathbf{S} - \mathbf{s}_n \to \mathbf{0}$. p. 348

10.2.8 For $n > K$, $|z|^n/n! \le C\,2^{-n}$ with $C = 2^K|z|^K/K!$. p. 348

10.2.9 On $[-r, r]$ take $M_n = r^n$. On $(-1, 1)$ the difference $1/(1-x) - s_N(x)$ is unbounded. p. 348

10.2.10 The series of the hint has partial sums that return to 1 and to 0 infinitely often. p. 348

10.2.11 The subsequence $\mathbf{x}_{n_k}$ is the partial-sum sequence of an absolutely convergent series and converges. A Cauchy sequence with a convergent subsequence converges. p. 348

10.3 Integral Test

10.3.1 (a) converges ($p = \frac{3}{2}$) and (b) diverges ($p = \frac{2}{3}$). p. 352

10.3.2 It converges, with sum between $\frac{\pi}{4}$ and $\frac{1}{2} + \frac{\pi}{4}$. p. 352

10.3.3 It converges exactly when $p > 1$. p. 352

10.3.4 $\dfrac{1}{50} \le R_4 \le \dfrac{1}{32}$. p. 352

10.3.5 $n = 1000$ terms suffice. For 10^{-6} the bound requires $n = 10^6$. p. 352

10.3.6 $d_n - d_{n+1} \ge 0$ and $\frac{1}{n} \le d_n \le 1$. p. 352

10.3.7 Either $f \ge 0$ everywhere and Theorem 10.3.2 applies or both the series and the integral diverge to $-\infty$. p. 352

10.3.8 The tents give $f(n) \ge 1/n$ and total area $\sum 1/n^3 < \infty$. The function is not nonincreasing. p. 352

10.3.9 $1 \le \sum 1/n^2 \le 2$. The integral and the sum differ by an amount between 0 and a_1. p. 352

10.3.10 $\frac{1}{2}\sum_{k=1}^{K} 2^k a_{2^k} \le \sum_{n=2}^{2^K} a_n$ and $\sum_{n=1}^{2^{K+1}-1} a_n \le \sum_{k=0}^{K} 2^k a_{2^k}$. For $a_n = n^{-p}$ the condensed series is geometric with ratio 2^{1-p}. p. 352

10.4 Comparison Tests

10.4.1 Converges. p. 356

10.4.2 Diverges. p. 356

10.4.3 Diverges. p. 356

10.4.4 Converges. p. 356

10.4.5 It converges. Only 3^n and 4^n matter, with comparison series $\sum(3/4)^n$. p. 356

10.4.6 $|a_n b_n| \le \frac{1}{2}(a_n^2 + b_n^2)$. The sequence $b_n = 1/n$ is square-summable. p. 356

10.4.7 $a_n/(1+a_n) \le a_n$ and eventually $a_n \le 2a_n/(1+a_n)$. p. 356

10.4.8 $\sum 1/n^2$ converges and $\sum 1/(n \ln n)$ diverges while both ratios tend to 0. With $c = 0$ and a divergent $\sum b_n$ the test gives no information. p. 356

10.4.9 The series $\sum a_n$ telescopes to at most $2\sqrt{r_0}$ and has $a_n/b_n = 1/\sqrt{r_{n-1}} \to \infty$. p. 356

10.5 Ratio and Root Tests

10.5.1 Converges, with $\rho = 0$. p. 359

10.5.2 Converges, with $\rho = \frac{1}{2}$. p. 360

10.5.3 Converges, with $\rho = 0$. p. 360

10.5.4 $\limsup = 2$, $\liminf = \frac{1}{8}$, $\lim a_n^{1/n} = \frac{1}{2}$. The series converges to 1. p. 360

10.5.5 $a_{n+1}/a_n = (1+1/n)^{-n} \to 1/e$. p. 360

10.5.6 It converges for $0 < x < e$ and diverges for $x \geq e$. p. 360

10.5.7 Both give $\rho = 1$. The series converges by comparison with $\sum 1/n^2$. p. 360

10.5.8 The first gives $a_n \leq a_1 r^{n-1}$. The harmonic series has every ratio below 1 and diverges. p. 360

10.5.9 $\{s_n\}$ is nondecreasing and $s_{2m} = \sum_{k\leq m} a_{2k-1} + \sum_{k\leq m} a_{2k} \to A + B$. p. 360

10.5.10 The partial sums are bounded by $Na_N/(p-1)$ plus the first N terms. For $1/n^2$, $n(1-n^2/(n+1)^2) \to 2$. p. 360

10.6 Alternating Series and Rearrangement

10.6.1 It converges conditionally. p. 364

10.6.2 It converges absolutely. p. 364

10.6.3 Absolutely for $p > 1$, conditionally for $0 < p \leq 1$, and it diverges for $p \leq 0$. p. 364

10.6.4 The terms of index 0 to 6, seven in all, give $\frac{265}{720} \approx 0.36806$. p. 364

10.6.5 $1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \frac{1}{3}, -\frac{1}{10}, -\frac{1}{12}$, with eighth partial sum $\frac{13}{120}$. p. 364

10.6.6 Comparison in one direction and Theorem 10.2.10 in the other. If exactly one diverged, $\sum a_n$ would diverge. p. 364

10.6.7 With finitely many negative terms the series and its absolute series differ in finitely many terms. p. 364

10.6.8 Both pieces converge, with sums $\pi^2/8$ and $\pi^2/24$. For $\sum(-1)^{n+1}/n$ both pieces diverge. p. 365

10.6.9 $0 \leq a_n^2 \leq a_n$ once $a_n \leq 1$. The series $\sum(-1)^n/\sqrt{n}$ converges and $\sum 1/n$ diverges. p. 365

10.6.10 The partial sums lie in $[0, 2c_k]$ during block k and tend to 0. The cubes sum to $\sum 6/k = \infty$. p. 365

10.6.11 The block sum is at most $2Ka_{n+1}$ in absolute value. For Theorem 10.6.2 take $b_k = (-1)^{k+1}$, with $|B_n| \leq 1$. p. 365

11.1 Power Series

11.1.1 $R = \infty$. The interval is $(-\infty, \infty)$. p. 372

11.1.2 $R = 2$. The ratios $|c_{n+1}/c_n|$ tend to 0 along even n and to ∞ along odd n. p. 372

11.1.3 $(-1, 1]$. p. 372

11.1.4 $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$. The sum is 2. p. 372

11.1.5 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)2^{4n+1}} \approx 0.49396$, from four terms with error below 5×10^{-7}. p. 372

11.1.6 The series in x^2 converges when $x^2 < R$ and diverges when $x^2 > R$. p. 372

11.1.7 $\sum 2c_{2m+1}x^{2m+1} = 0$ on $(-R, R)$ and its kth derivative at 0 is $k!$ times its kth coefficient. p. 372

11.1.8 $M_n = 1/n^3$ and $M_n = 1/n^2$ serve on every interval. p. 372

11.1.9 The original series converges on $[-1, 1]$ and the derived series on $[-1, 1)$. p. 372

11.1.10 $\int_0^1 f_n = 1 - (n+1)e^{-n} \to 1 \neq 0$. The maximum $f_n(1/n) = n/e$ is unbounded and the convergence is therefore not uniform. p. 372

11.2 Taylor and Maclaurin Series

11.2.1 $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \cdots$, for all real x. p. 376

11.2.2 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} = x - \frac{x^5}{2} + \frac{x^9}{24} - \cdots$, for all real x. p. 376

11.2.3 $\arctan \frac{1}{2} \approx 0.4636$, from six terms. p. 376

11.2.4 $x + x^2 + \dfrac{x^3}{3} + 0 \cdot x^4 + \cdots$. p. 376

11.2.5 $\sin^2 x = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots$, for all real x. p. 376

11.2.6 $(-1)^k f^{(k)}(-x) = -f^{(k)}(x)$ and hence $f^{(k)}(0) = 0$ for even k. p. 376

11.2.7 The derived series is the same series, reindexed. p. 376

11.2.8 That coefficient equals $(x+y)^n/n!$. p. 376

11.2.9 $\sum(-1)^n x^{2n}$ with $R=1$. The series must diverge at $x=\pm1$ because its terms have size 1. p. 376

11.2.10 $x=2$ lies outside $(-1,1]$ and the terms $2^n/n$ do not tend to 0. p. 376

11.3 Convergence of Taylor Series

11.3.1 $P_3(0.3)=0.2955$, with $|R_3|\le 0.3^4/4!=0.0003375$. p. 380

11.3.2 Equation (11.3.2) at $n=2$ with $M_2=1$. p. 380

11.3.3 $n=9$ suffices: five nonzero terms, through x^8. p. 380

11.3.4 $\ln(1.2)\approx 0.1823$ from four terms, with error below 0.000064. p. 380

11.3.5 Both keep every term through x^{13}: seven nonzero terms. p. 380

11.3.6 It is $\operatorname{Re}\exp(\exp(i\theta))=\operatorname{Re}(e^{\cos\theta}\exp(i\sin\theta))$. p. 381

11.3.7 $|R_n(x)|\le e^{|a|+|x-a|}|x-a|^{n+1}/(n+1)!\to 0$. p. 381

11.3.8 $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. p. 381

11.3.9 e^x and $e^x+\phi(x)$. Only e^x is analytic at 0. p. 381

11.3.10 All derivatives of g vanish at 0 from both sides. Since an analytic function with all derivatives zero at a point vanishes near that point, the zero set of a nonzero analytic function contains no interval. p. 381

11.4 Binomial Series and Applications of Taylor Series

11.4.1 $1+\dfrac{x}{3}-\dfrac{x^2}{9}+\cdots$, for $|x|<1$. p. 384

11.4.2 0.46128, from the five terms through $n=4$, with error below $u_5\approx 3.7\times10^{-7}$. p. 384

11.4.3 0.7468, from the terms through $n=6$, with error below 1.4×10^{-5}. p. 384

11.4.4 $\arcsin x=x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+\cdots$ for $|x|<1$. p. 385

11.4.5 $\binom{-1/2}{k}=(-1)^k\dfrac{(2k)!}{4^k(k!)^2}$. p. 385

11.4.6 $(m-k)\binom{m}{k}+k\binom{m}{k}=m\binom{m}{k}$. p. 385

11.4.7 $\frac{1}{24}$. For $f(x)=x$ and $g(x)=x^2$ the quotient $1/x$ has no limit. p. 385

11.4.8 The series converges at $x=1$ by Theorem 10.6.2 and Abel's theorem gives the sum $\lim_{x\to1^-}\sqrt{1+x}=\sqrt{2}$. p. 385

11.4.9 For $m>0$ the numbers ku_k decrease for $k>m$ and $m\sum u_k$ telescopes. For $-1<m<0$ the series at $x=1$ alternates with sizes decreasing to 0, and at $x=-1$ it has terms $u_k\ge u_1/k$. For $m\le-1$ the sizes satisfy $u_k\ge 1$. p. 385

11.4.10 $\frac{1}{2}$. L'Hôpital's rule needs two applications. p. 385

11.5 Analytic Functions and the Shift Operator

11.5.1 $\dfrac{1}{1-x}=\displaystyle\sum_{k=0}^{\infty}\frac{2^{k+1}}{3^{k+1}}\Big(x+\frac{1}{2}\Big)^k$ with radius $\frac{3}{2}$, larger than the guaranteed $\frac{1}{2}$. p. 389

11.5.2 $\sum_{k\ge1}(-1)^{k+1}h^k/k=\ln(1+h)=f(h)$. p. 389

11.5.3 For fixed m the inner geometric series is $1/(m(m-1))$ and these telescope to 1. p. 389

11.5.4 $e^{x+y}=\sum y^k e^x/k!=e^x e^y$. p. 389

11.5.5 The set of points of $(-R,R)$ beyond which f vanishes on a whole interval extends to both ends of $(-R,R)$. p. 389

11.5.6 The square sums are 1. Since the rows have $\sum_n|a_{mn}|=2$, $\sum_m\sum_n|a_{mn}|$ diverges. p. 389

11.5.7 $\limsup|c_k|^{1/k}=(1+b^2)^{-1/2}$. p. 389

Appendix D

Solutions

1.1 Statements and Connectives

1.1.1 (a) The hypothesis $3 < 2$ is false and the implication is vacuously true. (b) The hypothesis is true and the conclusion false, the one case in which an implication is false. (c) The hypothesis is false and the implication is true. (d) Both sides are true and the biconditional is true. p. 13

1.1.2 If $x > 3$ then $x > 0$ and $x^2 = x \cdot x > 3x > 9$. The implication is therefore true for every real x. By Theorem 1.1.10 the contrapositive is true as well. The converse is false at $x = -4$ because $x^2 = 16 > 9$ and $-4 \leq 3$. The negation $x > 3 \wedge x^2 \leq 9$ is false for every x because the implication is true for every x. p. 13

1.1.3 In the order of Table 1.1.1 the column of $P \implies Q$ is $\mathrm{T}, \mathrm{F}, \mathrm{T}, \mathrm{T}$. Applying the implication rule to that column and the column $\mathrm{T}, \mathrm{T}, \mathrm{F}, \mathrm{F}$ of P gives $\mathrm{T} \implies \mathrm{T} = \mathrm{T}$, $\mathrm{F} \implies \mathrm{T} = \mathrm{T}$, $\mathrm{T} \implies \mathrm{F} = \mathrm{F}$ and $\mathrm{T} \implies \mathrm{F} = \mathrm{F}$. The result agrees with the column of P in every row. p. 13

1.1.4 The statement $P \implies (Q \implies R)$ is false only when P is true and $Q \implies R$ is false, that is, when P and Q are true and R is false. The statement $(P \wedge Q) \implies R$ is false only when $P \wedge Q$ is true and R false, the same single row. The two statements are therefore equivalent.

With $P = Q = R = \mathrm{F}$ the implication $P \implies Q$ is true and $(P \implies Q) \implies R$ is the false implication $\mathrm{T} \implies \mathrm{F}$. In the same row $P \implies (Q \implies R)$ has a false hypothesis and is true. Implication is therefore not associative. A chain such as $P \implies Q \implies R$ must be bracketed. p. 13

1.1.5 (a) If $n = 4k$ then $n = 2(2k)$ is even. (b) The implication "if n is even then 4 divides n" fails at $n = 2$. (c) "P only if Q" is $P \implies Q$. If $n = 2k$ then $n^2 = 2(2k^2)$ is even. p. 13

1.1.6 By the second part of Theorem 1.1.10 applied to $\neg P$ and Q, $\neg P \implies Q \equiv \neg\neg P \vee Q \equiv P \vee Q$. By the third part applied to P and $\neg Q$, $\neg(P \implies \neg Q) \equiv P \wedge \neg\neg Q \equiv P \wedge Q$. Double negation is the first law of Theorem 1.1.11. p. 13

1.2 Quantifiers

1.2.1 (a) Given x take $y = -x$. (b) If one y worked for every x, then $x = 0$ would give $y = 0$ and $x = 1$ would give $y = -1$. This is a contradiction. (c) The counterexample $x = 0$ gives $0 \cdot y = 0 \neq 1$ for every y. (d) Take $x = 0$. p. 17

1.2.2 (a) The negation says some real number is an upper bound of $\mathbb{N}$. The original statement is the Archimedean property (Theorem 2.4.10) and is true. (b) The original is true with $x = 0$ since $y^2 \geq 0$ for every real y. Its negation fails at $x = 0$ because no real y has $y^2 < 0$. p. 18

1.2.3 Flipping the quantifier and negating the implication gives $\exists x, y \in I\ (x < y \wedge f(x) \geq f(y))$. With $x = -1$ and $y = 0$ we have $x < y$ and $f(x) = 1 \geq 0 = f(y)$. p. 18

1.2.4 For $n \in \mathbb{N}$, $1/n < \varepsilon$ is equivalent to $n > 1/\varepsilon$. With $\varepsilon = \frac{1}{100}$ the condition is $n > 100$. The least integer N that works is 100: $N = 99$ fails at $n = 100$, where $1/n = \varepsilon$. With $\varepsilon = 0.003$ the condition is $n > 1/0.003 = 333.\overline{3}$, that is, $n \geq 334$. The least N is 333. p. 18

1.2.5 We prove the negation of convergence with $\varepsilon = \frac{1}{4}$. Let N be given and choose $n = N + 4$. Then $n > N$ and $n \geq 4$, hence $1/n \leq \frac{1}{4}$ and $|1/n - \frac{1}{2}| = \frac{1}{2} - 1/n \geq \frac{1}{4} = \varepsilon$. p. 18

1.2.6 Flipping the outer quantifier and applying De Morgan's law gives $\forall x \in S\,(\neg P(x) \vee \neg \forall y \in S\,(P(y) \implies y = x))$. The inner negation is $\exists y \in S\ (P(y) \wedge y \neq x)$. In words: every element either fails P or is accompanied by a different element satisfying P. Over $\mathbb{R}$ both 2 and -2 satisfy $x^2 = 4$ and uniqueness fails. Over $[0, \infty)$ the number 2 satisfies it. If $y \geq 0$ and $y^2 = 4$ then $(y-2)(y+2) = 0$. Since $y + 2 > 0$, the factor $y - 2$ is 0 and $y = 2$. p. 18

1.3 Methods of Proof

1.3.1 Let $m = 2k + 1$ and $n = 2l + 1$ with integers k and l. Then $mn = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$ is odd and $m + n = 2(k + l + 1)$ is even. p. 22

1.3.2 By Lemma 1.3.3 the contrapositive reads: if n is even then n^2 is even. With $n = 2k$, $n^2 = 2(2k^2)$ is even. The original implication follows by Theorem 1.1.10. p. 22

1.3.3 (a) $x = -2$ gives $x^2 = 4 > 1$ and $x \leq 1$. (b) $9 = 3 \cdot 3$ is odd, greater than 1 and not prime. (c) $a = b = 1$ gives $\sqrt{2}$ on the left and 2 on the right. Here $\sqrt{2} \neq 2$ because $(\sqrt{2})^2 = 2 \neq 4$. p. 22

1.3.4 The difference $a^2 + b^2 - 2ab = (a-b)^2$ is a square and hence nonnegative. It is zero exactly when $a = b$. For $a, b \geq 0$ the real numbers $\sqrt{a}$ and $\sqrt{b}$ exist by Theorem 2.4.8 and the first part gives $a + b \geq 2\sqrt{a}\sqrt{b}$. The product $\sqrt{a}\sqrt{b}$ is nonnegative with square ab and equals $\sqrt{ab}$ by the uniqueness in Theorem 2.4.8. Dividing by 2 gives the inequality between the arithmetic and geometric means. p. 22

1.3.5 If $n = 2k$ then $3n^2 + n + 14 = 2(6k^2 + k + 7)$. If n is odd then n^2 and $3n^2$ are odd by Problem 1.3.1. The sum $3n^2 + n$ of two odd integers is even by the same problem. Adding 14 gives the even number $3n^2 + n + 14$. Every integer falls in one of the two cases by Lemma 1.3.3. p. 23

1.3.6 Existence: $2^3 = 8$. Uniqueness: if $x^3 = 8$ then $0 = x^3 - 8 = (x-2)(x^2 + 2x + 4)$. The second factor equals $(x+1)^2 + 3 \geq 3$ and is not zero. The first factor is therefore zero and $x = 2$. p. 23

1.4 Mathematical Induction

1.4.1 For $n = 1$ both sides equal 1. Suppose the formula is true for k. Then

$$\sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6},$$

because $2k^2 + 7k + 6 = (k+2)(2k+3)$. The result is the formula at $k + 1$. p. 29

1.4.2 For $n = 1$ both sides equal 1. If the formula is true for k, then

$$\sum_{j=1}^{k+1} j^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2,$$

which is the formula at $k+1$. By Theorem 1.4.6 the right-hand side is the square of $1+2+\cdots+n$. p. 29

1.4.3 For $n=1$, $4-1=3$. Suppose $3 \mid (4^k-1)$. Then $4^{k+1}-1=4(4^k-1)+3$. By Theorem 1.3.5 with $a=3$, $b=4^k-1$, $c=3$, $x=4$, $y=1$, the number 3 divides it. p. 29

1.4.4 For $n=0$ both sides equal 1. If the formula is true for $k \geq 0$, then

$$1+r+\cdots+r^{k+1}=\frac{1-r^{k+1}}{1-r}+r^{k+1}=\frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}=\frac{1-r^{k+2}}{1-r}.$$

With $r=3$ and $n=5$ the sum is $(1-3^6)/(1-3)=728/2=364$. p. 29

1.4.5 The inductive step is correct as an implication. The base case was never checked. In fact $1^2+1=2$ is even. By Example 1.3.13, n^2+n is even for every integer n. The claim is false everywhere. The step proves only that the claim would propagate from k to $k+1$ if it were ever true. p. 29

1.4.6 We use strong induction from the base 12 (Corollary 1.4.9). The four cases $12=4\cdot3+5\cdot0$, $13=4\cdot2+5\cdot1$, $14=4\cdot1+5\cdot2$ and $15=4\cdot0+5\cdot3$ are checked directly. Let $k \geq 15$ and suppose every integer m with $12 \leq m \leq k$ has a representation. Then $k+1 \geq 16$ and $12 \leq k+1-4 \leq k$. The representation $k+1-4=4a+5b$ gives $k+1=4(a+1)+5b$. The four base cases are needed because the step reaches back four places. p. 29

1.4.7 Division with remainder gives $120=1\cdot84+36$ and $84=2\cdot36+12$. Substituting back,

$$12=84-2\cdot36=84-2(120-84)=3\cdot84-2\cdot120.$$

The number 12 divides $84=12\cdot7$ and $120=12\cdot10$. Every common divisor of 84 and 120 divides $3\cdot84-2\cdot120=12$ by Theorem 1.3.5 and is at most 12. Hence $\gcd(84,120)=12$ with $x=3$ and $y=-2$. p. 29

1.4.8 Suppose $p \nmid a$. A positive common divisor of p and a is a positive divisor of p and is therefore 1 or p. It is not p because $p \nmid a$. Hence p and a are coprime. Euclid's lemma (Theorem 1.4.15) applied to $p \mid ab$ gives $p \mid b$. p. 29

1.4.9 Write $c=ak$ with k an integer. Since $b \mid ak$ with b and a coprime, Theorem 1.4.15 gives $b \mid k$. Writing $k=bl$ gives $c=abl$ and $ab \mid c$. Without the hypothesis the conclusion can fail: $2 \mid 2$ and $2 \mid 2$ while $2\cdot2=4 \nmid 2$. p. 29

1.5 Sets and Functions

1.5.1 $A \cup B=\{1,2,3,4,5\}$, $A \cap B=\{3,4\}$, $A \setminus B=\{1,2\}$ and $B^c=U \setminus B=\{1,2,6\}$. The product $A \times B$ consists of the pairs (a,b) with four choices of a and three of b, $4\cdot3=12$ pairs in all. p. 36

1.5.2 (a) $2n_1=2n_2$ gives $n_1=n_2$. The odd integer 1 is not a value by Lemma 1.3.3. (b) $g(-1)=g(1)$ while every $y \geq 0$ equals $g(\sqrt{y})$ by Theorem 2.4.8. (c) $n_1+1=n_2+1$ gives $n_1=n_2$. The number 1 is not a value because $n+1 \geq 2$. (d) For each y the equation $5-2x=y$ has exactly one solution $x=(5-y)/2$. p. 36

1.5.3 With $f(x)=(x-2)^2-1$, on $[0,3]$ the square $(x-2)^2$ takes every value from 0 (at $x=2$) to 4 (at $x=0$). Hence $f([0,3])=[-1,3]$. The equation $f(x)=0$ is $(x-1)(x-3)=0$ with solution set $\{1,3\}$. The condition $-1 \leq f(x) \leq 0$ is $0 \leq (x-2)^2 \leq 1$, that is, $|x-2| \leq 1$, giving $[1,3]$. p. 36

1.5.4 The equation $f(x) = 1$ would give $x + 2 = x - 1$. Hence f maps $\mathbb{R} \setminus \{1\}$ into itself. For $y \neq 1$, $y(x-1) = x+2$ is equivalent to $x(y-1) = y+2$ and to $x = (y+2)/(y-1)$. This x differs from 1 because $y+2 \neq y-1$. Each $y \neq 1$ has exactly one preimage, and $f^{-1}(y) = (y+2)/(y-1)$. The inverse has the same formula as f, and $f \circ f$ is the identity. p. 36

1.5.5 For $x \geq 0$, $(g \circ f)(x) = 1 - (\sqrt{x})^2 = 1 - x$. The formula $1 - x$ makes sense for every real x. The composition is defined only on the domain $[0, \infty)$ of f. The composition $(f \circ g)(x) = \sqrt{1 - x^2}$ is defined when $1 - x^2 \geq 0$, that is, on $[-1, 1]$. p. 36

1.5.6 If $f(x_1) = f(x_2)$, applying g gives $g(f(x_1)) = g(f(x_2))$, and injectivity of $g \circ f$ gives $x_1 = x_2$. For the example take $A = \{0\}$, $B = C = \mathbb{R}$, $f(0) = 0$ and $g(x) = x^2$. Every function on a one-element set is injective, $g \circ f$ among them. The function g has $g(-1) = g(1)$. p. 36

1.5.7 A value y lies in $f(S \cup T)$ exactly when $y = f(x)$ for some x in S or in T, that is, exactly when $y \in f(S) \cup f(T)$. If $y = f(x)$ with $x \in S \cap T$, then $y \in f(S)$ and $y \in f(T)$. In the example $S \cap T = \{0\}$ and $f(S \cap T) = \{0\}$ while $f(S) = f(T) = [0, 1]$. If f is injective and $y \in f(S) \cap f(T)$, write $y = f(s) = f(t)$ with $s \in S$ and $t \in T$. Injectivity gives $s = t \in S \cap T$. Hence $y \in f(S \cap T)$. p. 37

2.1 Number Systems

2.1.1 With $x = 0.\overline{27}$ we have $100x = 27.\overline{27}$. Subtracting gives $99x = 27$ and hence $x = 27/99 = 3/11$. p. 43

2.1.2 Long division of 5 by 12 gives the digits 4 (remainder 2), 1 (remainder 8), 6 (remainder 8) and the remainder 8 then recurs. Hence $5/12 = 0.41\overline{6}$. As a check, $\frac{41}{100} + \frac{6}{900} = \frac{123+2}{300} = \frac{5}{12}$. p. 43

2.1.3 (a) The conditions $-3 \leq x$ and $x < 5$ describe $[-3, 5)$. (b) The set contains -1 and 1 and misses 0. It is therefore the union $(-\infty, 0) \cup (0, \infty)$ of two intervals and no interval itself. (c) $3 - x > 1$ is equivalent to $2 > x$. The set is $(-\infty, 2)$. p. 43

2.1.4 If $m = 3k+1$ then $m^2 = 3(3k^2+2k)+1$. If $m = 3k+2$ then $m^2 = 3(3k^2+4k+1)+1$. In both cases m^2 leaves remainder 1 on division by 3. Hence if 3 divides m^2 then $m = 3k$. Suppose $(m/n)^2 = 3$ with $m, n \in \mathbb{N}$ and n least. Then $m^2 = 3n^2$ and 3 divides m. Writing $m = 3k$ gives $9k^2 = 3n^2$ and $n^2 = 3k^2$. Hence 3 divides n as well. Then $m/n = k/(n/3)$ has a smaller denominator. This is a contradiction. p. 43

2.1.5 Suppose $s+t = r$ is rational. Then $r^2 = s^2+2st+t^2 = 5+2st$ and $st = (r^2-5)/2$ is rational. Its square is $(st)^2 = s^2t^2 = 6$. Because 6 is not the square of an integer, Proposition 2.1.4 shows that no rational number has square 6. This contradiction proves that $s + t$ is irrational. p. 43

2.1.6 Long division gives the remainders $1, 3, 2, 6, 4, 5, 1, \ldots$ and the digits $1, 4, 2, 8, 5, 7$ before the remainder 1 recurs. Hence $1/7 = 0.\overline{142857}$ with period $6 \leq 7$. As a check, $142857 \cdot 7 = 999999$ gives $142857/999999 = 1/7$. p. 43

2.1.7 If $n \geq 2$ then n is a product of primes by Theorem 1.4.10. Each of them divides n and is therefore 2 or 5. Hence $n = 2^a 5^b$ with integers $a, b \geq 0$. The case $n = 1$ has this form with $a = b = 0$. Let c be the larger of a and b. Then

$$\frac{m}{n} = \frac{m\,2^{c-a}5^{c-b}}{2^c 5^c} = \frac{N}{10^c}$$

with $N = m\,2^{c-a}5^{c-b}$ an integer. Dividing an integer by 10^c moves the decimal point c places to the left. The expansion therefore terminates after at most c digits. p. 43

2.2 Field and Order Axioms

2.2.1 By Proposition 2.2.4(iii), (F1) and (F2), $a(-b) = a((-1)b) = (-1)(ab) = -(ab)$. p. 47

2.2.2 Because $\mathbb{N}$ has no additive identity and no additive inverses, (F4) and (F5) fail and $x + 5 = 3$ has no solution in $\mathbb{N}$. In $\mathbb{Z}$ every axiom except the multiplicative half of (F5) is true. The number 2 has no inverse because $2x$ is even for every integer x and $2x = 1$ has no integer solution. p. 47

2.2.3 By (F3) and (F1), $(a+b)^2 = (a+b)a + (a+b)b = a^2 + ba + ab + b^2$. By (F1), $ba = ab$. By (F4) and (F3), $ab + ab = 1 \cdot ab + 1 \cdot ab = (1+1)ab = 2ab$. The last step is where $2 = 1 + 1$ enters. With (F2) to regroup the sum, $(a+b)^2 = a^2 + 2ab + b^2$. p. 47

2.2.4 By Proposition 2.2.5(vi), $a^2 < b^2$. Since $a \geq 0$ we have $a^2 \cdot a \leq b^2 \cdot a$ (equality when $a = 0$, (O4) otherwise). Since $b^2 > 0$, (O4) applied to $a < b$ gives $b^2 a < b^2 b$. Combining, $a^3 \leq ab^2 < b^3$. p. 47

2.2.5 With $a = c = -3$ and $b = d = -1$ we have $a < b$ and $c < d$ but $ac = 9$ and $bd = 1$. Now suppose $0 \leq a < b$ and $0 \leq c < d$. Then $b > 0$ and (O4) gives $bc < bd$. Also $ac \leq bc$, with equality when $c = 0$ and by (O4) when $c > 0$. Hence $ac \leq bc < bd$. p. 47

2.2.6 By Proposition 2.2.5(iii), $(a-b)^2 \geq 0$ with equality exactly when $a = b$. Expanding gives $a^2 + b^2 \geq 2ab$. Multiplying by the positive number $(ab)^{-1}$ gives $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ with equality exactly when $a = b$. p. 47

2.2.7 Suppose $a > b$. Then $\varepsilon = (a-b)/2$ is positive and the hypothesis gives $a \leq b + (a-b)/2$. Multiplying by 2 gives $2a \leq 2b + a - b = a + b$ and hence $a \leq b$ in contradiction to $a > b$. Hence $a \leq b$. This argument is used repeatedly in Chapter 5 to show that two limits are equal. p. 47

2.3 Inequalities and Absolute Value

2.3.1 $x^2 - x - 6 = (x-3)(x+2)$. The two factors have opposite signs exactly on $(-2, 3)$ by Lemma 2.3.2. The product is negative exactly there. p. 53

2.3.2 $3x - 2 > 7$ gives $x > 3$. $3x - 2 < -7$ gives $x < -\frac{5}{3}$. The solution set is $(-\infty, -\frac{5}{3}) \cup (3, \infty)$. p. 53

2.3.3 The critical points are -3, 1 and 2. On $(-\infty, -3)$ the signs are $(+)(-)/(-)$, positive. On $(-3, 1)$ and on $(1, 2)$ they are $(+)(+)/(-)$, negative. On $(2, \infty)$ the quotient is positive. Because the inequality is weak the zeros -3 and 1 of the numerator are included. The zero 2 of the denominator is excluded. The solution set is $[-3, 1] \cup [1, 2) = [-3, 2)$. p. 53

2.3.4 $\dfrac{1}{x} < x$ is equivalent to $x - \dfrac{1}{x} = \dfrac{(x-1)(x+1)}{x} > 0$. The critical points are -1, 0 and 1. The signs on the four intervals are $-$, $+$, $-$, $+$. The solution set is $(-1, 0) \cup (1, \infty)$. p. 53

2.3.5 Since $|2x - 6| = 2|x - 3|$, the condition $|2x - 6| < 0.1$ is equivalent to $|x - 3| < 0.05$. Any $\delta \leq 0.05$ works. A larger δ admits $x = 3 + 0.05$ with $|2x - 6| = 0.1$. The largest value is $\delta = 0.05$. p. 53

2.3.6 By Corollary 2.3.9, $1 - |x| \leq \big||x| - 1\big| \leq |x - 1| < \frac{1}{2}$, which gives $|x| > \frac{1}{2}$ and in particular $x \neq 0$. Then

$$\left|\frac{1}{x} - 1\right| = \frac{|1 - x|}{|x|} \leq \frac{|x - 1|}{1/2} = 2|x - 1|,$$

using Proposition 2.3.6(ii) for the quotient and Proposition 2.2.5(iv) to invert $|x| > \frac{1}{2}$. The inequality is strict for $x \neq 1$. Both sides are 0 at $x = 1$. p. 53

2.3.7 For $n = 1$ both sides equal $1 + x$. Assume the inequality for n. Since $1 + x > 0$,

$$(1+x)^{n+1} \geq \Big(1 + nx + \frac{n(n-1)}{2}x^2\Big)(1+x) = 1 + (n+1)x + \Big(\frac{n(n-1)}{2} + n\Big)x^2 + \frac{n(n-1)}{2}x^3.$$

The last term is nonnegative because $x \geq 0$. The coefficient of x^2 is $\dfrac{n(n-1)}{2} + n = \dfrac{(n+1)n}{2}$. Hence $(1+x)^{n+1} \geq 1 + (n+1)x + \dfrac{(n+1)n}{2}x^2$, the statement for $n+1$. p. 53

2.3.8 Let the sides be $a, b > 0$ with $ab = 16$. By Theorem 2.3.12, $16 \leq \left(\frac{a+b}{2}\right)^2$. Since $\frac{a+b}{2}$ and 4 are positive, Proposition 2.2.5(vi) gives $\frac{a+b}{2} \geq 4$ and hence the perimeter bound $2(a+b) \geq 16$. Equality requires $a = b$. With $ab = 16$ this gives $a = b = 4$. p. 53

2.4 Completeness

2.4.1 The number 3 belongs to the set and is an upper bound. Hence sup = max = 3. The number -1 is a lower bound. For $\varepsilon > 0$ the element $\min(-1 + \frac{\varepsilon}{2}, 3)$ of the set is less than $-1 + \varepsilon$. The infimum form of Lemma 2.4.4 gives inf $= -1$. There is no minimum since -1 is not in the set. p. 61

2.4.2 Since $\frac{1}{n} \leq 1 = \frac{1}{1}$, $\max S = 1 = \sup S$. Every element is positive. Hence 0 is a lower bound. For $\varepsilon > 0$, Theorem 2.4.10 gives n with $\frac{1}{n} < \varepsilon = 0 + \varepsilon$. The infimum form of Lemma 2.4.4 gives $\inf S = 0$. It is not attained because $\frac{1}{n} > 0$ for every n. p. 61

2.4.3 For even n the element is $1 + \frac{1}{n} \leq 1 + \frac{1}{2} = \frac{3}{2}$, with equality at $n = 2$. For odd n it is $-1 + \frac{1}{n} \leq 0$. Hence $\max S = \sup S = \frac{3}{2}$. Every element exceeds -1. Given $\varepsilon > 0$, Theorem 2.4.10 gives $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$. The number $n = 2k + 1$ is odd and $\frac{1}{n} < \frac{1}{k} < \varepsilon$. The element $-1 + \frac{1}{n}$ is therefore less than $-1 + \varepsilon$. The infimum form of Lemma 2.4.4 shows that $\inf S = -1$, a number that S never attains. p. 61

2.4.4 Every $x \in S$ lies in T and satisfies $x \leq \sup T$. Thus $\sup T$ is an upper bound of S. The supremum $\sup S$ exists by Assumption 2.4.5. Because it is the least upper bound, $\sup S \leq \sup T$. p. 61

2.4.5 Each $b \in B$ is an upper bound of A. Hence $\sup A$ exists by Assumption 2.4.5 and $\sup A \leq b$ for every $b \in B$. Then $\sup A$ is a lower bound of B. The infimum $\inf B$ exists by Proposition 2.4.6, and $\sup A \leq \inf B$ because the infimum is the greatest lower bound. p. 61

2.4.6 The number $\sqrt{a}\sqrt{b}$ is nonnegative and its square is $(\sqrt{a})^2(\sqrt{b})^2 = ab$ by (F1) and (F2). The uniqueness in Theorem 2.4.8 gives $\sqrt{ab} = \sqrt{a}\sqrt{b}$. If $a < b$ then $(\sqrt{a})^2 < (\sqrt{b})^2$ with both roots nonnegative. The converse half of Proposition 2.2.5(vi) gives $\sqrt{a} < \sqrt{b}$. By Theorem 2.3.12, $ab \leq \left(\frac{a+b}{2}\right)^2$. The monotonicity of square roots proved above extends to $\leq$ and gives $\sqrt{ab} \leq \sqrt{\left(\frac{a+b}{2}\right)^2} = \frac{a+b}{2}$. p. 61

2.4.7 The kth truncation is $t_k = \dfrac{3}{10} + \cdots + \dfrac{3}{10^k} = \dfrac{1 - 10^{-k}}{3}$, as in Worked Example 2.4.2. Hence $\frac{1}{3} - t_k = \dfrac{1}{3 \cdot 10^k} > 0$ and $\frac{1}{3}$ is an upper bound. Given $\varepsilon > 0$ choose k with $10^k > k > \dfrac{1}{3\varepsilon}$ by Theorem 2.4.10 and Bernoulli's inequality. Then $t_k > \frac{1}{3} - \varepsilon$. By Lemma 2.4.4, $0.\overline{3} = \sup_k t_k = \frac{1}{3}$. p. 61

3.1 Vectors in the Line and the Plane

3.1.1 Componentwise, $\mathbf{u}+\mathbf{v}=(2-3,\,-1+4)=(-1,3)$ and $3\mathbf{u}-2\mathbf{v}=(6,-3)-(-6,8)=(12,-11)$. Adding $-\mathbf{u}$ to both sides of $\mathbf{u}+\mathbf{w}=\mathbf{v}$ and using Proposition 3.1.4 gives $\mathbf{w}=\mathbf{v}-\mathbf{u}=(-5,5)$. p. 67

3.1.2 The displacement is $\overrightarrow{PQ}=(6,-3)$. The points are $P+\frac{1}{3}\overrightarrow{PQ}=(-1,3)+(2,-1)=(1,2)$ and $P+\frac{2}{3}\overrightarrow{PQ}=(-1,3)+(4,-2)=(3,1)$. p. 67

3.1.3 The centroid is $\frac{1}{3}(6,6)=(2,2)$. The midpoint of BC is $M=(3,3)$. Two thirds of the way from A to M is $A+\frac{2}{3}(M-A)=(2,2)$, the centroid. p. 67

3.1.4 By Definition 3.1.3 the left side is $((c+d)v_1,(c+d)v_2)$ and the right side is $(cv_1+dv_1,\,cv_2+dv_2)$. By commutativity of multiplication and distributivity in Assumption 2.2.1, $(c+d)v_1=v_1(c+d)=v_1c+v_1d=cv_1+dv_1$. The second components agree in the same way. Two pairs with equal components are equal. p. 67

3.1.5 By Proposition 3.1.4, $\overrightarrow{MN}=N-M=\frac{1}{2}(A+C)-\frac{1}{2}(A+B)=\frac{1}{2}(C-B)=\frac{1}{2}\overrightarrow{BC}$. A positive multiple of a vector points the same way and the factor $\frac{1}{2}$ halves the length. p. 67

3.1.6 Move the origin so that each point P acquires the coordinates $P-\mathbf{a}$ while vectors are unchanged. (a) $(P-\mathbf{a})+\mathbf{v}$ is the old value minus $\mathbf{a}$: a point. (b) The coefficients 2 and -1 add to 1: a point. (c) The coefficients add to 2 and the value changes by $-2\mathbf{a}$: neither. (d) $Q-P$ is unchanged and so is $\mathbf{v}$: a vector. p. 67

3.1.7 The endpoint $2+v=5$ gives $v=3$. The displacement $-3\mathbf{v}$ is $(2,-9)$ with endpoint $2-9=-7$. The sum in Definition 3.1.8 is defined only for two displacements with the same base point. Here the base points 2 and 5 differ. p. 67

3.2 Vector Spaces

3.2.1 (a) It contains $\mathbf{0}$. If $x+2y=0$ and $x'+2y'=0$ then $(x+x')+2(y+y')=0$ and $cx+2cy=0$. It is a subspace. (b) It misses $\mathbf{0}$. (c) It contains $(1,0)$ and $(0,-1)$ and not their sum $(1,-1)$. (d) It contains $(1,1)$ and not $2(1,1)=(2,2)$ since $2\neq 4$. p. 74

3.2.2 The differences of consecutive vectors are both $(3,3,3)$. Hence $(7,8,9)-(4,5,6)=(4,5,6)-(1,2,3)$. Thus $(1,2,3)-2(4,5,6)+(7,8,9)=\mathbf{0}$ is a combination with nonzero coefficients and the list is dependent. p. 74

3.2.3 A combination $a+b(1+x)+c(1+x+x^2)$ has coefficients $a+b+c$, $b+c$, c of 1, x, x^2. Matching gives $c=-2$, $b=1-c=3$, $a=5-b-c=4$. As a check, $4+3+3x-2-2x-2x^2=5+x-2x^2$. p. 75

3.2.4 Since U and W both contain $\mathbf{0}$, so does $U\cap W$. If $\mathbf{u},\mathbf{v}\in U\cap W$, then $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ lie in U and in W by closure of each, hence in $U\cap W$. For the example let U be the x-axis and W the y-axis. Their union contains $(1,0)$ and $(0,1)$ and not $(1,1)$. p. 75

3.2.5 Suppose $a+bx+c|x|=0$ for every real x. At $x=0$ this gives $a=0$. At $x=1$ and $x=-1$ it gives $b+c=0$ and $-b+c=0$. Adding gives $c=0$ and then $b=0$. p. 75

3.2.6 The zero polynomial and every sum and multiple of polynomials vanishing at 1 vanish at 1. If $p=a_0+a_1x+a_2x^2+a_3x^3\in W$, then $p(x)=p(x)-p(1)=a_1(x-1)+a_2(x^2-1)+a_3(x^3-1)$. The three polynomials span W. If $b_1(x-1)+b_2(x^2-1)+b_3(x^3-1)=0$, the coefficients of x, x^2, x^3 give $b_1=b_2=b_3=0$ by Example 3.2.19. The list is a basis and $\dim W=3$. p. 75

3.2.7 Since $c_1 \neq 0$, $\mathbf{v}_1 = c_1^{-1}(\mathbf{w} - \sum_{i\geq 2} c_i\mathbf{v}_i)$ lies in the span of the new list. That span contains every $\mathbf{v}_i$ and hence V (Proposition 3.2.17). If $a\mathbf{w} + \sum_{i\geq 2} a_i\mathbf{v}_i = \mathbf{0}$, then $ac_1\mathbf{v}_1 + \sum_{i\geq 2}(ac_i + a_i)\mathbf{v}_i = \mathbf{0}$. Independence of the old basis gives $ac_1 = 0$, hence $a = 0$, and then every $a_i = 0$. p. 75

3.2.8 The span of x^2, x^2+x, x^2+1 contains $x = (x^2+x)-x^2$ and $1 = (x^2+1)-x^2$. It contains $1, x, x^2$ and all of $\mathcal{P}_2$ (Proposition 3.2.17). Three spanning vectors of the three-dimensional space $\mathcal{P}_2$ form a basis by Theorem 3.2.27(2). A polynomial $a_0+a_1x+a_2x^2$ vanishes at 0 exactly when $a_0 = 0$. These polynomials form the subspace $W = \operatorname{span}(x, x^2)$ of dimension 2. Any three of them are linearly dependent by Theorem 3.2.27(1) applied in W. No three of them form a basis of $\mathcal{P}_2$. p. 75

3.3 Linear Maps and Matrices

3.3.1 (a) and (d) have components that are fixed combinations of x and y; they are given by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$. (b) sends $\mathbf{0}$ to $(1,0)$. (c) sends $(1,0)$ to $(1,0)$ and $(2,0)$ to $(4,0) \neq 2(1,0)$. p. 82

3.3.2 The columns are $T\mathbf{e}_1 = (1,0)$, $T\mathbf{e}_2 = (1,1)$, $T\mathbf{e}_3 = (0,1)$. The kernel equations give $y = -x$ and $z = x$. Hence $\ker T = \operatorname{span}\big((1,-1,1)\big)$ of dimension 1. The image contains $(1,0)$ and $(0,1)$, hence all of $\mathbb{R}^2$, of dimension 2. The sum is $3 = \dim \mathbb{R}^3$. p. 82

3.3.3 Here $ad - bc = 6 - 5 = 1$ and Theorem 3.3.19 gives $A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$. The solution of $A(x,y) = (4,7)$ is $A^{-1}(4,7) = (12-7,\, -20+14) = (5,-6)$. As a check, $10-6 = 4$ and $25-18 = 7$. p. 82

3.3.4 The images $0, 1, 2x$ of $1, x, x^2$ give the columns of the first matrix. Squaring it gives the second. The vector (a_0, a_1, a_2) is sent to $(2a_2, 0, 0)$. This image is zero exactly when $a_2 = 0$. The kernel is $\mathcal{P}_1$, the polynomials $a_0 + a_1x$. p. 82

3.3.5 M sends $1, x, x^2$ to x, x^2, x^3, whose coordinate vectors in $1, x, x^2, x^3$ are the stated columns. If $xp(x) = 0$ for all x, all coefficients of xp vanish and these are the coefficients of p. Hence $\ker M = \{0\}$. The image is $\operatorname{span}(x, x^2, x^3)$, of dimension 3, and $0 + 3 = 3 = \dim \mathcal{P}_2$. p. 82

3.3.6 If $\sum c_iT\mathbf{v}_i = \mathbf{0}$ then $T(\sum c_i\mathbf{v}_i) = \mathbf{0}$. The kernel is $\{\mathbf{0}\}$ by Theorem 3.3.10. Hence $\sum c_i\mathbf{v}_i = \mathbf{0}$ and independence gives every $c_i = 0$. p. 82

3.3.7 Write $F_{c,s} = R_{C,S}F_{1,0}$ and $F_{c',s'} = R_{C',S'}F_{1,0}$ by Proposition 3.3.23(3). Multiplying out, $F_{1,0}R_{C',S'}F_{1,0} = \begin{pmatrix} C' & S' \\ -S' & C' \end{pmatrix} = R_{C',-S'}$. Hence $F_{c,s}F_{c',s'} = R_{C,S}(F_{1,0}R_{C',S'}F_{1,0}) = R_{C,S}R_{C',-S'}$, a rotation by Proposition 3.3.23(1). p. 82

3.3.8 The equation $T(x,y) = \mathbf{0}$ means $y = 0$. The kernel is the x-axis $\operatorname{span}(\mathbf{e}_1)$. Every value $(y,0)$ lies on the x-axis and $T\mathbf{e}_2 = \mathbf{e}_1$. Hence the image is the x-axis as well. If $\ker S = \operatorname{im} S$ for an operator on $\mathbb{R}^3$, then Theorem 3.3.11 gives $2\dim\ker S = 3$. No integer satisfies this equation. p. 82

3.4 Linear Functionals and the Dual Space

3.4.1 (a) is $ax + by$ with $a = 3$, $b = -5$, and (d) with $a = b = 0$. (b) sends $\mathbf{0}$ to 1. (c) sends $(1,1)$ to 1 and $(2,2)$ to $4 \neq 2$. p. 86

3.4.2 For $\mathbf{u}_1^* = ax + by$ the conditions $a + b = 1$ and $a + 2b = 0$ give $b = -1$ and $a = 2$. For $\mathbf{u}_2^*$ the conditions $a + b = 0$ and $a + 2b = 1$ give $b = 1$ and $a = -1$. As a check, $\mathbf{u}_1^*(1, 2) = 0$ and $\mathbf{u}_2^*(1, 2) = 1$. p. 86

3.4.3 E_{-1} takes the values $1, -1, 1, -1$ on $1, x, x^2, x^3$, and Theorem 3.4.8 gives the stated combination. With p' as in Section 3.3, $p'(0) = a_1$ is the coefficient of x. The functional is $\mathbf{e}_1^*$. p. 86

3.4.4 If $\lambda \neq 0$ then $\lambda(\mathbf{v}) = t \neq 0$ for some $\mathbf{v}$, and $\lambda((s/t)\mathbf{v}) = s$ for every real s. The image is $\mathbb{R}$, of dimension 1. By Theorem 3.3.11 $\dim \ker \lambda = 2 - 1 = 1$. A one-dimensional subspace is the span of one nonzero vector and so a line through the origin. p. 87

3.4.5 The stated p and q have degree at most 1 and the required values. If $\alpha E_a + \beta E_b = 0$, applying it to p gives $\alpha = 0$ and to q gives $\beta = 0$. The two functionals are independent in the two-dimensional space $\mathcal{P}_1^*$ (Theorem 3.4.8) and form a basis by Theorem 3.2.27(2). p. 87

3.4.6 At $p = 2$ the functional is $\mathbf{v} \mapsto 5\mathbf{v}$, with value -15 at $(2, -3)$. At $p = 0$ it is $\mathbf{v} \mapsto \mathbf{v}$, with value 5 at $(0, 5)$. By Theorem 3.4.5(3), $f(p) = \omega_p((p, 1)) = p^2 + 1$. p. 87

3.4.7 Let

$$p_0(x) = \frac{(x - a_1)(x - a_2)}{(a_0 - a_1)(a_0 - a_2)},$$

and define p_1, p_2 by permuting the indices. Each p_i lies in $\mathcal{P}_2$ and $p_i(a_j) = \delta_{ij}$. If $\sum_j c_j E_{a_j} = 0$, applying it to p_i gives $c_i = 0$. The three functionals are independent in the three-dimensional space $\mathcal{P}_2^*$ and form a basis by Theorem 3.2.27(2). p. 87

3.4.8 Since $\lambda \neq 0$ some $\mathbf{u}$ has $\lambda(\mathbf{u}) \neq 0$. The vector $\mathbf{v} = \mathbf{u}/\lambda(\mathbf{u})$ has $\lambda(\mathbf{v}) = 1$. For every $\mathbf{w} \in V$ the vector $\mathbf{w} - \lambda(\mathbf{w})\mathbf{v}$ lies in $\ker \lambda = \ker \mu$. Applying μ gives $\mu(\mathbf{w}) = \lambda(\mathbf{w})\,\mu(\mathbf{v})$. Hence $\mu = c\lambda$ with $c = \mu(\mathbf{v})$. In $\mathbb{R}^2$ the statement says that two functionals with the same kernel have the same level lines with different values attached to them. p. 87

3.5 Inner Product and Isometries of the Plane

3.5.1 $\mathbf{u} \cdot \mathbf{v} = 2 - 3 = -1$, $|\mathbf{u}| = \sqrt{4 + 1}$, $|\mathbf{v}| = \sqrt{1 + 9}$, and the ratio is $-1/\sqrt{50} = -1/(5\sqrt{2})$. The dot product is not 0. p. 94

3.5.2 The vector (x, y) is orthogonal to $(3, 4)$ when $3x + 4y = 0$, that is $(x, y) = t(-4, 3)$. Its length $5|t|$ equals 1 for $t = \pm\frac{1}{5}$. p. 94

3.5.3 The edge vectors from $(1, 1)$ are $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (1, 4)$ with $\det(\mathbf{v}, \mathbf{w}) = 12 - 1 = 11$. By Theorem 3.5.15 the parallelogram has area 11, and the triangle has half of it. p. 94

3.5.4 A has the form of Definition 3.3.21 with $(c, s) = (\frac{3}{5}, \frac{4}{5})$ and $c^2 + s^2 = 1$. Its determinant is $\frac{9+16}{25} = 1$. For B, the determinant is $\frac{-25-144}{169} = -1$, and B is the case $\varepsilon = -1$ of Theorem 3.5.11 with $c = \frac{5}{13}$ and $s = \frac{12}{13}$. Then $c' = \sqrt{9/13} = 3/\sqrt{13}$ and $s' = 2/\sqrt{13}$, and $B = F_{c',s'}$. As a check, $B(3, 2) = \frac{1}{13}(15 + 24,\ 36 - 10) = (3, 2)$. p. 94

3.5.5 By Proposition 3.5.3(5) the two squares are $|\mathbf{u}|^2 \pm 2\,\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$. Adding them cancels the dot products. Geometrically, the squares of the two diagonals of a parallelogram add to the sum of the squares of its four sides. p. 94

3.5.6 With $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (1, 1, 1)$, $\mathbf{u} \cdot \mathbf{v} = a + b + c$, $|\mathbf{u}|^2 = a^2 + b^2 + c^2$ and $|\mathbf{v}|^2 = 3$. Squaring the Cauchy–Schwarz inequality gives the claim. Equality holds exactly when $\mathbf{u}$ is a multiple of $\mathbf{v}$, that is $a = b = c$. p. 94

3.5.7 Squaring, equality is equivalent to $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|$ by Proposition 3.5.3(5). If $\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = 0\mathbf{u}$ and equality holds. Otherwise equality in Theorem 3.5.4 gives $\mathbf{u} = t\mathbf{v}$, and then $t|\mathbf{v}|^2 = |t||\mathbf{v}|^2$ gives $t \geq 0$. Conversely, if $\mathbf{u} = t\mathbf{v}$ with $t \geq 0$, both sides equal $(1+t)|\mathbf{v}|$. p. 94

3.5.8 By linearity $|T\mathbf{v} - T\mathbf{w}| = |T(\mathbf{v}-\mathbf{w})| = |\mathbf{v}-\mathbf{w}|$ and $T\mathbf{0} = \mathbf{0}$. By Theorem 3.5.11 T is some $R_{c,s}$ or $F_{c,s}$, with determinant 1 or -1 by Proposition 3.5.14(6). If $\det T = 1$ it is not a reflection and so is a rotation. p. 94

3.5.9 Here $\det S = 1\cdot 1 - 1\cdot 0 = 1$. By Proposition 3.5.14(4), $\det(S\mathbf{v}, S\mathbf{w}) = \det(\mathbf{v},\mathbf{w})$ for all $\mathbf{v},\mathbf{w}$. The sign is unchanged and so is the area by Theorem 3.5.15. The vector $S\mathbf{e}_2 = (1,1)$ has length $\sqrt{2}$ and $\mathbf{e}_2$ has length 1. An isometry fixing $\mathbf{0}$ is some $R_{c,s}$ or $F_{c,s}$ (Theorem 3.5.11) and has determinant ± 1 by Proposition 3.5.14(6). The shear S shows that the converse fails. p. 94

3.6 Normed Spaces

3.6.1 $\|\mathbf{x}\|_1 = 1+2+2 = 5$, $\|\mathbf{x}\|_2 = \sqrt{9} = 3$ and $\|\mathbf{x}\|_\infty = 2$. Then $2 \leq 3 \leq 5 \leq 3\cdot 2$. Also $3 \leq 2\sqrt{3}$ since $9 \leq 12$. Finally $5 \leq 3\sqrt{3}$ since $25 \leq 27$. p. 103

3.6.2 Here $f(x) = (x-1)^2 - 1$. On $[-1,2]$ the number $x-1$ lies in $[-2,1]$ and $(x-1)^2$ lies in $[0,4]$. Hence $-1 \leq f(x) \leq 3$ with $f(-1) = 3$ and $f(1) = -1$. The largest absolute value is 3. p. 103

3.6.3 Here $A(x,y) = (x+3y,\, 2x)$. If $\|(x,y)\|_\infty \leq 1$ the components satisfy $|x+3y| \leq 4$ and $|2x| \leq 2$. Hence $\|A\| \leq 4$. Equality comes from $A(1,1) = (4,2)$. For the 1-norm, $|x+3y| + |2x| \leq 3|x| + 3|y| = 3\|(x,y)\|_1$. Hence $\|A\| \leq 3$. Equality comes from $A\mathbf{e}_1 = (1,2)$ with 1-norm 3. p. 103

3.6.4 Let $r = \max_i \sum_j |a_{ij}|$. If $\|\mathbf{x}\|_\infty \leq 1$, then $\left|\sum_j a_{ij}x_j\right| \leq \sum_j |a_{ij}| \leq r$ for every i. Hence $\|A\| \leq r$. Let row k attain r and put $x_j = 1$ when $a_{kj} \geq 0$ and $x_j = -1$ otherwise. Then $\|\mathbf{x}\|_\infty \leq 1$ and the kth component of $A\mathbf{x}$ is $\sum_j |a_{kj}| = r$. Hence $\|A\| = r$. p. 103

3.6.5 (N1) The maximum is nonnegative and is 0 only if $x = y = 0$. (N2) Each of the three absolute values scales by $|c|$. (N3) For each of the functionals $\lambda(x,y) = x$, y or $x+y$, $|\lambda(\mathbf{u}+\mathbf{v})| \leq |\lambda(\mathbf{u})| + |\lambda(\mathbf{v})| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. The largest of the three left sides obeys the same bound. The unit ball is the part of the square $|x| \leq 1$, $|y| \leq 1$ between the lines $x+y = 1$ and $x+y = -1$. These lines cut the corners $(1,1)$ and $(-1,-1)$ from the square and leave the six stated vertices. p. 103

3.6.6 In the 1-norm both $\|\mathbf{e}_1 \pm \mathbf{e}_2\|_1$ equal 2. The left side is 8 and the right side is 4. In the ∞-norm both $\|\mathbf{e}_1 \pm \mathbf{e}_2\|_\infty$ equal 1. The left side is 2 and the right side is 4. On $[0,1]$ the norms $\|1+x\|_\infty = 2$ and $\|1-x\|_\infty = 1$ give the left side 5 against $2\cdot 1 + 2\cdot 1 = 4$. The proof of Problem 3.5.5 uses only (5) of Proposition 3.5.3, and the proof of (5) uses only (1) and (2). Hence a norm of the form $\sqrt{\langle \mathbf{v},\mathbf{v}\rangle}$ satisfies the parallelogram law. p. 103

3.6.7 By Lemma 2.3.8, $|\lambda(f)| \leq |f(0)| + 2|f(1)| \leq 3\|f\|_\infty$. Hence $\|\lambda\| \leq 3$. The function $f(x) = 1 - 2x$ has $\|f\|_\infty = 1$ because $|1-2x| \leq 1$ on $[0,1]$, and $\lambda(f) = 1 - 2(-1) = 3$. p. 103

3.6.8 On $[0,1]$, $|a+bx| \leq |a| + |b|$. For the other bound, $|a| = |p(0)| \leq \|p\|_\infty$ and $|b| = |p(1) - p(0)| \leq 2\|p\|_\infty$. For $p(x) = 1 - 2x$ the coefficient norm is 3 and the sup norm is 1. The constants of Theorem 3.6.15 for $\mathcal{P}_1$ are explicit here. On all of $\mathcal{P}$ no constant exists.

The derivative of $a + bx$ is the constant b, of sup norm $|b|$. The bound $|b| \leq 2\|p\|_\infty$ gives $\|T\| \leq 2$. The polynomial $1 - 2x$ has sup norm 1 and derivative -2 of sup norm 2. Hence $\|T\| = 2$. p. 103

3.6.9 The rows of $R_{c,s}$ are $(c,-s)$ and (s,c) with absolute sums $\frac{3}{5}+\frac{4}{5}=\frac{7}{5}$. By Problem 3.6.4 the operator norm is $\frac{7}{5}$. As a check, $R_{c,s}(1,-1)=(c+s,\,s-c)=(\frac{7}{5},\frac{1}{5})$. p. 103

3.6.10 (a) Each term satisfies (N2) and (N3). The sum is 0 only when $\|f\|_\infty=0$. (b) The function $x(1-x)$ is nonzero and has value 0 at both ends. (c) Doubling f multiplies the value by 4. (d) A positive multiple of a norm is a norm. p. 103

3.6.11 Let $m\|\mathbf{v}\|\le\|\mathbf{v}\|'\le M\|\mathbf{v}\|$. If $\|T\mathbf{v}\|\le K\|\mathbf{v}\|$, then $\|T\mathbf{v}\|\le(K/m)\|\mathbf{v}\|'$. If $\|T\mathbf{v}\|\le K\|\mathbf{v}\|'$, then $\|T\mathbf{v}\|\le KM\|\mathbf{v}\|$. For the identity of $\mathcal{P}$, a bound $\|p\|_c\le K\|p\|_\infty$ would give $3^n\le K$ for every n. Since $3^n\ge 1+2n$ (Theorem 2.3.11), the Archimedean property excludes it. p. 103

3.6.12 On $[0,1]$ the number $1-x$ lies in $[0,1]$. Hence $\|(1-x)^n\|_\infty=1$, attained at 0. The derivative is $-n(1-x)^{n-1}$ with value $-n$ at 0. No M has $n\le M$ for every n. Hence λ is unbounded. For the second part, h_n has $h_n'(0)=1$ and $0\le h_n\le 1/n$ on $[0,1]$. The polynomial $r=q-q'(0)h_n$ has $r'(0)=q'(0)-q'(0)=0$ and $\|q-r\|_\infty=|q'(0)|\,\|h_n\|_\infty\le|q'(0)|/n$. Choosing n with $|q'(0)|/n<\varepsilon$ (Theorem 2.4.10) completes the proof. For a bounded functional μ and $\mathbf{k}\in\ker\mu$, $|\mu(\mathbf{v})|=|\mu(\mathbf{v}-\mathbf{k})|\le\|\mu\|\,\|\mathbf{v}-\mathbf{k}\|$. Hence a vector arbitrarily close to $\ker\mu$ lies in it. The polynomial $1-x$ is arbitrarily close to $\ker\lambda$ and has $\lambda(1-x)=-1$. p. 103

4.1 Functions and Their Graphs

4.1.1 The square root needs $x^2-9\ge 0$ and the denominator needs $\sqrt{x^2-9}\neq 0$. Together these give $x^2>9$, that is, $|x|>3$. The natural domain is $(-\infty,-3)\cup(3,\infty)$. p. 108

4.1.2 $f(-x)=\dfrac{-x^3}{1+x^2}=-f(x)$. Hence f is odd. For g, $g(1)=2$ and $g(-1)=0$. Hence g is neither even nor odd. For h, $h(-x)=|-x|-(-x)^4=|x|-x^4=h(x)$. Hence h is even. p. 108

4.1.3 Let f and g share a domain symmetric about 0. If both are odd, then $(fg)(-x)=f(-x)g(-x)=(-f(x))(-g(x))=f(x)g(x)$. Hence fg is even. If f is even and g is odd, then $(fg)(-x)=f(x)(-g(x))=-(fg)(x)$. Hence fg is odd. p. 108

4.1.4 The domain of g is $\{x:0\le 2x-6\le 4\}=\{x:3\le x\le 5\}=[3,5]$. The point $(2,7)$ of $\Gamma(f)$ corresponds to the x with $2x-6=2$, namely $x=4$ with $g(4)=f(2)+1=8$. The point is $(4,8)$. p. 108

4.1.5 For $x<-1$ both $x-1$ and $x+1$ are negative and $f(x)=-(x-1)-(x+1)=-2x$. For $-1\le x\le 1$ the first is nonpositive and the second nonnegative with $f(x)=-(x-1)+(x+1)=2$. For $x>1$ both are positive and $f(x)=(x-1)+(x+1)=2x$. p. 109

4.1.6 The natural domain is $\mathbb{R}\setminus\{3\}$. For a given y the equation $y(x-3)=2x+1$ is $(y-2)x=3y+1$. When $y=2$ it reads $0=7$ and has no solution. Hence f never takes the value 2. When $y\neq 2$ it has the single solution $x=\dfrac{3y+1}{y-2}$. This x is not 3 because $x=3$ would give $3y-6=3y+1$. Every $y\neq 2$ is therefore the value of f at exactly one point. Hence f is injective with range $\mathbb{R}\setminus\{2\}$ and $f^{-1}(y)=\dfrac{3y+1}{y-2}$. p. 109

4.1.7 Let $g:S\to T$, $f:T\to\mathbb{R}$ and $x<y$ in S. If both are increasing, then $g(x)\le g(y)$ and applying the increasing f gives $f(g(x))\le f(g(y))$ (when $g(x)=g(y)$ the values are equal). If both are decreasing, then $g(x)\ge g(y)$ and the decreasing f reverses this inequality to $f(g(x))\le f(g(y))$. In both cases $f\circ g$ is increasing (Definitions 1.5.20 and 4.1.17). p. 109

4.2 Polynomials and Rational Functions

4.2.1 Long division gives $x^3-2x^2+4=(x-3)(x^2+x+3)+13$. By Theorem 4.2.5 the remainder is $f(3)=27-18+4=13$, in agreement. p. 115

4.2.2 The value at 1 is $1-6+11-6=0$. Hence $x-1$ is a factor and division gives $(x-1)(x^2-5x+6)$. The quadratic is $(x-2)(x-3)$ and the factorisation is $(x-1)(x-2)(x-3)$. p. 115

4.2.3 $3x^2-12x+7=3(x^2-4x)+7=3\big[(x-2)^2-4\big]+7=3(x-2)^2-5$. The square is nonnegative and vanishes only at $x=2$. The least value is -5 at $x=2$. p. 115

4.2.4 The value at 2 is $8+4k-8+4=4k+4$ and vanishes when $k=-1$. Then $x^3-x^2-4x+4=x^2(x-1)-4(x-1)=(x-1)(x^2-4)=(x-1)(x-2)(x+2)$ by grouping. p. 115

4.2.5 $x^2+x-6=(x-2)(x+3)$. By Proposition 4.2.20 with $p=5$, $a=2$ and $b=-3$, $A=5/5=1$ and $B=5/(-5)=-1$. The quotient is $\dfrac{1}{x-2}-\dfrac{1}{x+3}$. p. 115

4.2.6 Since the leading coefficient and the constant term are both 1, the only candidates are ±1. The values are $1-3+1=-1$ at 1 and $-1+3+1=3$ at -1, neither zero. By Theorem 4.2.11 there is no rational root. p. 115

4.2.7 By Eq. (4.3.4) with $a=x^2+2$ and $b=2x$, the product is $(x^2+2)^2-4x^2=x^4+4x^2+4-4x^2=x^4+4$. Each factor has discriminant $4-8=-4<0$ and is irreducible by Theorem 4.2.15. p. 116

4.2.8 The number $\sqrt{2}$ is a root of $p(x)=x^2-2$. If it were rational, say r/s in lowest terms, then Theorem 4.2.11 would give $r\mid 2$ and $s\mid 1$ and hence $\sqrt{2}\in\{\pm1,\pm2\}$. None of these squares to 2. Hence $\sqrt{2}$ is irrational. Theorem 2.1.3 proves the same fact directly. p. 116

4.3 Algebra Toolkit

4.3.1 $\dfrac{1}{\sqrt[3]{x^2}}=\dfrac{1}{x^{2/3}}=x^{-2/3}$; $x\sqrt{x}=x^1x^{1/2}=x^{3/2}$; $\dfrac{(x^{1/2})^5}{x^3}=x^{5/2-3}=x^{-1/2}$. p. 121

4.3.2 By the second special product, $(2x-3)^2=4x^2-12x+9$. By Eq. (4.3.4), $(x^2+1)(x^2-1)=x^4-1$. With $a=2x$ and $b=3$ in $a^3-b^3=(a-b)(a^2+ab+b^2)$, $8x^3-27=(2x-3)(4x^2+6x+9)$. p. 121

4.3.3 $\dfrac{1}{\sqrt{5}-\sqrt{3}}\cdot\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\dfrac{\sqrt{5}+\sqrt{3}}{5-3}=\dfrac{\sqrt{5}+\sqrt{3}}{2}$. p. 121

4.3.4 Squaring gives $x^2-2x-3=0$, that is $(x-3)(x+1)=0$. At $x=3$, $\sqrt{9}=3$ and the equation holds. At $x=-1$ the left side is $\sqrt{1}=1$ and the right side is -1. The candidate -1 is therefore extraneous and the solution is $x=3$. p. 121

4.3.5 $\dfrac{x}{x-1}-2=\dfrac{x-2(x-1)}{x-1}=\dfrac{2-x}{x-1}$. A quotient is positive exactly when numerator and denominator have the same sign. Both are positive when $1<x<2$. Both cannot be negative because that would need $x>2$ and $x<1$ at once. The solution set is $(1,2)$. p. 121

4.3.6 By the second special product, $0\le(\sqrt{a}-\sqrt{b})^2=a-2\sqrt{a}\sqrt{b}+b=a+b-2\sqrt{ab}$. Rearranging gives the inequality. Equality holds exactly when $\sqrt{a}=\sqrt{b}$, that is, when $a=b$. p. 121

4.3.7 Since both sides are symmetric in a and b, we may assume $a\ge b$. Then $\sqrt{a}\ge\sqrt{b}$ and $\sqrt{a}\sqrt{b}\ge\sqrt{b}\sqrt{b}=b$. Hence $(\sqrt{a}-\sqrt{b})^2=a-2\sqrt{a}\sqrt{b}+b\le a-2b+b=a-b$. The claim

follows because both sides of it are nonnegative and taking square roots preserves the order of nonnegative numbers. p. 121

4.4 Exponential and Logarithmic Functions

4.4.1 $27^{-2/3} = (3^3)^{-2/3} = 3^{-2} = \frac{1}{9}$ and $16^{3/4} = (2^4)^{3/4} = 2^3 = 8$. Since $2^{-5} = \frac{1}{32}$, $\log_2 \frac{1}{32} = -5$. Since $9^{3/2} = 3^3 = 27$, $\log_9 27 = \frac{3}{2}$. p. 129

4.4.2 By Eq. (4.4.8), $2\ln x = \ln x^2$ and $\frac{1}{2}\ln(x^2+1) = \ln\sqrt{x^2+1}$. The product and quotient laws combine the three terms into $\ln \dfrac{x^2\sqrt{x^2+1}}{x+1}$. All arguments are positive for $x > 0$. p. 129

4.4.3 Taking ln gives $(x+1)\ln 5 = 2x \ln 3$, that is, $x(2\ln 3 - \ln 5) = \ln 5$. Since $2\ln 3 - \ln 5 = \ln \frac{9}{5} > 0$, $x = \dfrac{\ln 5}{\ln(9/5)}$. p. 129

4.4.4 The argument $(x-1)(x-3)$ is positive when both factors have the same sign: $x < 1$ or $x > 3$. The natural domain is $(-\infty, 1) \cup (3, \infty)$. p. 129

4.4.5 Both logarithms require $x > 2$. By Eq. (4.4.7) the equation is $\log_2(x(x-2)) = 3$, that is, $x^2 - 2x = 8$ or $(x-4)(x+2) = 0$. The candidate -2 violates $x > 2$. The solution is $x = 4$. p. 129

4.4.6 By Theorem 4.4.24, $\log_a b = \ln b / \ln a$ and $\log_b a = \ln a / \ln b$, where neither $\ln a$ nor $\ln b$ is zero. The product is 1. p. 129

4.4.7 Theorem 2.3.11 gives $(1+x)^n \geq 1+nx$ for $x \geq -1$. With $x = \frac{1}{n}$ the right side is 2. p. 129

4.4.8 Since $1 < 3 < 4$ and $\log_2$ is strictly increasing, $0 < \log_2 3 < 2$. Suppose $\log_2 3 = p/q$ with integers $p, q \geq 1$. Then $2^{p/q} = 3$, and raising to the power q gives $2^p = 3^q$ by Theorem 4.4.10. The left side is even and the right side is odd. This is a contradiction. p. 129

4.5 Trigonometric Functions

4.5.1 $\sin \frac{7\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}$ and $\cos \frac{5\pi}{4} = -\cos \frac{\pi}{4} = -\frac{\sqrt{2}}{2}$ by the half-turn. Because tangent is odd, $\tan(-\frac{\pi}{3}) = -\dfrac{\sqrt{3}/2}{1/2} = -\sqrt{3}$. p. 136

4.5.2 By Theorem 4.5.9, $\tan(\alpha+\beta) = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$. Dividing numerator and denominator by $\cos\alpha\cos\beta \neq 0$ gives $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$. p. 136

4.5.3 $\sin \frac{5\pi}{12} = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$. p. 136

4.5.4 By Eq. (4.5.4) the equation is $2\cos^2 x - \cos x - 1 = 0$, that is $(2\cos x + 1)(\cos x - 1) = 0$. By Theorem A.1.14(3) each point of the unit circle is $P(x)$ for exactly one $x \in [0, 2\pi)$. The only point with first coordinate 1 is $(1, 0) = P(0)$. The points with first coordinate $-\frac{1}{2}$ are $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. By Lemma A.1.15, $P(\frac{2\pi}{3}) = F_{0,1}P(\frac{\pi}{3}) = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and the half-turn gives $P(\frac{4\pi}{3}) = -P(\frac{\pi}{3}) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The solutions are 0, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. p. 136

4.5.5 $\cos \frac{2\pi}{3} = -\frac{1}{2}$ with $\frac{2\pi}{3} \in [0, \pi]$, and $\tan(-\frac{\pi}{4}) = -1$ with $-\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Next $\sin \frac{5\pi}{6} = \frac{1}{2}$ and $\arcsin \frac{1}{2} = \frac{\pi}{6}$. Finally let $\theta = \arccos \frac{3}{5} \in [0, \pi]$. Then $\sin\theta \geq 0$ and $\sin\theta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$. p. 136

4.5.6 In the form $A\sin(B(x-C))+D$ we have $A=3$, $B=2$, $C=\frac{\pi}{6}$ and $D=1$. The amplitude is 3, the period $2\pi/2=\pi$, the shift $\frac{\pi}{6}$ to the right and the range $[1-3,1+3]=[-2,4]$. p. 136

4.5.7 Let $\theta=\arccos x\in[0,\pi]$. Then $\cos\theta=x$ and $\sin\theta\geq 0$ on $[0,\pi]$. Hence $\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-x^2}$. Let $\varphi=\arcsin x$ with $x\in(-1,1)$. Then $\varphi\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\cos\varphi>0$. Hence $\cos\varphi=\sqrt{1-x^2}$. Hence $\tan\varphi=x/\sqrt{1-x^2}$. p. 137

4.5.8 Let $\theta=\frac{\pi}{2}-\arccos x$. Since $\arccos x\in[0,\pi]$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. By Theorem 4.5.7(4), $\sin\theta=\cos(\arccos x)=x$. By Definition 4.5.14 these two properties say $\theta=\arcsin x$. Rearranging gives the claim. p. 137

4.5.9 By the remark on the origin of the trigonometric functions, $R_{c,s}$ is multiplication by $c+is=\frac{1}{5}(3+4i)$. Then $\frac{1}{5}(3+4i)(1+2i)=\frac{1}{5}(3+6i+4i+8i^2)=\frac{1}{5}(-5+10i)=-1+2i$, the point $(-1,2)$. The matrix gives $R_{3/5,4/5}(1,2)=(\frac{3}{5}-\frac{8}{5},\ \frac{4}{5}+\frac{6}{5})=(-1,2)$. p. 137

4.5.10 By the rules, $\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2=\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2=-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2=-\mathbf{e}_1$. Hence $\mathbf{v}B=x\mathbf{e}_2-y\mathbf{e}_1$, the vector $(-y,x)$. The same rules give $B\mathbf{e}_1=-\mathbf{e}_1B$ and $B\mathbf{e}_2=-\mathbf{e}_2B$. Hence $B\mathbf{v}=-\mathbf{v}B$. For $\theta=\pi$ the rotor is $R=\cos\frac{\pi}{2}-B\sin\frac{\pi}{2}=-B$ with $R^{-1}=B$ because $(-B)B=-B^2=1$. Then $R\mathbf{v}R^{-1}=-B\mathbf{v}B=\mathbf{v}BB=-\mathbf{v}$, the half turn $R_{-1,0}$ of Section 3.3. p. 137

4.6 Finite Sums and Binomial Coefficients

4.6.1 Put $k=n-3$. Then $n=k+3$, $n-2=k+1$ and the lower limit $n=3$ becomes $k=0$. The sum is $\sum_{k=0}^{\infty}(k+1)x^{k+3}$. Both sides open with x^3. p. 145

4.6.2 By Eq. (4.3.5), $(2k-1)^3=8k^3-12k^2+6k-1$. By Theorem 4.6.5 and Theorem 4.6.8 the sum is

$$2n^2(n+1)^2-2n(n+1)(2n+1)+3n(n+1)-n.$$

Taking out the factor n leaves $2n^3+4n^2+2n-(4n^2+6n+2)+3n+3-1=2n^3-n$. The sum is $n^2(2n^2-1)$. As a check, $n=2$ gives $1+27=28=4\cdot 7$. p. 145

4.6.3 By Theorem 4.6.5 the sum is $\dfrac{n(n+1)(2n+1)}{6}+\dfrac{n(n+1)}{2}=\dfrac{n(n+1)(2n+1+3)}{6}=\dfrac{n(n+1)(n+2)}{3}$. p. 145

4.6.4 $(2n+1)!=(2n+1)(2n)(2n-1)!$ gives $(2n+1)(2n)$. Similarly $n!=n(n-1)(n-2)!$ gives $n(n-1)$. In the third, $((n+1)!)^2=(n+1)\,n!\,(n+1)!$ and $(n+2)!=(n+2)(n+1)!$. The ratio is $\dfrac{n+1}{n+2}$. p. 145

4.6.5 For $k\geq 1$ the numerator is $(-1)(-2)\cdots(-k)=(-1)^k k!$. Dividing by $k!$ gives $(-1)^k$. The case $k=0$ is the convention $\binom{-1}{0}=1$. Next $\binom{-2}{3}=\dfrac{(-2)(-3)(-4)}{3!}=\dfrac{-24}{6}=-4$. p. 146

4.6.6 Theorem 4.6.14 with $a=b=1$ gives $2^n=\sum_{k=0}^{n}\binom{n}{k}$. With $a=1$ and $b=-1$ it gives $0^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k$, and $0^n=0$ for $n\geq 1$. p. 146

4.6.7 By Theorem 4.6.14 and Eq. (4.6.17),

$$\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}=1+\sum_{k=1}^{n}\frac{1}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k}.$$

Each of the k factors $\frac{n-j}{n}$ in the second fraction is at most 1. Hence the kth term is at most $1/k!$. Since $k!\geq 2^{k-1}$ (each factor from 2 to k is at least 2), the kth term is at most 2^{1-k}. By

Eq. (4.6.11) with $r = \frac{1}{2}$,

$$\left(1+\frac{1}{n}\right)^n \le 1+\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k = 1+2\left(1-\frac{1}{2^n}\right) < 3.$$

p. 146

4.6.8 With $r = 2$, $\sum_{k=0}^{9} 2^k = \dfrac{1-2^{10}}{1-2} = 1023$. Next $\sum_{k=1}^{n} 3^{-k} = \frac{1}{3}\sum_{k=0}^{n-1} 3^{-k} = \frac{1}{3}\cdot\dfrac{1-3^{-n}}{1-\frac{1}{3}} = \dfrac{1-3^{-n}}{2}$.

p. 146

5.1 Limits of Sequences

5.1.1 Over a common denominator, $a_n - 3 = \dfrac{3n+1-3n-6}{n+2} = -\dfrac{5}{n+2}$ and $|a_n-3| = 5/(n+2)$. The inequality $5/(n+2) < 0.01$ is equivalent to $n+2 > 500$, that is $n > 498$. The condition holds for every $n > 498$ and fails at $n = 498$ with $|a_{498} - 3| = 0.01$ exactly. The smallest admissible N is therefore 498.

p. 157

5.1.2 Dividing by n^3 gives $\dfrac{1+2/n^2}{4-1/n^3}$. Since $1/n \to 0$, the product law gives $1/n^2 \to 0$ and $1/n^3 \to 0$. The numerator then tends to 1 and the denominator to $4 \neq 0$. The quotient law gives the limit $\frac{1}{4}$.

For the second, $(n+1)/n = 1 + 1/n \to 1$ by the sum law. Two applications of the product law give $(1+1/n)^3 \to 1^3 = 1$.

p. 157

5.1.3 Rationalising,

$$\sqrt{n^2+3n}-n = \frac{(n^2+3n)-n^2}{\sqrt{n^2+3n}+n} = \frac{3n}{\sqrt{n^2+3n}+n} = \frac{3}{\sqrt{1+3/n}+1}.$$

Since $1 \le \sqrt{1+3/n} \le 1+3/n$ (square both sides of the right-hand inequality to check it) and $1+3/n \to 1$, Theorem 5.1.11 gives $\sqrt{1+3/n} \to 1$. The denominator tends to 2 and the limit is $\frac{3}{2}$. Because both terms of the original difference tend to infinity, the limit cannot be read off directly.

p. 157

5.1.4 Let $\varepsilon > 0$ and choose an integer $N \ge 1/\varepsilon^2$ by the Archimedean property. For $n > N$ we have $n > 1/\varepsilon^2$. Since the square root is increasing, $\sqrt{n} > 1/\varepsilon$ and therefore $|1/\sqrt{n} - 0| = 1/\sqrt{n} < \varepsilon$.

p. 157

5.1.5 Since $0 < 2^n \le 3^n$, we have $3^n \le 2^n + 3^n \le 2\cdot 3^n$. Since the nth root is increasing on $[0,\infty)$,

$$3 \le \left(2^n+3^n\right)^{1/n} \le 3\cdot 2^{1/n}.$$

By Proposition 5.1.15, $2^{1/n} \to 1$ and the right-hand side tends to 3. Since the left-hand side is the constant 3, Theorem 5.1.11 gives the limit 3.

p. 157

5.1.6

1. Bounds. $a_1 = 1$. If $1 \le a_n < 3$, then $\frac{1}{3} < 1/a_n \le 1$ and therefore $1 < 2 \le a_{n+1} < \frac{8}{3} < 3$.

2. Monotonicity. $a_2 = 2 \ge a_1$. If $a_n \ge a_{n-1}$, then, both being positive, $1/a_n \le 1/a_{n-1}$ and $a_{n+1} - a_n = 1/a_{n-1} - 1/a_n \ge 0$. Induction gives a nondecreasing sequence.

By Theorem 5.1.18 the sequence converges to some L with $1 \le L \le 3$ (Theorem 5.1.10). The limit laws applied to the recursion give $L = 3 - 1/L$, that is $L^2 - 3L + 1 = 0$, with roots $\frac{1}{2}(3 \pm \sqrt{5})$. Since the root $\frac{1}{2}(3-\sqrt{5}) \approx 0.38$ is less than 1, $L = \frac{1}{2}(3+\sqrt{5})$.

p. 157

5.1.7 Since $(1+1/n)^{2n} = a_n \cdot a_n$ with $a_n = (1+1/n)^n \to e$, the product law gives e^2. Likewise $(1+1/n)^{n+1} = a_n(1+1/n)$ with $1+1/n \to 1$ has limit $e \cdot 1 = e$. p. 157

5.1.8 $a_{2k} = 1 + \dfrac{1}{2k} \to 1$ and $a_{2k-1} = -1 + \dfrac{1}{2k-1} \to -1$ by the sum law. If $\{a_n\}$ converged to some L, both subsequences would converge to L by Theorem 5.1.23 and uniqueness (Theorem 5.1.6) would force $1 = L = -1$. Hence the sequence diverges, although it is bounded. p. 157

5.1.9 From $|x| = |(x-y)+y| \le |x-y| + |y|$ we get $|x| - |y| \le |x-y|$. Exchanging x and y gives $|y| - |x| \le |x-y|$. Together, $\big||x| - |y|\big| \le |x-y|$. Given $\varepsilon > 0$, choose N with $|a_n - L| < \varepsilon$ for $n > N$. Then $\big||a_n| - |L|\big| \le |a_n - L| < \varepsilon$ for $n > N$.

For $a_n = (-1)^n$ the absolute values are constantly 1 and converge while $\{a_n\}$ diverges. When $L = 0$ the converse does hold because $|a_n - 0| = \big||a_n| - 0\big|$. p. 157

5.1.10 A convergent sequence is Cauchy by Theorem 5.1.28. With $\varepsilon = \frac{1}{2}$ there is an N such that $|a_m - a_n| < \frac{1}{2}$ for all $m, n > N$. Since two distinct integers differ by at least 1, $a_m = a_n$ for all $m, n > N$. In particular $a_n = a_{N+1}$ for every $n > N$. p. 157

5.1.11 The coordinates tend to 0 and to 2. Hence $\mathbf{v}_n \to (0, 2)$ by the coordinate example. The difference is $\mathbf{v}_n - \mathbf{v} = \left(\frac{1}{n},\ -\frac{5}{n+3}\right)$. Since $5n > n+3$ for $n \ge 1$, the second coordinate is the larger in absolute value and $\|\mathbf{v}_n - \mathbf{v}\|_\infty = 5/(n+3)$. The condition $5/(n+3) < 0.01$ reads $n > 497$. At $n = 497$ the two sides are equal. For the 1-norm the condition $\frac{1}{n} + \frac{5}{n+3} < 0.01$ is $n^2 - 597n - 300 > 0$, that is $n > \frac{1}{2}(597 + \sqrt{357609}) = 597.50\ldots$. It holds for every $n \ge 598$ and fails at $n = 597$. Both norms give the same limit with different indices as Theorem 3.6.15 predicts. p. 157

5.1.12 With $\varepsilon = 1$ there is an N with $\|\mathbf{v}_n - \mathbf{v}_{N+1}\| < 1$ for $n > N$. Then $\|\mathbf{v}_n\| < 1 + \|\mathbf{v}_{N+1}\|$ for $n > N$ by (N3). The largest of this number and $\|\mathbf{v}_1\|, \ldots, \|\mathbf{v}_N\|$ bounds every term. For the second statement let $\varepsilon > 0$, choose N' with $\|\mathbf{v}_m - \mathbf{v}_n\| < \varepsilon/2$ for $m, n > N'$ and then k with $n_k > N'$ and $\|\mathbf{v}_{n_k} - \mathbf{v}\| < \varepsilon/2$. For $n > N'$, $\|\mathbf{v}_n - \mathbf{v}\| \le \|\mathbf{v}_n - \mathbf{v}_{n_k}\| + \|\mathbf{v}_{n_k} - \mathbf{v}\| < \varepsilon$. By the second statement the Cauchy sequence of Example 5.1.36 has no convergent subsequence in c_{00}. p. 157

5.2 Limits of Functions

5.2.1 Let $\varepsilon > 0$ and put $\delta = \varepsilon/5$. If $0 < |x-2| < \delta$, then $|(5x-3)-7| = |5x-10| = 5|x-2| < 5\delta = \varepsilon$. p. 165

5.2.2 Since $x^3 + 1 = (x+1)(x^2 - x + 1)$, for $x \ne -1$ the quotient equals $x^2 - x + 1$ with limit $1 + 1 + 1 = 3$ at -1 by Corollary 5.2.14.

Since $x^2 - x - 6 = (x-3)(x+2)$ and $x^2 - 9 = (x-3)(x+3)$, for $x \ne 3$ the quotient equals $(x+2)/(x+3)$ with limit $\frac{5}{6}$ at 3. p. 165

5.2.3 Let $\varepsilon > 0$ and put $\delta = \min\{4, 2\varepsilon\}$. If $0 < |x-4| < \delta$, then $x > 0$ and

$$|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2} \le \frac{|x-4|}{2} < \frac{2\varepsilon}{2} = \varepsilon.$$

p. 165

5.2.4 For $x > 3$, $|x-3| = x - 3$ and $f(x) = 1$. For $x < 3$, $|x-3| = -(x-3)$ and $f(x) = -1$. Since the one-sided limits 1 and -1 differ, Theorem 5.2.11 shows that the two-sided limit does not exist. p. 165

5.2.5 Since $t = 3x \to 0$ with $t \neq 0$ as $x \to 0$, Theorem 5.2.18 gives $\sin 3x/(3x) \to 1$ and the first limit is 3. For the second, since $\sin x/x \to 1$ and $\cos x \to 1 \neq 0$ (Proposition 5.2.17), the product and quotient laws give $1 \cdot 1 = 1$. p. 165

5.2.6 For $x \neq 0$, $-|x| \leq x \sin(1/x) \leq |x|$ because $|\sin(1/x)| \leq 1$. Since both bounds tend to 0, Theorem 5.2.15 gives the limit 0. p. 165

5.2.7 For $2 < x < 3$ the numerator exceeds 3 and the denominator lies in $(0, 1)$. Hence $(x+1)/(x-2) > 3/(x-2) > M$ for any $M > 0$ once $x - 2 < 3/M$. The right-hand limit is therefore ∞. For $1 < x < 2$ the numerator exceeds 2 and the denominator is negative. Hence $(x+1)/(x-2) < 2/(x-2) < -M$ once $2 - x < 2/M$. The left-hand limit is therefore $-\infty$. At infinity, $\dfrac{3x+1}{x^2+1} = \dfrac{3/x + 1/x^2}{1 + 1/x^2} \to \dfrac{0}{1} = 0$. p. 165

5.2.8 With $t = x/2$ we have $x^{10}/e^{x/2} = 2^{10}\, t^{10}/e^t$ and $t \to \infty$ as $x \to \infty$. Since $t^{10}/e^t \to 0$ by Theorem 5.2.26 with $p = 10$, the limit is $2^{10} \cdot 0 = 0$.

For $x > 1$, $(\ln x)^2/x = \left(\ln x/x^{1/2}\right)^2$. Since the bracket tends to 0 by Theorem 5.2.26 with $q = \frac{1}{2}$, the product law gives 0. p. 165

5.2.9 Let $x_n = 1/(2\pi n)$ and $y_n = 1/((2n+1)\pi)$. Both are nonzero and tend to 0. Along them $\cos(1/x_n) = \cos(2\pi n) = 1$ and $\cos(1/y_n) = \cos((2n+1)\pi) = -1$. If the limit were some L, Theorem 5.2.7 would give both $L = 1$ and $L = -1$. p. 165

5.3 Continuity

5.3.1 Since both pieces are polynomials, f is continuous away from 1. At 1 the value $f(1) = 2 + k$ equals the right-hand limit while the left-hand limit is $1 - k$. Continuity requires $1 - k = 2 + k$, that is $k = -\frac{1}{2}$. p. 171

5.3.2 Since $x^2 - 4 = (x-2)(x+2)$ and $x^2 - x - 2 = (x-2)(x+1)$, the function equals $(x+2)/(x+1)$ for $x \neq 2, -1$. The limit $\frac{4}{3}$ at 2 makes 2 a removable discontinuity. At -1 the numerator tends to 1 while the denominator tends to 0 with opposite signs on the two sides. The one-sided limits are therefore $\pm\infty$ and -1 is an infinite discontinuity. p. 171

5.3.3 Since $e^{\sin x + \cos x}$ is continuous by Theorems 5.3.6 and 5.3.10, its limit at π is its value $e^{0+(-1)} = e^{-1}$. For the second, $\pi \sin x/(2x) \to \pi/2$ by Theorem 5.2.18. Since the cosine is continuous at $\pi/2$, Theorem 5.3.7 gives the limit $\cos(\pi/2) = 0$. p. 171

5.3.4 Let $\varepsilon > 0$. Choose δ_1 with $|f(x) - f(a)| < \varepsilon/2$ for $|x - a| < \delta_1$ and δ_2 with $|g(x) - g(a)| < \varepsilon/2$ for $|x - a| < \delta_2$, with x in the common domain. For $\delta = \min\{\delta_1, \delta_2\}$ and $|x - a| < \delta$, the triangle inequality (Lemma 2.3.8) gives

$$|(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < \varepsilon.$$

p. 171

5.3.5 By Problem 5.2.6, $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at 0. For g, the positive sequences $1/(2\pi n)$ and $1/(\frac{\pi}{2} + 2\pi n)$ tend to 0 while g tends to 0 along the first and to 1 along the second. The right-hand limit therefore fails to exist as a number. Because $|g| \leq 1$, it is not infinite either. Since g is odd, the same holds on the left. The discontinuity is oscillating. p. 171

5.3.6 The floor function equals $m-1$ on $[m-1, m)$ and m on $[m, m+1)$. Its one-sided limits at m are therefore the two different numbers $m-1$ and m. The floor function is continuous from the right at m and has a jump of 1 there. p. 171

5.3.7 $|f|$ is the composition of the continuous function $|\cdot|$ with f and is continuous by Theorem 5.3.6. The identity $\max\{u, v\} = \frac{1}{2}(u + v + |u - v|)$, checked separately for $u \geq v$ and $u < v$, expresses $\max\{f, g\}$ through sums, a constant multiple and the absolute value of $f - g$, all continuous. p. 171

5.3.8 Let $a \in \mathbb{R}$. For each n the interval $(a, a + 1/n)$ contains a rational r_n and an irrational s_n by Theorem 2.4.12. Both sequences tend to a while $\chi_{\mathbb{Q}}(r_n) = 1$ and $\chi_{\mathbb{Q}}(s_n) = 0$ for every n. Continuity at a would force both sequences of values to converge to $\chi_{\mathbb{Q}}(a)$ by Theorem 5.3.5. The constant sequences 1 and 0 have no common limit. The same two sequences approach a from the right and show by Theorem 5.2.7 that the right-hand limit is no number. The bound $0 \leq \chi_{\mathbb{Q}} \leq 1$ excludes $\pm\infty$. The intervals $(a - 1/n, a)$ give the same conclusion on the left. The discontinuity is oscillating. p. 171

5.3.9 Since $|f(x) - f(0)| = |f(x)| \leq |x|$, the choice $\delta = \varepsilon$ proves continuity at 0. For $a \neq 0$ take rationals $r_n \to a$ and irrationals $s_n \to a$ (Theorem 2.4.12). Then $f(r_n) = r_n \to a$ and $f(s_n) = 0$. Since $a \neq 0$ the two limits differ and Theorem 5.3.5 shows that f is not continuous at a. p. 171

5.3.10 By Theorem 5.3.6 the product f^2 is continuous. Hence Q maps $C([0,1])$ into itself. At every x, $|f(x)^2 - g(x)^2| = |f(x) + g(x)|\,|f(x) - g(x)| \leq (\|f\|_\infty + \|g\|_\infty)\|f - g\|_\infty$. The right side is an upper bound of the values on the left. The supremum is therefore at most this bound (Definition 2.4.3). Given $\varepsilon > 0$ put $\delta = \min\{1, \varepsilon/(2\|f\|_\infty + 1)\}$. If $\|g - f\|_\infty < \delta$, then $\|g\|_\infty < \|f\|_\infty + 1$ by Proposition 3.6.4(3) and $\|g^2 - f^2\|_\infty < (2\|f\|_\infty + 1)\delta \leq \varepsilon$. This is Definition 3.6.17 at f. Finally $Q(2f) = 4f^2$ differs from $2Q(f) = 2f^2$ whenever $f \neq 0$. p. 171

5.4 Theorems on Continuous Functions

5.4.1 The polynomial $f(x) = x^5 + x - 1$ is continuous with $f(0) = -1$ and $f(1) = 1$. Since 0 lies strictly between them, Theorem 5.4.2 gives $c \in [0, 1]$ with $f(c) = 0$. Because f does not vanish at 0 or 1, the point c lies in $(0, 1)$. p. 177

5.4.2 Since $f(1) = -2$ and $f(2) = 1$, a root lies in $[1, 2]$. The three midpoint values $f(1.5) = -0.75$, $f(1.75) = 0.0625$ and $f(1.625) = -0.359375$ shrink the interval to $[1.5, 2]$, then to $[1.5, 1.75]$ and finally to $[1.625, 1.75]$. The positive root $\sqrt{3} = 1.7320\ldots$ lies in this interval. p. 177

5.4.3 Since the function is continuous on a closed bounded interval, Theorem 5.4.6 guarantees both extremes. Since $x^2 - 1 \leq 3$ and $1 - x^2 \leq 1$ for $|x| \leq 2$, we have $0 \leq f(x) \leq 3$. The value 3 is attained only at ± 2 and the value 0 only at ± 1. p. 177

5.4.4 (a) Both extremes exist because the interval is closed and bounded and the function is continuous. Since $x^2 \geq 0$ with equality at 0 and $x^2 \leq 4$ on $[-1, 2]$ with equality at 2, they are 0 and 4.

(b) Since $1/x \geq 1$ on $(0, 1]$ with equality at 1, the minimum is 1. Because $1/x \to \infty$ as $x \to 0^+$, the function is unbounded and has no maximum. The theorem does not apply because $(0, 1]$ is not closed.

(c) The minimum is 0 at 0. No point of $[0, 1)$ attains the supremum 1. Again the interval is not closed. p. 177

5.4.5 The function $g(x) = e^x + x - 3$ is continuous with $g(0) = -2$ and $g(1) = e - 2 > 0$ because $e > 2$ (Theorem 5.1.20). A solution lies in $(0, 1)$ by Theorem 5.4.2. As a sum of two strictly increasing functions, g is strictly increasing and vanishes at most once. p. 177

5.4.6 Each function is a composition of continuous functions defined at the point. Each limit is therefore the value there: $\arctan 1 = \pi/4$, $\sqrt{4+5} = 3$ and $\ln(1+0) = 0$. p. 177

5.4.7 Dividing by the leading coefficient, we may assume $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ with n odd. Since $p(x)/x^n = 1 + c_{n-1}/x + \cdots + c_0/x^n \to 1$ as $x \to \pm\infty$, there is an $R > 0$ with $p(x)/x^n > \frac{1}{2}$ for $|x| \geq R$. Because n is odd, $R^n > 0$ and $(-R)^n < 0$. Hence $p(R) > 0$ and $p(-R) < 0$. Theorem 5.4.2 on $[-R, R]$ gives a root. p. 178

5.4.8 The function g is continuous on $[0, \frac{1}{2}]$ with $g(0) + g(\frac{1}{2}) = f(\frac{1}{2}) - f(0) + f(1) - f(\frac{1}{2}) = f(1) - f(0) = 0$. If $g(0) = 0$, then $c = 0$ works. Otherwise $g(0)$ and $g(\frac{1}{2})$ have opposite signs and Theorem 5.4.2 gives c with $g(c) = 0$. p. 178

5.4.9 Injectivity means $f(x) \neq f(y)$ whenever $x \neq y$. If f is neither strictly increasing nor strictly decreasing, there are pairs $x_1 < y_1$ and $x_2 < y_2$ in I with $f(x_1) < f(y_1)$ and $f(x_2) > f(y_2)$. For $t \in [0, 1]$ let $x(t) = (1-t)x_1 + tx_2$ and $y(t) = (1-t)y_1 + ty_2$. Both lie in the interval I and satisfy $x(t) < y(t)$ for every t. The function $h(t) = f(y(t)) - f(x(t))$ is continuous with $h(0) > 0$ and $h(1) < 0$. By Theorem 5.4.2, $h(t) = 0$ for some t. Then $f(x(t)) = f(y(t))$ with $x(t) \neq y(t)$, contradicting injectivity. p. 178

5.5 Uniform Continuity

5.5.1 Since $|f(x) - f(y)| = |4x - 4y| = 4|x - y|$, the function is Lipschitz with constant 4 and Proposition 5.5.3 gives $\delta = \varepsilon/4$. p. 182

5.5.2 For $x, y \in [-2, 2]$, $|x^2+xy+y^2| \leq x^2+|x||y|+y^2 \leq 4+4+4 = 12$. Hence $|x^3-y^3| \leq 12|x-y|$ and Proposition 5.5.3 gives $\delta = \varepsilon/12$. No smaller constant works because $x^2 + xy + y^2 \to 12$ as $x, y \to 2$. p. 182

5.5.3 Let $x \geq y \geq 0$. Squaring shows $\sqrt{x} \leq \sqrt{y} + \sqrt{x-y}$ because the square of the right-hand side is $x + 2\sqrt{y(x-y)} \geq x$. Hence $0 \leq \sqrt{x} - \sqrt{y} \leq \sqrt{x-y}$. By symmetry $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$ for all $x, y \geq 0$. Given $\varepsilon > 0$, $\delta = \varepsilon^2$ gives $|\sqrt{x} - \sqrt{y}| < \sqrt{\varepsilon^2} = \varepsilon$ whenever $|x - y| < \delta$. Because the function is not Lipschitz near 0, Proposition 5.5.3 is unavailable on all of $[0, \infty)$. p. 182

5.5.4 Over a common denominator, $f(x) - f(y) = \dfrac{y^2 - x^2}{(1+x^2)(1+y^2)}$. From $(1 - |x|)^2 \geq 0$ we get $|x| \leq \frac{1}{2}(1 + x^2) \leq \frac{1}{2}(1 + x^2)(1 + y^2)$. The same bound holds for $|y|$. Adding the two bounds gives $|x + y| \leq |x| + |y| \leq (1 + x^2)(1 + y^2)$ and therefore $|f(x) - f(y)| \leq |x - y|$. p. 182

5.5.5 Both sequences lie in $(0, 1)$ and $x_n - y_n = e^{-n}(1 - e^{-1}) \to 0$ by Proposition 5.1.14 with $r = e^{-1}$. Their images satisfy $\ln x_n - \ln y_n = -n - (-n - 1) = 1$. Hence $\varepsilon_0 = 1$ satisfies the condition of Proposition 5.5.6. p. 182

5.5.6 Rationalising, $x_n - y_n = \dfrac{x_n^2 - y_n^2}{x_n + y_n} = \dfrac{\pi/2}{x_n + y_n} \leq \dfrac{\pi/2}{2\sqrt{n\pi}} \to 0$. Meanwhile $\sin(y_n^2) = \sin(n\pi) = 0$ and $\sin(x_n^2) = \sin(n\pi + \pi/2) = \pm 1$. Proposition 5.5.6 applies with $\varepsilon_0 = 1$. p. 182

5.5.7 Given $\varepsilon > 0$ take δ from Definition 5.5.1 and then N with $|x_m - x_n| < \delta$ for $m, n > N$. For those m, n, $|f(x_m) - f(x_n)| < \varepsilon$. The sequence $x_n = 1/(n + 1)$ in $(0, 1)$ converges to 0 and is therefore Cauchy (Theorem 5.1.28). The sequence $1/x_n = n + 1$ is unbounded and so fails to be Cauchy. Hence $1/x$ is not uniformly continuous on $(0, 1)$. p. 182

5.5.8 Let δ correspond to $\varepsilon = 1$, let ℓ be the length of I and fix $x_0 \in I$. Choose an integer $m > \ell/\delta$. Any $x \in I$ can be joined to x_0 by m equal steps $x_0 = z_0, z_1, \dots, z_m = x$ of length $|x - x_0|/m < \delta$ with every z_j in the interval I. Since each step changes f by less than 1, we have $|f(x) - f(x_0)| < m$ and $|f(x)| < |f(x_0)| + m$ for every $x \in I$. p. 182

6.1 Derivative as a Best Linear Approximation

6.1.1 The numerator of $Q(h)$ is $3(a^2+2ah+h^2)-5a-5h-3a^2+5a = 6ah+3h^2-5h$. Dividing by $h \neq 0$,

$$Q(h) = 6a + 3h - 5 \to 6a - 5 \quad (h \to 0).$$

Hence $f'(a) = 6a - 5$. p. 190

6.1.2 For $h \neq 0$ with $a + h \neq 0$,

$$Q(h) = \frac{1}{h}\Big(\frac{1}{a+h} - \frac{1}{a}\Big) = \frac{1}{h} \cdot \frac{a-(a+h)}{a(a+h)} = \frac{-1}{a(a+h)}.$$

Since the function $h \mapsto a(a+h)$ is continuous with value $a^2 \neq 0$ at $h = 0$ it follows that $Q(h) \to -1/a^2$. p. 190

6.1.3 For $h \neq 0$ with $a + h > 0$,

$$Q(h) = \frac{\sqrt{a} - \sqrt{a+h}}{h\sqrt{a}\sqrt{a+h}} = \frac{a-(a+h)}{h\sqrt{a}\sqrt{a+h}(\sqrt{a}+\sqrt{a+h})} = \frac{-1}{\sqrt{a}\sqrt{a+h}(\sqrt{a}+\sqrt{a+h})}.$$

By continuity of the square root (Corollary 5.4.13) the denominator tends to $\sqrt{a}\cdot\sqrt{a}\cdot 2\sqrt{a} = 2a^{3/2}$ and hence $f'(a) = -1/(2a^{3/2})$. p. 190

6.1.4 Expanding gives $(2+h)^3 = 8+12h+6h^2+h^3$. Hence $f(2) = 8$, $L = 12$ and $r(h) = 6h^2+h^3$. Since $r(h)/h = 6h + h^2 \to 0$, the remainder is $o(h)$ and $f'(2) = 12$ by Theorem 6.1.9. With $h = 0.01$ the linear part gives $8+12(0.01) = 8.12$. The exact value $8.12+6(0.0001)+0.000001 = 8.120601$ exceeds the estimate by the remainder $r(0.01) = 0.000601$. p. 190

6.1.5 $f(1) = 0$. For $0 < h < 1$, $f(1+h) = (1+h)^2 - 1 = 2h + h^2$ and $Q(h) = 2 + h \to 2$. For $-1 < h < 0$, $f(1+h) = 1-(1+h)^2 = -2h - h^2$ and $Q(h) = -2 - h \to -2$. Since the one-sided derivatives 2 and -2 differ, Proposition 6.1.14 shows that f is not differentiable at 1. The graph has a corner there. p. 190

6.1.6 For $h \neq 0$, $Q(h) = (h^2 \sin(1/h) - 0)/h = h\sin(1/h)$. Since $|\sin(1/h)| \leq 1$, the quotient satisfies $-|h| \leq Q(h) \leq |h|$ with both bounds tending to 0. The squeeze theorem (Section 5.2) gives $Q(h) \to 0$ and hence $f'(0) = 0$. Compared with Example 6.1.18 the extra factor of x turns a remainder that is not $o(h)$ into one that is. p. 190

6.1.7 Continuity at 2 requires $2a + b = f(2) = 4$. The right-hand derivative of x^2 at 2 is $\lim_{h\to 0^+}(4+h) = 4$. With $2a + b = 4$ the left-hand quotient is $(a(2+h) + b - 4)/h = a$. Equal one-sided derivatives give $a = 4$ and then $b = 4 - 8 = -4$. p. 190

6.1.8 By Worked Example 6.1.1 the slope at 4 is $1/(2\sqrt{4}) = \frac{1}{4}$ and the tangent line is $y = 2 + \frac{1}{4}(x-4)$. At $x = 4.1$ it gives $2 + \frac{1}{4}(0.1) = 2.025$. The true value $2.024845\ldots$ differs from this estimate by the $o(h)$ remainder of Theorem 6.1.9 at $h = 0.1$. p. 190

6.1.9 Taking $x = 0$ gives $|f(0)| \le 0$ and hence $f(0) = 0$. For $h \neq 0$,

$$|Q(h)| = \frac{|f(h)|}{|h|} \le \frac{h^2}{|h|} = |h|.$$

Given $\varepsilon > 0$, take $\delta = \varepsilon$. Then $0 < |h| < \delta$ implies $|Q(h) - 0| < \varepsilon$. By Definition 6.1.2, $f'(0) = 0$. p. 190

6.1.10 For $h \neq 0$,

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2} \cdot \frac{f(a+h) - f(a)}{h} + \frac{1}{2} \cdot \frac{f(a-h) - f(a)}{-h} = \tfrac{1}{2}Q(h) + \tfrac{1}{2}Q(-h).$$

Since both $Q(h)$ and $Q(-h)$ tend to $f'(a)$ as $h \to 0$, their average tends to $f'(a)$. For $f(x) = |x|$ at $a = 0$ the numerator $|h| - |-h|$ is 0 for every h and the symmetric limit is 0. Example 6.1.16 shows that $|x|$ is not differentiable at 0. p. 191

6.2 Differential Operator and 1-Forms

6.2.1 By Eq. (6.2.5), $(5x\,\mathrm{d}x)_2(3) = 5 \cdot 2 \cdot 3 = 30$. By Eq. (6.2.4), $(\mathrm{d}x)_{-7}(0.4) = 0.4$. The value of $\mathrm{d}x$ does not depend on the point. p. 198

6.2.2 (a) is multiplication by -4 and (c) is multiplication by 0. Map (b) fails homogeneity because $\lambda(2 \cdot 1) = 4$ while $2\lambda(1) = 2$. Map (d) is not linear because it sends 0 to $-1 \neq 0$. p. 198

6.2.3 By Eq. (6.2.2) with $f'(p) = 2p$, $(\mathrm{d}f)_p(\mathbf{v}) = 2p\mathbf{v}$. The increment is $(p+\mathbf{v})^2 - p^2 = 2p\mathbf{v} + \mathbf{v}^2$. The difference $\mathbf{v}^2$ is $o(\mathbf{v})$ because its ratio $\mathbf{v}$ with $\mathbf{v}$ tends to 0. p. 199

6.2.4 By linearity $\omega_p(1) = \frac{1}{2}\omega_p(2) = 3p^2$. Hence $\omega = 3x^2\,\mathrm{d}x$ by Theorem 6.2.8. Since the derivative of x^3 is $3x^2$ (Worked Example 6.1.2), Eq. (6.2.6) gives $\mathrm{d}(x^3) = 3x^2\,\mathrm{d}x = \omega$. p. 199

6.2.5 By Eq. (6.2.8), $g^*\omega = (x^2 + 1) \cdot 2x\,\mathrm{d}x$. At $p = 1$ on $\mathbf{v} = 3$ it gives $2 \cdot 1 \cdot 2 \cdot 3 = 12$. As a check, $g(1) = 2$, $g'(1) \cdot 3 = 6$ and $\omega_2(6) = 2 \cdot 6 = 12$. p. 199

6.2.6 By Eq. (6.2.6) and the derivatives found in Section 6.1, $\mathrm{d}y = 3x^2\,\mathrm{d}x$ and $\mathrm{d}z = 2x\,\mathrm{d}x$. Since $3x^2\,\mathrm{d}x = \frac{3}{2}x \cdot 2x\,\mathrm{d}x$ with $2x \neq 0$ on $(0, \infty)$, it follows that $\mathrm{d}y = \frac{3}{2}x\,\mathrm{d}z$ and the quotient is $\frac{3}{2}x$. Since $y = z^{3/2}$ on $(0, \infty)$, the quotient $\frac{3}{2}x = \frac{3}{2}z^{1/2}$ is the derivative of $z^{3/2}$ that Theorem 6.4.6 of Section 6.4 gives. p. 199

6.2.7 The difference quotients $((x + h + c) - (x + c))/h = 1$ and $(k(x + h) - kx)/h = k$ give $\tau' = 1$ and $\sigma' = k$. By Eq. (6.2.8), $\tau^*(f(u)\,\mathrm{d}u) = f(x + c) \cdot 1\,\mathrm{d}x$ and $\sigma^*(f(u)\,\mathrm{d}u) = f(kx) \cdot k\,\mathrm{d}x$. Pointwise, a displacement $\mathbf{v}$ at p is sent by τ to the displacement $\mathbf{v}$ at $p + c$ and by σ to the displacement $k\mathbf{v}$ at kp. The factor k in the second pullback is this stretching. p. 199

6.2.8 By linearity $\omega_p(-1) = -\omega_p(1)$. Since both numbers are non-negative by hypothesis, $\omega_p(1) = 0$. Then $\omega_p(\mathbf{v}) = \omega_p(1)\,\mathbf{v} = 0$ for every $\mathbf{v}$. The map $\mathbf{v} \mapsto |\mathbf{v}|$ is non-negative and takes the value 1 at $\mathbf{v} = 1$. If it were the value at p of a 1-form, the first part would force it to be zero. Chapter 8 meets this map again as the arc-length density, an integrand that is not a 1-form. p. 199

6.2.9 Pointwise $(f + h)^3 = f^3 + 3f^2h + (3fh^2 + h^3)$. The map $h \mapsto 3f^2h$ is linear with $\|3f^2h\|_\infty \le 3\|f\|_\infty^2\|h\|_\infty$. For $\|h\|_\infty < 1$ the remainder satisfies $\|3fh^2 + h^3\|_\infty \le (3\|f\|_\infty + 1)\|h\|_\infty^2$. This is at most $\varepsilon\|h\|_\infty$ once $\|h\|_\infty < \min\{1, \varepsilon/(3\|f\|_\infty + 1)\}$. Hence $\mathrm{d}G_f h = 3f^2h$. For $f(x) = x$ and $h(x) = 1 - x$ the function $\mathrm{d}G_f h$ is $3x^2(1 - x)$ with value $3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$ at $\frac{1}{2}$. Since $\|f\|_\infty = 1$ the operator norm is at most 3. The constant function $h = 1$ has norm 1 and $\|3x^2\|_\infty = 3$. The operator norm is 3. p. 199

6.2.10 Write $L = \mathrm{d}F_{\mathbf{a}}$. Applying the linear map T gives $TF(\mathbf{a}+\mathbf{h}) = TF(\mathbf{a}) + (T\circ L)\mathbf{h} + TR(\mathbf{h})$. The map $T \circ L$ is linear and bounded by Proposition 3.6.20(3). Given $\varepsilon > 0$, choose δ for the tolerance $\varepsilon/(\|T\| + 1)$ in Definition 6.2.14 for F. For $\|\mathbf{h}\| < \delta$, Proposition 3.6.20(1) gives $\|TR(\mathbf{h})\| \le \|T\|\,\|R(\mathbf{h})\| \le \varepsilon\|\mathbf{h}\|$. Hence $T \circ L$ is the differential of $T \circ F$. For $V = W = U = \mathbb{R}$ and $T(y) = cy$, Proposition 6.2.16 turns the identity into $(cf)'(a)h = c\,f'(a)h$ for every h. Taking $h = 1$ gives the rule. p. 199

6.3 Rules of Differentiation

6.3.1 The first is $4\cdot 5x^4 - 3\cdot 2x + 0 = 20x^4 - 6x$. For the second, Eq. (6.3.2) gives $2x(x^3 - 2x) + (x^2+1)(3x^2-2) = 2x^4 - 4x^2 + 3x^4 + x^2 - 2 = 5x^4 - 3x^2 - 2$. p. 205

6.3.2 The first was computed in the example after Eq. (6.3.6): $((x+1)-(x-1))/(x+1)^2 = 2/(x+1)^2$. For the second, $(3x^{-4})' = -12x^{-5}$ by Eq. (6.3.4). p. 205

6.3.3 By Eq. (6.3.9), $(u^7)' = 7u^6u' = 7(x^2 - 3x + 1)^6(2x - 3)$. For the second, $(u^{-3})' = -3u^{-4}u' = -3(1 + x^2)^{-4}\cdot 2x$. p. 205

6.3.4 By Eq. (6.3.6), $\mathrm{d}y = (1-x)^3\cdot 2x\,\mathrm{d}x + x^2\cdot 3(1-x)^2(-1)\,\mathrm{d}x$. Factoring $x(1-x)^2$ leaves $2(1-x) - 3x = 2 - 5x$ and gives $\mathrm{d}y = x(1-x)^2(2-5x)\,\mathrm{d}x$. p. 205

6.3.5 f is continuous and, as a sum of the strictly increasing functions x^5 and $2x$, strictly increasing with $f'(x) = 5x^4 + 2 > 0$. Since $f(1) = 4$ and $f'(1) = 7$, Eq. (6.3.16) gives $(f^{-1})'(4) = 1/7$. p. 205

6.3.6 By Eq. (6.2.8), $g^*\omega = (x^2+1)^3\cdot 2x\,\mathrm{d}x$. By the chain rule and Eq. (6.2.6), $\mathrm{d}(\tfrac14(x^2+1)^4) = (x^2+1)^3\cdot 2x\,\mathrm{d}x$. The two agree by Theorem 6.3.5 for $F(u) = \tfrac14 u^4$ with $\mathrm{d}F = u^3\,\mathrm{d}u$. p. 205

6.3.7 On $(0,\infty)$, $f(x) = x^2$ and $f'(x) = 2x = 2|x|$. On $(-\infty,0)$, $f(x) = -x^2$ and $f'(x) = -2x = 2|x|$. At 0, $Q(h) = h|h|/h = |h| \to 0$ and hence $f'(0) = 0 = 2|0|$. p. 205

6.3.8 For $n = 1$ both sides are f'. Assume $(f^n)' = nf^{n-1}f'$. By Eq. (6.3.2),

$$(f^{n+1})' = (f\cdot f^n)' = f'\,f^n + f\cdot nf^{n-1}f' = (n+1)f^nf'.$$

The statement follows for every $n \ge 1$ by Theorem 1.4.2. In the language of forms, $\mathrm{d}(f^n) = nf^{n-1}\,\mathrm{d}f$. p. 205

6.3.9 By Eq. (6.3.2), $(fg)' = f'g + fg'$. Since each term is a product of differentiable functions, linearity and the product rule give

$$(fg)'' = (f'g)' + (fg')' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''.$$

The coefficients $1, 2, 1$ are the binomial coefficients of the second row. The general Leibniz formula for $(fg)^{(n)}$ has the binomial coefficients of row n. p. 206

6.3.10 By Problem 6.1.6, $f'(0) = 0$. Also $g'(f(0)) = g'(0) = 0$. The chain rule gives $(g\circ f)'(0) = 0\cdot 0 = 0$. Directly, $(g\circ f)(h) = h^4\sin^2(1/h)$ and $|(g\circ f)(h)/h| \le |h|^3 \to 0$. The inner increment $f(h) - f(0) = h^2\sin(1/h)$ is zero at the points $h = 1/(n\pi)$ that accumulate at 0. The naive argument divides by this increment and fails at those points. The proof of Theorem 6.3.4 never divides by it. p. 206

6.4 Derivatives of the Elementary Functions

6.4.1 $(e^{3x}\sin x)' = 3e^{3x}\sin x + e^{3x}\cos x$. By Theorem 6.4.5 and the chain rule, $(\ln u)' = u'/u = 2x/(x^2+1)$. p. 213

6.4.2 $(\tan 5x)' = \sec^2(5x)\cdot 5$. By the product rule, $(x^2e^{-x})' = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x}$. p. 213

6.4.3 By Theorem 6.4.6 and the chain rule, $(2^{x^2})' = 2^{x^2}\ln 2 \cdot 2x$. For the second, $\left((1+\sin^2 x)^{1/2}\right)' = \frac{1}{2}(1+\sin^2 x)^{-1/2}\cdot 2\sin x\cos x$. p. 213

6.4.4 $(\arctan\sqrt{x})' = \dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$. For the second, $\dfrac{1}{\sqrt{1-x^2/4}}\cdot\dfrac{1}{2} = \dfrac{1}{\sqrt{4-x^2}}$ by $2\sqrt{1-x^2/4} = \sqrt{4-x^2}$. p. 213

6.4.5 With $u = \sin x\ln x$, the product rule gives $u' = \cos x\ln x + \sin x\cdot\frac{1}{x}$. Hence $(e^u)' = e^u u' = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$. p. 213

6.4.6 $(\ln|\cos x|)' = -\sin x/\cos x = -\tan x$. By the product rule, $(\sec x\tan x)' = \sec x\tan x\cdot\tan x + \sec x\cdot\sec^2 x = \sec x(\tan^2 x + \sec^2 x)$. Both forms follow from Eq. (6.2.6). p. 213

6.4.7 Since $e^0 = 1$ and $(e^x)' = 1$ at 0, the first tangent is $y = 1 + x$. Since $\ln 1 = 0$ and $(\ln x)' = 1$ at 1, the second is $y = x - 1$. Exchanging x and y in $y = 1 + x$ gives $x = 1 + y$, the same line as $y = x - 1$. Because the logarithm is the inverse of the exponential, its graph is the reflection of the graph of e^x in $y = x$ and the tangent lines reflect with it, as in Theorem 6.3.9. Lemma 6.4.2 states that the curve $y = e^x$ lies above its tangent at 0. p. 213

6.4.8 The cosine is continuous and strictly decreasing on $(0,\pi)$ with derivative $-\sin y \neq 0$ there. For $y = \arccos x$ with $-1 < x < 1$, Eq. (6.3.16) gives $(\arccos x)' = -1/\sin y$. Since the sine is positive on $(0,\pi)$ with $\sin^2 y = 1 - \cos^2 y = 1 - x^2$, it follows that $\sin y = \sqrt{1-x^2}$. Adding the derivative of arcsin from Theorem 6.4.10 gives 0. A function with zero derivative on an interval is constant there (Corollary 7.2.2). At $x = 0$ the sum is $0 + \frac{\pi}{2}$. p. 213

6.4.9 Since $\ln 1 = 0$, the quotient $\ln(1+h)/h$ is the difference quotient of ln at 1. It tends to $(\ln)'(1) = 1/1 = 1$ by Theorem 6.4.5. For $h > 0$, $(1+h)^{1/h} = e^{\ln(1+h)/h}$ with an exponent that tends to 1. Since the exponential is continuous at 1 (Theorem 5.3.10), the limit is $e^1 = e$ (Theorem 5.3.7). p. 213

6.4.10 For a constant c the chain rule gives $(\sin(x+c))' = \cos(x+c) = \sin(x+c+\frac{\pi}{2})$ by the angle-sum formula. The case $n = 0$ is trivial. If $(\sin x)^{(n)} = \sin(x + \frac{n\pi}{2})$, then applying this with $c = \frac{n\pi}{2}$ gives $(\sin x)^{(n+1)} = \sin(x + \frac{(n+1)\pi}{2})$. Induction (Theorem 1.4.2) completes the proof. p. 213

6.5 Implicit Differentiation and Related Rates

6.5.1 On the circle $y' = -x/y = 3/4$ at $(-3,4)$. On the hyperbola, $y\,dx + x\,dy = 0$ gives $y' = -y/x = -3/2$ at $(2,3)$. p. 216

6.5.2 Applying d gives $e^y\,dy + y\,dx + x\,dy = 0$ and hence $y' = -y/(e^y + x)$. At $(0,1)$ this is $-1/e$. p. 216

6.5.3 $2x - 2yy' = 0$ gives $y' = x/y$. Then

$$y'' = \frac{y - xy'}{y^2} = \frac{y - x^2/y}{y^2} = \frac{y^2 - x^2}{y^3} = -\frac{1}{y^3},$$

by $x^2 - y^2 = 1$. p. 216

6.5.4 At the point, $xy = \pi/6$ and $\sin(\pi/6) = \frac{1}{2} = x$. Differentiating gives $\cos(xy)(y + xy') = 1$ and hence $y' = (1/\cos(xy) - y)/x$. With $\cos(\pi/6) = \sqrt{3}/2$ this is $(2/\sqrt{3} - \pi/3)/\frac{1}{2} = 4/\sqrt{3} - 2\pi/3 \approx 0.215$. The point-slope form gives the tangent line. p. 216

6.5.5 $V' = 4\pi r^2 r' = 4\pi \cdot 25 \cdot 2 = 200\pi$. p. 216

6.5.6 Differentiating $s^2 = a^2 + b^2$ gives $ss' = aa' + bb'$. At the instant $s = \sqrt{900 + 1600} = 50$ and hence $s' = (30 \cdot 60 + 40 \cdot 80)/50 = 5000/50 = 100$ km/h. p. 216

6.5.7 From $6s = 1.8(x + s)$, $4.2s = 1.8x$ and $s = \frac{3}{7}x$. Differentiating, $s' = \frac{3}{7}x' = \frac{3}{7} \cdot 1.5 = \frac{9}{14}$. The rate does not depend on the position. p. 217

6.5.8 Differentiating $\tan\theta = h/4$ gives $\sec^2\theta\,\theta' = h'/4$. At $h = 3$, $\tan\theta = \frac{3}{4}$ and $\sec^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$. Hence $\theta' = \frac{0.5}{4} \cdot \frac{16}{25} = 0.08$ radians per second. p. 217

6.5.9 Applying d gives $\dfrac{2x}{a^2}\,dx + \dfrac{2y}{b^2}\,dy = 0$. By Proposition 6.5.2 the slope at (x_0, y_0) is $m = -b^2x_0/(a^2y_0)$. The tangent line $y - y_0 = m(x - x_0)$, multiplied by y_0/b^2, becomes

$$\frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = -\frac{x_0 x}{a^2} + \frac{x_0^2}{a^2}.$$

Rearranging and using $x_0^2/a^2 + y_0^2/b^2 = 1$ gives the stated equation. p. 217

7.1 Mean Value Theorem

7.1.1 The secant slope is $(f(3) - f(0))/3 = (15 - 0)/3 = 5$. Solving $f'(c) = 2c + 2 = 5$ gives the point $c = \frac{3}{2}$ of $(0, 3)$. p. 223

7.1.2 The function is continuous on $[0, 4]$ and differentiable on $(0, 4)$ with $f'(x) = 1/(2\sqrt{x})$. The theorem asks for differentiability only on the open interval. The secant slope is $(2 - 0)/4 = \frac{1}{2}$, and $1/(2\sqrt{c}) = \frac{1}{2}$ gives $\sqrt{c} = 1$ and $c = 1$. p. 223

7.1.3 The secant slope is $(\frac{1}{3} - 1)/2 = -\frac{1}{3}$ and $f'(c) = -1/c^2$. The equation $-1/c^2 = -\frac{1}{3}$ gives $c^2 = 3$ with the root $c = \sqrt{3}$ in $(1, 3)$. p. 223

7.1.4 The sine is continuous and differentiable everywhere, and $\sin 0 = \sin\pi = 0$. Its derivative $\cos x$ vanishes in $(0, \pi)$ only at $c = \pi/2$. p. 223

7.1.5 The secant slope is $(2 - 1)/3 = \frac{1}{3}$. For $x < 1$ we have $f(x) = 1 - x$ and $f'(x) = -1$. For $x > 1$ we have $f'(x) = 1$. At $x = 1$ the one-sided difference quotients have limits -1 and 1 and $f'(1)$ does not exist. No point of $(0, 3)$ has $f'(c) = \frac{1}{3}$. The failing hypothesis is differentiability on $(0, 3)$. p. 223

7.1.6 Since $f(\pi/2) - f(0) = 1$ and $g(\pi/2) - g(0) = -1$, Eq. (7.1.2) reads $1 \cdot (-\sin c) = (-1)\cos c$, that is $\sin c = \cos c$. In $(0, \pi/2)$ the only solution is $c = \pi/4$. p. 223

7.1.7 Let $f(x) = x^5 + x - 1$. Since $f(0) = -1 < 0 < 1 = f(1)$, the Intermediate Value Theorem gives a root in $(0, 1)$. If f had two roots $r < s$, Rolle's theorem on $[r, s]$ would give c with $f'(c) = 0$, but $f'(x) = 5x^4 + 1 \geq 1$ for every x. Hence the root is unique. p. 223

7.1.8 Both endpoint values equal 1 because $\sqrt[3]{-1} = -1$ and $\sqrt[3]{1} = 1$. The cube root $u(x) = \sqrt[3]{x}$ is the inverse of the continuous strictly increasing function $t \mapsto t^3$, whose derivative $3t^2$ is nonzero

for $t \neq 0$. For $x \neq 0$, Theorem 6.3.9 gives $u'(x) = 1/(3u(x)^2)$ and the chain rule (Theorem 6.3.4) gives

$$f'(x) = 2u(x)\,u'(x) = \frac{2}{3u(x)} = \frac{2}{3\sqrt[3]{x}} \neq 0.$$

At 0 the difference quotient $f(h)/h = 1/\sqrt[3]{h}$ is unbounded as $h \to 0$ and $f'(0)$ does not exist. Rolle's theorem requires differentiability at every point of $(-1, 1)$, and that hypothesis fails at 0. p. 223

7.1.9 Suppose $r < s$ in I and $f(r) = f(s)$. The interval $[r, s]$ lies in I, on which f is differentiable and hence continuous (Theorem 6.1.11). Rolle's theorem gives $c \in (r, s)$ with $f'(c) = 0$, contrary to hypothesis. Hence distinct points have distinct values. p. 223

7.1.10 A polynomial of degree 1, $p(x) = ax + b$ with $a \neq 0$, has the single root $-b/a$. Suppose every polynomial of degree $n - 1 \geq 1$ has at most $n - 1$ real roots and let p have degree n. If p had $n + 1$ distinct roots $r_0 < r_1 < \cdots < r_n$, Rolle's theorem on each $[r_{i-1}, r_i]$ would give $c_i \in (r_{i-1}, r_i)$ with $p'(c_i) = 0$ for $i = 1, \ldots, n$. These n points are distinct because the intervals are disjoint. A polynomial p' of degree $n - 1$ with n roots contradicts the induction hypothesis. The statement follows by induction (Theorem 1.4.2). p. 223

7.2 Consequences of the Mean Value Theorem

7.2.1 The derivative $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$ is negative on $(-\infty, -1)$, positive on $(-1, 0)$, negative on $(0, 1)$ and positive on $(1, \infty)$. Theorem 7.2.4 gives the stated intervals. The first derivative test gives local minima at ± 1 with value $1 - 2 = -1$ and a local maximum at 0 with value 0. p. 231

7.2.2 The critical points are the roots 1 and 3 of $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$. Since $f''(x) = 6x - 12$, we have $f''(1) = -6 < 0$ and $f''(3) = 6 > 0$. Hence $f(1) = 1 - 6 + 9 + 1 = 5$ is a local maximum and $f(3) = 27 - 54 + 27 + 1 = 1$ a local minimum. p. 231

7.2.3 Both $\ln x$ and $x - 1$ tend to 0 and the derivative of $x - 1$ is $1 \neq 0$. By Theorem 7.2.16 the limit equals $\lim_{x \to 1}(1/x)/1 = 1$. p. 231

7.2.4 The form is $\frac{0}{0}$ and $(x^3)' = 3x^2 \neq 0$ for $x \neq 0$. One application of Theorem 7.2.16 gives

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{\tan^2 x}{3x^2} = \frac{1}{3}\left(\lim_{x \to 0} \frac{\tan x}{x}\right)^2.$$

Another application gives $\tan x/x \to \sec^2 0/1 = 1$. The limit is $\frac{1}{3}$. p. 231

7.2.5 For $x > 0$, $x^x = e^{x \ln x}$. By Worked Example 7.2.4, $x \ln x \to 0$ as $x \to 0^+$. The continuity of the exponential gives $x^x \to e^0 = 1$. p. 231

7.2.6 For $x > 0$, $(\ln x)^2/\sqrt{x} = \left(\ln x/x^{1/4}\right)^2$ with $\ln x/x^{1/4} \to 0$ by Theorem 5.2.26 with $q = \frac{1}{4}$. The square of a quantity tending to 0 tends to 0. p. 231

7.2.7 The first is $x^3/e^x \to 0$ by Theorem 5.2.26. For the second, the logarithm is $\ln(1 + 2/x)/(1/x)$ of type $\frac{0}{0}$ with derivative quotient

$$\frac{-2/x^2}{1 + 2/x} = \frac{2}{1 + 2/x} \to 2.$$

Hence the limit is e^2. p. 231

7.2.8 The derivative $1/(1+t^2)$ of the arctangent lies in $(0,1]$ for every t. With $M=1$, Eq. (7.2.6) gives the inequality. p. 231

7.2.9 Let $u=\ln(1+x)$, defined for $x>-1$. By Worked Example 7.2.2, $1+x=e^u\geq 1+u$, that is $x\geq\ln(1+x)$. Equality holds only when $u=0$, that is when $x=0$. p. 231

7.2.10 For $x>0$, $(x+\sin x)/x=1+\sin x/x$ and $|\sin x/x|\leq 1/x\to 0$. The limit is therefore 1. The derivative quotient $(1+\cos x)/1$ takes the value 2 at every $x=2k\pi$ and 0 at every $x=(2k+1)\pi$ and has no limit. The rule assumes the existence of $\lim f'/g'$ and draws no conclusion otherwise. There is no contradiction. p. 231

7.2.11 The left inequality is Worked Example 7.2.2. Applying the same result at $-x$ gives $e^{-x}\geq 1-x$ with a positive right side for $x<1$. Taking reciprocals of two positive numbers reverses the inequality: $e^x=1/e^{-x}\leq 1/(1-x)$. p. 231

7.2.12 By the product and chain rules, $\big(f(x)e^{-x}\big)'=f'(x)e^{-x}-f(x)e^{-x}=0$ for every x. Corollary 7.2.2 on the interval $\mathbb{R}$ shows that $f(x)e^{-x}$ is constant with value $f(0)$ at 0. Multiplying by e^x gives the claim. p. 231

7.3 Differentiation as a Linear Operator

7.3.1 $D1=0$, $Dx=1$ and $Dx^2=2x$ give the columns $(0,0,0)$, $(1,0,0)$ and $(0,2,0)$ of A. Then D^2 sends 1 and x to 0 and x^2 to 2 as the matrix A^2 states. Finally $D^3=0$ on $\mathcal{P}_2$. p. 240

7.3.2 The function $1/x$ has derivative $-1/x^2$. By Eq. (7.3.2) the solutions are $1/x$ plus an element of $\ker D$ on the union. By Theorem 7.3.8 those elements are the functions constant on each of the two intervals. p. 240

7.3.3 Theorem 7.3.19 gives $f(x)=f(0)\cos x+f'(0)\sin x=\cos x-2\sin x$ as the only function that satisfies the conditions. p. 240

7.3.4 Since $De^x=e^x$ and $D(xe^x)=e^x+xe^x$, the columns are $(1,0)$ and $(1,1)$. The nth power of this matrix is $\begin{pmatrix}1&n\\0&1\end{pmatrix}$ by induction because multiplying it by the matrix raises n by one. Its second column gives $D^n(xe^x)=ne^x+xe^x=(x+n)e^x$. The same result follows from Eq. (7.3.12) with only the terms $k=0,1$ of D^kx nonzero. p. 240

7.3.5 With $D^ke^x=e^x$ and $D^kx^3=x^3,3x^2,6x,6,0$ for $k=0,1,2,3,4$,

$$D^5(x^3e^x)=e^x\left(x^3+\binom{5}{1}3x^2+\binom{5}{2}6x+\binom{5}{3}6\right)=e^x(x^3+15x^2+60x+60).$$

p. 240

7.3.6 By Eq. (7.3.5), $J(3x^2-4x+1)=x^3-2x^2+x$. For $p=x^2+5$, $Dp=2x$ and $JDp=x^2=x^2+5-5=p-p(0)$. p. 240

7.3.7 The first claim is proved in the example after Definition 7.3.2. The function $f(x)=|x|^3$ equals x^3 for $x\geq 0$ and $-x^3$ for $x<0$. Hence $f'(x)=3x|x|$ for $x\neq 0$. At 0, $|f(h)/h|=h^2\to 0$ and therefore $f'(0)=0$ and $f'(x)=3x|x|$ everywhere. By the first claim the second derivative $f''(x)=6|x|$ exists and is continuous. Hence $f\in C^2(\mathbb{R})$. Since $6|x|$ has no derivative at 0, $f\notin C^3(\mathbb{R})$. p. 240

7.3.8 Differentiating gives $a\lambda e^{\lambda x}+b\mu e^{\mu x}=0$. Subtracting λ times the original identity leaves $b(\mu-\lambda)e^{\mu x}=0$. Since $\mu\neq\lambda$ and $e^{\mu x}>0$, $b=0$. Then $ae^{\lambda x}=0$ gives $a=0$. p. 240

7.3.9 For $n = 1$ the claim is $Df = Df$. If it holds for n, then Eq. (7.3.3) gives the claim for $n+1$: $D(f^{n+1}) = D(f^n \cdot f) = nf^{n-1}(Df)f + f^n Df = (n+1)f^n Df$. p. 240

7.3.10 Let $u = f' + f$ and $v = f' - f$. Then $u' = f'' + f' = f + f' = u$ and $v' = f'' - f' = f - f' = -v$. By Theorem 7.3.18, $u(x) = u(0)e^x$ and $v(x) = v(0)e^{-x}$. Subtracting gives $2f = u - v$ and the stated form

$$f(x) = \tfrac{1}{2}(f'(0) + f(0))e^x - \tfrac{1}{2}(f'(0) - f(0))e^{-x}.$$

In operator language, $D^2 - \mathrm{id} = (D - \mathrm{id})(D + \mathrm{id})$ and the proof solves the two first-order factors in turn. p. 240

7.3.11 On $[0,1]$, $0 \le x^n \le 1$ with value 1 at $x = 1$. Hence $\|g_n\|_\infty = 1/\sqrt{n}$. The derivative is $Dg_n = \sqrt{n}\, x^{n-1}$ with sup norm $\sqrt{n}$. The C^1 norm is the sum $1/\sqrt{n} + \sqrt{n}$. The quotient of the two norms of g_n is $1 + n$. No constant M has $\|f\|_{C^1} \le M\|f\|_\infty$ for every f. The norms are therefore not equivalent. p. 240

7.4 Taylor's Theorem

7.4.1 The derivatives $1/(1+x)$, $-1/(1+x)^2$, $2/(1+x)^3$ have values 1, -1, 2 at 0 with $\ln 1 = 0$. Dividing by $1!, 2!, 3!$ gives $P_3(x) = x - x^2/2 + x^3/3$. p. 247

7.4.2 With $t = \tan x$, $t' = 1 + t^2$, $t'' = 2tt' = 2t + 2t^3$ and $t''' = (2 + 6t^2)t'$. At 0 these are 1, 0 and 2 with $\tan 0 = 0$. Hence $P_3(x) = x + \frac{2}{3!}x^3 = x + x^3/3$. p. 247

7.4.3 Here $f(9) = 3$, $f'(9) = \frac{1}{6}$ and $f''(9) = -\frac{1}{4} \cdot 9^{-3/2} = -\frac{1}{108}$, so

$$P_2(9.2) = 3 + \frac{0.2}{6} - \frac{1}{216}(0.2)^2 \approx 3.0331481.$$

On $[9, 9.2]$, $|f'''(t)| = \frac{3}{8}t^{-5/2} \le \frac{3}{8} \cdot \frac{1}{243} = \frac{1}{648}$. Corollary 7.4.9 gives $|R_2| \le \frac{1}{648} \cdot \frac{(0.2)^3}{6} = \frac{0.008}{3888} \approx 2.06 \times 10^{-6}$. p. 247

7.4.4 Since $1 < e^c < 2$, the remainder lies strictly between $(0.5)^{n+1}/(n+1)!$ and $2(0.5)^{n+1}/(n+1)!$. For $n = 5$ the upper bound is $2 \cdot \frac{1}{64}/720 \approx 4.3 \times 10^{-5} < 10^{-4}$ and $n = 5$ suffices. For $n = 4$ the lower bound is $\frac{1}{32}/120 \approx 2.6 \times 10^{-4} > 10^{-4}$. The error of P_4 therefore exceeds 10^{-4}. A lower order has a larger error because $R_{n-1}(0.5) = R_n(0.5) + (0.5)^n/n!$. Hence $n = 5$ is the smallest order that works. p. 247

7.4.5 The Maclaurin polynomial of order 2 of $\sin x$ is x and $\sin''' = -\cos$. For $x \neq 0$, Eq. (7.4.6) gives $\sin x - x = -\cos(c)\, x^3/6$ for some c with $|\cos c| \le 1$. At $x = 0$ both sides are 0. p. 247

7.4.6 By Theorem 7.4.6 with $n = 3$, $\sin x = x - x^3/6 + r(x)$ with $r(x)/x^3 \to 0$. Hence $(\sin x - x)/x^3 = -\frac{1}{6} + r(x)/x^3 \to -\frac{1}{6}$. p. 247

7.4.7 The function $g(x) = \cos x - 1 + x^2/2$ has $g(0) = 0$, $g'(x) = x - \sin x$ with $g'(0) = 0$ and $g''(x) = 1 - \cos x \ge 0$. By Theorem 7.2.12, $g(x) \ge g(0) + g'(0)x = 0$ for every x. p. 247

7.4.8 Using the matrices of Problem 7.3.1, the identity matrix plus $hA + \frac{h^2}{2}A^2$ has columns $(1,0,0)$, $(h,1,0)$ and $(h^2, 2h, 1)$. The third column is the coordinate vector of $h^2 + 2hx + x^2 = (x+h)^2$, in agreement with Theorem 7.4.12. p. 247

7.4.9 By Eq. (7.4.6) with $n = 1$, $f(a \pm h) = f(a) \pm f'(a)h + \frac{1}{2}f''(c_\pm)h^2$ with $c_\pm$ between a and $a \pm h$. Adding, the terms in $f'(a)$ cancel and

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{f''(c_+) + f''(c_-)}{2}.$$

As $h \to 0$, $c_\pm \to a$ and the continuity of f'' at a gives the limit $f''(a)$. p. 247

7.4.10 Suppose $e = p/q$ with positive integers p, q. By Worked Example 7.4.2 with $n = q$, $0 < e - P_q(1) < 3/(q+1)!$. Multiply by $q!$:

$$0 < q!\, e - \sum_{k=0}^{q} \frac{q!}{k!} < \frac{3}{q+1}.$$

The middle quantity is an integer because $q!\, e = (q-1)!\, p$ and each $q!/k!$ is an integer. For $q \geq 2$ the right side is at most 1 and no integer lies strictly between 0 and 1. For $q = 1$ the number e would be an integer, contrary to $2 < e < 3$. Either way the assumption fails. p. 247

7.4.11 For $x \neq 0$ the chain rule gives $f'(x) = 2x^{-3}e^{-1/x^2}$. If $f^{(k)}(x) = q_k(1/x)e^{-1/x^2}$, then $f^{(k+1)}(x) = (-x^{-2}q_k'(1/x) + 2x^{-3}q_k(1/x))e^{-1/x^2}$, of the same form with $q_{k+1}(u) = -u^2 q_k'(u) + 2u^3 q_k(u)$.

For every integer $m \geq 1$, $x^{-m}e^{-1/x^2} \to 0$ as $x \to 0$: with $t = 1/x^2$, $|x^{-m}e^{-1/x^2}| = t^{m/2}/e^t \to 0$ as $t \to \infty$ by Theorem 5.2.26. The case $m = 0$ is immediate. Hence $q(1/x)e^{-1/x^2} \to 0$ for every polynomial q.

Now suppose $f^{(k)}(0) = 0$, as is the case for $k = 0$. The difference quotient $f^{(k)}(h)/h = h^{-1}q_k(1/h)e^{-1/h^2}$ at 0 has the form $q(1/h)e^{-1/h^2}$ and tends to 0. Hence $f^{(k+1)}(0) = 0$. By induction every derivative exists everywhere and is continuous at 0 by the limit above. Hence $f \in C^\infty(\mathbb{R})$. Every Taylor polynomial at 0 is 0. The sum $\sum_{k\geq 0} h^k f^{(k)}(0)/k!$ of the Taylor series is therefore 0 although $f(h) = e^{-1/h^2} > 0$ for $h \neq 0$. p. 247

8.1 Antiderivatives

8.1.1

$$\int (2x^3 - 5x + 7)\,\mathrm{d}x = \frac{2x^4}{4} - \frac{5x^2}{2} + 7x + C = \frac{x^4}{2} - \frac{5x^2}{2} + 7x + C.$$

The derivative of $\frac{x^4}{2} - \frac{5x^2}{2} + 7x$ equals the integrand $2x^3 - 5x + 7$ (Definition 8.1.1). p. 253

8.1.2 Since $\dfrac{\mathrm{d}}{\mathrm{d}x}\arctan(2x) = \dfrac{2}{1+4x^2}$,

$$\int \frac{1}{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\arctan(2x) + C.$$

p. 253

8.1.3 The general antiderivative is $F(x) = x^3 + C$. Then $F(1) = 1 + C = -1$ gives $C = -2$ and hence $F(x) = x^3 - 2$. p. 253

8.1.4 Both F and G are antiderivatives of $f = F'$ on I. By Theorem 8.1.3 there is a constant C with $G(x) = F(x) + C$ for every $x \in I$. Because $G(x_0) = F(x_0) + C$ and $F(x_0) = G(x_0)$, the constant is $C = 0$. Hence $G = F$ on I. p. 253

8.1.5 By Proposition 8.1.5 the function $2\ln|x|$ is an antiderivative of $\frac{2}{x}$ on each of the two intervals. By Theorem 8.1.3 applied on each interval, $F(x) = 2\ln x + C_2$ for $x > 0$ and $F(x) = 2\ln(-x) + C_1$ for $x < 0$. The condition $F(1) = 0$ gives $C_2 = 0$ and the condition $F(-1) = 2\ln 1 + C_1 = 1$ gives $C_1 = 1$. p. 253

8.1.6 For $x > 0$ we have $F(x) = \frac{1}{2}x^2$ near x and hence $F'(x) = x = |x|$. For $x < 0$ we have $F(x) = -\frac{1}{2}x^2$ near x and hence $F'(x) = -x = |x|$. At $x = 0$ the difference quotient is

$$\frac{F(h) - F(0)}{h} = \frac{\frac{1}{2}h|h|}{h} = \tfrac{1}{2}|h| \qquad (h \neq 0),$$

which tends to $0 = |0|$ as $h \to 0$. Hence $F'(x) = |x|$ for every real x. p. 253

8.1.7 Suppose such an F exists. On the interval $(0, 1)$ it is an antiderivative of the constant 1. Theorem 8.1.3 gives a constant C_2 with $F(x) = x + C_2$ there. In the same way $F(x) = -x + C_1$ on $(-1, 0)$. Since F is differentiable at 0 it is continuous there (Theorem 6.1.11). Letting $x \to 0^+$ and $x \to 0^-$ gives $C_2 = F(0) = C_1$. Hence $F(x) = |x| + F(0)$ for every $x \in (-1, 1)$. By Example 8.1.2 this function has no derivative at 0. This is a contradiction. The sign function is therefore not the derivative of any function on $(-1, 1)$. p. 253

8.1.8 By the chain rule and Theorem 6.4.10,

$$G'(x) = -\frac{1}{1 + x^{-2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2 + 1} \qquad (x \neq 0),$$

which is $F'(x)$. The difference $G - F$ has zero derivative on each of the intervals $(0, \infty)$ and $(-\infty, 0)$ and is constant on each of them by Theorem 8.1.3. At $x = 1$ it is $-\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$. At $x = -1$ it is $\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. The two constants differ. The domain $\mathbb{R} \setminus \{0\}$ of G is a union of two intervals with one independent constant on each (Theorem 7.3.8). On $\mathbb{R}$ itself G is not an antiderivative because it is undefined at 0. p. 253

8.1.9 For $x \neq 0$ the product and chain rules give $F'(x) = 2x \sin(1/x) + x^2 \cos(1/x) \cdot (-1/x^2) = 2x \sin(1/x) - \cos(1/x)$. At 0 the difference quotient is

$$\frac{F(h) - F(0)}{h} = h \sin \frac{1}{h}, \qquad \left|h \sin \frac{1}{h}\right| \leq |h|,$$

which tends to 0. Hence $F'(0) = 0$. The points $x_k = 1/(2k\pi)$ tend to 0 and $f(x_k) = 0 - \cos(2k\pi) = -1$ for every k. If f were continuous at 0, then $f(x_k) \to f(0) = 0$ by Theorem 5.3.5. Hence f is not continuous at 0, although F is an antiderivative of f on $\mathbb{R}$. Continuity is therefore sufficient for an antiderivative (Corollary 8.3.8) and not necessary. p. 253

8.2 Definite Integral

8.2.1 Traversing $[1, 3]$ from 3 to 1 reverses every displacement (Definition 8.2.2). By the sign-reversal rule (Theorem 8.2.15), $\displaystyle\int_3^1 7 \, \mathrm{d}x = -\int_1^3 7 \, \mathrm{d}x = -7(3 - 1) = -14.$ p. 262

8.2.2 The integrand is continuous and the integral exists by Theorem 8.2.13. With n steps of width $\frac{1}{n}$ and $c_k = \frac{k}{n}$,

$$\sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \frac{1}{n} = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{(n+1)^2}{4n^2} \longrightarrow \frac{1}{4}.$$

p. 262

8.2.3 Suppose $f(p) > 0$ for some $p \in [a, b]$. By continuity there is a closed interval $[\alpha, \beta] \subseteq [a, b]$ with $\alpha < \beta$ and $p \in [\alpha, \beta]$ on which $f > \frac{1}{2}f(p)$. By additivity and the order property of

Theorem 8.2.15,

$$\int_a^b f = \int_a^\alpha f + \int_\alpha^\beta f + \int_\beta^b f \geq 0 + \tfrac{1}{2}f(p)(\beta-\alpha) + 0 > 0,$$

which contradicts the hypothesis. Hence $f = 0$ on $[a, b]$. p. 262

8.2.4 In a Riemann sum of f the only non-zero terms come from steps whose sample point is p. At most two steps contain p, each of width at most $\|P\|$. Hence $0 \leq S(f) \leq 2\|P\|$. Given $\varepsilon > 0$, put $\delta = \varepsilon/2$. Every sum with mesh $< \delta$ then satisfies $|S(f) - 0| < \varepsilon$. The integral is therefore 0.

Let $\tilde{g}$ equal g except at p and put $c = \tilde{g}(p) - g(p)$. On every tagged partition the two Riemann sums satisfy $S(\tilde{g}) = S(g) + c\,S(f)$ because they differ only in terms with sample point p. Let $J = \int_a^b g$ and $\varepsilon > 0$. Take δ_1 for g and $\varepsilon/2$, and take $\delta_2 = \varepsilon/(4(|c|+1))$. For mesh below both,

$$|S(\tilde{g}) - J| \leq |S(g) - J| + |c|\,2\|P\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\tilde{g}$ is integrable with $\int_a^b \tilde{g} = J$. p. 262

8.2.5 At $x = 0$ every f_n vanishes. For $x > 0$ choose $N \geq 2/x$ by Theorem 2.4.10. For $n \geq N$ the point x lies in $[\frac{2}{n}, 1]$ and $f_n(x) = 0$. Hence $f_n(x) \to 0$ for every x. By additivity and Worked Example 8.2.1,

$$\int_0^1 f_n\,\mathrm{d}x = n^2 \int_0^{1/n} x\,\mathrm{d}x + n^2 \int_{1/n}^{2/n} \Big(\frac{2}{n} - x\Big)\mathrm{d}x = \frac{1}{2} + \frac{1}{2} = 1.$$

The function f_n increases on $[0, \frac{1}{n}]$, decreases on $[\frac{1}{n}, \frac{2}{n}]$ and is 0 afterwards. Its maximum is $f_n(\frac{1}{n}) = n$. The last statement of Theorem 8.2.17 requires $\|f_n - f\|_\infty \to 0$. Here $\|f_n - 0\|_\infty = n$ grows without bound. The hypothesis therefore fails although the sequence converges to 0 at every point. Pointwise convergence does not allow the limit to pass through the integral. p. 262

8.2.6 Since $f(a) \leq f(x) \leq f(b)$ on $[a, b]$, the function is bounded. On the step $[x_{k-1}, x_k]$ every value lies between the two attained values $f(x_{k-1})$ and $f(x_k)$. Hence $M_k = f(x_k)$ and $m_k = f(x_{k-1})$. Because each difference $f(x_k) - f(x_{k-1})$ is non-negative and $\Delta x_k \leq \|P\|$, the sum telescopes to the bound

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\Delta x_k \leq \|P\| \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \|P\|(f(b) - f(a)).$$

Given $\varepsilon > 0$ put $\delta = \varepsilon/(f(b) - f(a) + 1)$. Every partition with mesh $< \delta$ has $U(f,P) - L(f,P) < \varepsilon$. Lemma 8.2.10 then shows that f is integrable. p. 263

8.2.7 The function $|f|$ is the constant 1 and has integral 1 by Example 8.2.5. Every step of every partition contains rational and irrational points (Theorem 2.4.12). Rational sample points give the Riemann sum $\sum_k \Delta x_k = 1$ and irrational ones give -1, on the same partition and for every mesh. No number lies within 1 of both 1 and -1. The definition therefore fails for $\varepsilon = 1$ and f is not integrable. The inequality of Theorem 8.2.15(iv) compares two integrals and so presupposes that $\int f$ exists. An integrable $|f|$ does not make f integrable. p. 263

8.2.8 For $x \in (0, 1]$ there is exactly one positive integer n with $n \leq 1/x < n+1$ (Lemma 2.4.11), that is, with $\frac{1}{n+1} < x \leq \frac{1}{n}$. Hence f is defined on $[0, 1]$. If $0 < x < y \leq 1$, the integer belonging to x is at least the one belonging to y and therefore $f(x) \leq f(y)$. Also $f(0) = 0 \leq f(y)$. Hence f is nondecreasing. For $n \geq 2$ the value at $\frac{1}{n}$ is $\frac{1}{n}$. Every $x \in (\frac{1}{n}, \frac{1}{n-1}]$ has $f(x) = \frac{1}{n-1}$. The

right-hand limit $\frac{1}{n-1}$ differs from $f(\frac{1}{n})$ and f is discontinuous at $\frac{1}{n}$. By Problem 8.2.6 f is integrable. It has infinitely many discontinuities and is therefore not piecewise continuous in the sense of Definition 8.2.12. The hypothesis of Theorem 8.2.13 is sufficient for integrability and not necessary. p. 263

8.3 Fundamental Theorem of Calculus

8.3.1 $\displaystyle\int_0^\pi \sin x\,\mathrm{d}x = \big[-\cos x\big]_0^\pi = -\cos\pi - (-\cos 0) = 1 + 1 = 2.$ p. 269

8.3.2 With $u = x^2$, $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_0^u \cos t\,\mathrm{d}t = \cos(u)\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos(x^2)\cdot 2x = 2x\cos(x^2)$. p. 269

8.3.3 Reversing the limits gives $\displaystyle\int_x^5 3t\sin t\,\mathrm{d}t = -\int_5^x 3t\sin t\,\mathrm{d}t$. By Theorem 8.3.4, $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_x^5 3t\sin t\,\mathrm{d}t = -3x\sin x$. p. 269

8.3.4 By Eq. (8.3.12), $F'(x) = (3x)^2\cdot 3 - x^2\cdot 1 = 27x^2 - x^2 = 26x^2$. Evaluating first, $F(x) = \big[\frac{1}{3}t^3\big]_x^{3x} = 9x^3 - \frac{1}{3}x^3 = \frac{26}{3}x^3$ with derivative $26x^2$. p. 270

8.3.5 Differentiating both sides gives $\dfrac{f(x)}{x^2} = \dfrac{1}{\sqrt{x}}$ by Corollary 8.3.8. Hence $f(x) = x^{3/2}$. At $x = a$ the integral is 0 and the identity reads $6 = 2\sqrt{a}$ with solution $a = 9$. Conversely with these choices $\int_9^x t^{-1/2}\,\mathrm{d}t = 2\sqrt{x} - 6$ by Eq. (8.3.4) and the identity is true for every $x > 0$. p. 270

8.3.6 The function $G(x) = \int_a^x f(t)\,\mathrm{d}t$ is identically 0 on $[a, b]$. By Theorem 8.3.4 its derivative at every $x \in [a, b]$, one-sided at the endpoints, is $f(x)$. Since the derivative of the zero function is 0, $f(x) = 0$ for every x. If f and g have the same area-so-far function, linearity gives $\int_a^x (f - g) = 0$ for every x. The first part applied to $f - g$ gives $f = g$. p. 270

8.3.7 The integrand is piecewise continuous and every integral exists by Theorem 8.2.13. Changing the integrand at the single point 0 does not change any of these integrals (Problem 8.2.4). For $-1 \le x \le 0$, $F(x) = \int_{-1}^x (-1)\,\mathrm{d}t = -(x + 1) = |x| - 1$. For $0 < x \le 1$ additivity gives $F(x) = -1 + \int_0^x 1\,\mathrm{d}t = x - 1$. Hence $F(x) = |x| - 1$. For $x, y \in [-1, 1]$, $|F(x) - F(y)| = \big|\int_y^x \operatorname{sgn} t\,\mathrm{d}t\big| \le |x - y|$ by Theorem 8.2.15(iv). Hence F is continuous. The function $|x| - 1$ has no derivative at 0 (Example 8.1.2). Theorem 8.3.4 assumes f continuous on $[a, b]$. The sign function is discontinuous at 0. Away from 0 the integrand is continuous and $F' = \operatorname{sgn}$ there, as the proof of Theorem 8.3.4 shows. p. 270

8.3.8 The choice $g = f$ gives $\int_a^b f(x)^2\,\mathrm{d}x = 0$. The function f^2 is continuous and non-negative. By Problem 8.2.3 it vanishes at every point of $[a, b]$. Hence $f = 0$. p. 270

8.3.9 Since $|\sin| \le 1$, $\|g_n\|_\infty \le \frac{1}{n} \to 0$. By the chain rule $g_n'(x) = n\cos(n^2 x)$. Its absolute value is at most n and equals n at $x = 0$. Hence $\|g_n'\|_\infty = n$. By Theorem 8.3.10, $J(g_n') = g_n - g_n(0) = g_n$. In the sup norm the sequence (g_n) converges to 0 while $\|g_n'\|_\infty = n$. Differentiation is therefore not continuous at 0. The bounded map J sends the large functions g_n' back to the small functions g_n as $\|Jh\|_\infty \le \|h\|_\infty$ on $[0, 1]$ requires. The estimate is far from equality here because g_n' oscillates and its positive and negative parts cancel in the integral. p. 270

8.3.10 By Corollary 8.3.8 and the chain rule, $G'(x) = f(x) + f(-x)\cdot(-1) = f(x) - f(-x) = 0$ for every real x because f is even. By Corollary 7.2.2 G is constant on $\mathbb{R}$ with value $G(0) = 2F(0) = 0$. Hence $F(-x) = -F(x)$ for every x. p. 270

8.4 Indefinite Integrals and the Substitution Method

8.4.1 With $u = 3x^2+4x$ the differential $\mathrm{d}u = (6x+4)\,\mathrm{d}x = 2(3x+2)\,\mathrm{d}x$ gives $(3x+2)\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. Then

$$\int (3x+2)(3x^2+4x)^4\,\mathrm{d}x = \frac{1}{2}\int u^4\,\mathrm{d}u = \frac{u^5}{10} + C = \frac{(3x^2+4x)^5}{10} + C.$$

p. 273

8.4.2 Let $u = 1 - r^3$. Then $\mathrm{d}u = -3r^2\,\mathrm{d}r$ gives $9r^2\,\mathrm{d}r = -3\,\mathrm{d}u$ and

$$\int \frac{9r^2}{\sqrt{1-r^3}}\,\mathrm{d}r = -3\int u^{-1/2}\,\mathrm{d}u = -3\cdot 2u^{1/2} + C = -6\sqrt{1-r^3} + C.$$

p. 273

8.4.3 Let $u = \sin(x/3)$. The differential $\mathrm{d}u = \frac{1}{3}\cos(x/3)\,\mathrm{d}x$ gives $\cos(x/3)\,\mathrm{d}x = 3\,\mathrm{d}u$. Then

$$\int \sin^5\!\left(\frac{x}{3}\right)\cos\!\left(\frac{x}{3}\right)\mathrm{d}x = 3\int u^5\,\mathrm{d}u = \frac{3u^6}{6} + C = \frac{1}{2}\sin^6\!\left(\frac{x}{3}\right) + C.$$

p. 273

8.4.4 Put $g(x) = \alpha x + \beta$ with $\mathrm{d}g = \alpha\,\mathrm{d}x$. The integrand $f(\alpha x+\beta)\,\mathrm{d}x$ is $\frac{1}{\alpha}\,g^*(f(u)\,\mathrm{d}u)$. By Theorem 8.4.2 and Proposition 8.1.6 its antiderivatives are $\frac{1}{\alpha}F(\alpha x+\beta) + C$. Differentiating confirms the result: $\dfrac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\alpha}F(\alpha x+\beta) = \frac{1}{\alpha}f(\alpha x+\beta)\cdot\alpha$. p. 273

8.4.5 With $u = \ln x$ and $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$,

$$\int \frac{\ln x}{x}\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

p. 273

8.4.6 With $u = \ln x$ and $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ the integral becomes $\int \dfrac{1}{u}\,\mathrm{d}u = \ln|u| + C$. On $(1,\infty)$ the value $u = \ln x$ is positive and the result is $\ln(\ln x) + C$. p. 273

8.4.7 The function g is continuous on I (Theorem 6.1.11). If it took a positive and a negative value, the Intermediate Value Theorem (Theorem 5.4.2) would give a zero between them. Hence g has one sign on I. Its image $K = g(I)$ is an interval (Corollary 5.4.4) contained in $(0,\infty)$ or in $(-\infty,0)$. On K the function $f(u) = 1/u$ is continuous with antiderivative $\ln|u|$ (Proposition 8.1.5). The integrand is $f(g(x))\,g'(x)$. Theorem 8.4.2 gives the general antiderivative $\ln|g(x)| + C$ on I. p. 273

8.4.8 On $(1,\infty)$ the substitution of Problem 8.4.6 gives $\ln(\ln x)+C_2$. On $(0,1)$ put $u = \ln x$ with values in the interval $(-\infty,0)$, on which $1/u$ has the antiderivative $\ln|u| = \ln(-u)$. Theorem 8.4.2 gives $\ln(-\ln x) + C_1$. Both formulas are $\ln|\ln x|$. The difference of two antiderivatives on the union has zero derivative and is constant on each of the two intervals by Theorem 7.3.8. The two constants are unrelated. p. 273

8.4.9 Take $I = \mathbb{R}$, $g(x) = x^2$ with values in $K = [0,\infty)$, and $f = \cos$, continuous on K with antiderivative sin. By Theorem 8.4.2 the general antiderivative is $\sin(x^2) + C$. The proof of the theorem shows $\mathrm{d}(F\circ g) = g^*(\mathrm{d}F)$ by the chain rule for differentials and uses no inverse of g. Injectivity becomes necessary only when an antiderivative found after a substitution $x = \varphi(t)$ has to be expressed through x as in trigonometric substitution (Theorem 9.1.4). p. 274

8.5 Definite Integral Substitutions and the Area Between Curves

8.5.1 Let $u = g(x) = x^2 + 1$, with $g^*(\mathrm{d}u) = \mathrm{d}g = 2x\,\mathrm{d}x$. The integrand is $\frac{1}{2}g^*(u^3\,\mathrm{d}u)$. By Eq. (8.5.3) its integral equals the integral over the image interval oriented from $g(0) = 1$ to $g(1) = 2$. Then

$$\int_0^1 x(x^2+1)^3\,\mathrm{d}x = \frac{1}{2}\int_1^2 u^3\,\mathrm{d}u = \frac{1}{2}\left[\tfrac{u^4}{4}\right]_1^2 = \frac{16-1}{8} = \frac{15}{8}.$$

Alternatively antidifferentiate in x first to get $\int x(x^2+1)^3\,\mathrm{d}x = \frac{(x^2+1)^4}{8}$ and evaluate without converting the limits: $\left[\frac{(x^2+1)^4}{8}\right]_0^1 = \frac{16-1}{8} = \frac{15}{8}$. p. 278

8.5.2 Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$. As $x : 0 \to \frac{\pi}{2}$, $u : 0 \to 1$. Then

$$\int_0^{\pi/2} \sin^2 x \cos x\,\mathrm{d}x = \int_0^1 u^2\,\mathrm{d}u = \left[\tfrac{u^3}{3}\right]_0^1 = \frac{1}{3}.$$

p. 278

8.5.3 The curves meet where $4 - x^2 = x + 2$. This equation reduces to $x^2 + x - 2 = 0$ with roots $x = -2$ and $x = 1$. Since the parabola lies above the line on $[-2, 1]$,

$$A = \int_{-2}^1 ((4-x^2) - (x+2))\,\mathrm{d}x = \int_{-2}^1 (2 - x - x^2)\,\mathrm{d}x = \left[2x - \tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_{-2}^1 = \frac{9}{2}.$$

p. 278

8.5.4 The curves meet where $x^3 = x$, at $x = -1, 0, 1$. On $[0, 1]$ the line is on top and the area there is $\int_0^1 (x - x^3)\,\mathrm{d}x = \frac{1}{4}$. On $[-1, 0]$ the cubic is on top. The substitution $x = -t$ shows that $\int_{-1}^0 (x^3 - x)\,\mathrm{d}x = \int_0^1 (t - t^3)\,\mathrm{d}t = \frac{1}{4}$. The total area is $\frac{1}{2}$. p. 278

8.5.5 Apply Theorem 8.5.2 with $g(t) = \frac{\pi}{2} - t$, $g'(t) = -1$ and $f(x) = \sin^n x$ on $[0, \frac{\pi}{2}]$. Since $g(0) = \frac{\pi}{2}$ and $g(\frac{\pi}{2}) = 0$,

$$-\int_0^{\pi/2} \sin^n\left(\tfrac{\pi}{2} - t\right)\mathrm{d}t = \int_{\pi/2}^0 \sin^n x\,\mathrm{d}x = -\int_0^{\pi/2} \sin^n x\,\mathrm{d}x.$$

With $\sin(\frac{\pi}{2} - t) = \cos t$ the left side is $-\int_0^{\pi/2} \cos^n t\,\mathrm{d}t$. p. 278

8.5.6 By additivity (Theorem 8.2.15), $\int_a^{a+p} f = \int_a^0 f + \int_0^p f + \int_p^{a+p} f$. In the last integral apply Theorem 8.5.2 with $g(t) = t + p$ and $g'(t) = 1$ on the interval from 0 to a: $\int_0^a f(t+p)\,\mathrm{d}t = \int_p^{a+p} f(x)\,\mathrm{d}x$. Periodicity gives $f(t+p) = f(t)$. Hence $\int_p^{a+p} f = \int_0^a f = -\int_a^0 f$. The first and last terms cancel and $\int_0^p f$ remains. p. 278

8.5.7 Write I for the left side. Apply Theorem 8.5.2 with $g(t) = \pi - t$, $g'(t) = -1$, $g(0) = \pi$ and $g(\pi) = 0$:

$$I = \int_\pi^0 (\pi - t)\, f(\sin(\pi - t))\,(-1)\,\mathrm{d}t = \int_0^\pi (\pi - t)\, f(\sin t)\,\mathrm{d}t = \pi\int_0^\pi f(\sin t)\,\mathrm{d}t - I.$$

Hence $2I = \pi \int_0^\pi f(\sin x)\,\mathrm{d}x$. The integrand $\dfrac{\sin x}{1+\cos^2 x} = \dfrac{\sin x}{2-\sin^2 x}$ is $f(\sin x)$ for the function $f(s) = s/(2-s^2)$, continuous on $[0,1]$. With $u = \cos x$, decreasing from 1 to -1,

$$\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x = -\int_1^{-1} \frac{\mathrm{d}u}{1+u^2} = \big[\arctan u\big]_{-1}^{1} = \frac{\pi}{2}.$$

The required integral is $\dfrac{\pi}{2}\cdot\dfrac{\pi}{2} = \dfrac{\pi^2}{4}$. p. 278

8.5.8 The hypotheses of Theorem 8.5.2 are that g' be continuous on $[-1,2]$ and $f(u) = e^u$ continuous on the range $[0,4]$ of g. Both hold. Hence

$$\int_{-1}^{2} 2x\,e^{x^2}\,\mathrm{d}x = \int_1^4 e^u\,\mathrm{d}u = e^4 - e.$$

Splitting at $x = 0$ shows the cancellation. On $[-1,0]$ the substitution decreases and gives $\int_1^0 e^u\,\mathrm{d}u$. On $[0,2]$ it increases and gives $\int_0^4 e^u\,\mathrm{d}u$. By additivity the sum is $\int_1^0 + \int_0^1 + \int_1^4 = \int_1^4$. The integrals over $[0,1]$ in the two orientations cancel. Injectivity is needed only in a substitution $x = \varphi(t)$ by a bijection, to express t through x in an indefinite integral (Theorem 9.1.4). Neither a substitution $u = g(x)$ nor a definite integral needs it. p. 279

8.5.9 By Theorem 8.3.5, $\int_{-1}^{1} \dfrac{\mathrm{d}x}{1+x^2} = \arctan 1 - \arctan(-1) = \dfrac{\pi}{2}$. Theorem 8.5.2 needs a function g with g' continuous on the whole interval $[-1,1]$. The function $1/x$ is not even defined at 0. The computation applies the theorem outside its hypotheses. On $[\varepsilon,1]$ and $[-1,-\varepsilon]$ the substitution is valid. As x runs over these two pieces, u runs from $1/\varepsilon$ to 1 and from -1 to $-1/\varepsilon$. Neither piece of the u-line lies between -1 and 1. p. 279

8.6 Arc Length

8.6.1 Here $f' = 2x(x^2+1)^{1/2}$ gives $1 + (f')^2 = 1 + 4x^2(x^2+1) = (2x^2+1)^2$. Then

$$L = \int_0^1 (2x^2+1)\,\mathrm{d}x = \Big[\tfrac{2x^3}{3} + x\Big]_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

p. 284

8.6.2 Here $f' = \frac{x^2}{2} - \frac{1}{2x^2}$ gives $1 + (f')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$ and

$$L = \int_1^2 \left(\tfrac{x^2}{2} + \tfrac{1}{2x^2}\right)\mathrm{d}x = \Big[\tfrac{x^3}{6} - \tfrac{1}{2x}\Big]_1^2 = \frac{17}{12}.$$

p. 284

8.6.3 Here $f' = \frac{x}{4} - \frac{1}{x}$ gives $1 + (f')^2 = \left(\frac{x}{4} + \frac{1}{x}\right)^2$ and

$$L = \int_1^2 \left(\tfrac{x}{4} + \tfrac{1}{x}\right)\mathrm{d}x = \Big[\tfrac{x^2}{8} + \ln x\Big]_1^2 = \frac{3}{8} + \ln 2.$$

p. 284

8.6.4 For a partition P let $\Lambda(P)$ be the length of the inscribed polygon. Adding one point to P replaces a segment by two segments with the same endpoints. By the triangle inequality in the plane the two are together at least as long as the one. Hence $\Lambda(P') \geq \Lambda(P)$ whenever P'

contains P. The partition $\{a, b\}$ gives the chord length Λ_0. Let P_n divide $[a, b]$ into n equal steps. Each P_n contains $\{a, b\}$ and therefore $\Lambda(P_n) \geq \Lambda_0$. By the derivation following Definition 8.6.2, $\Lambda(P_n) \to L$ as $n \to \infty$. The inequality passes to the limit: $L \geq \Lambda_0$. p. 284

8.6.5 By the chain rule $g'(x) = f'(x - h)$, continuous on $[a + h, b + h]$. With $u = x - h$ and $\mathrm{d}u = \mathrm{d}x$, Theorem 8.5.2 gives

$$\int_{a+h}^{b+h} \sqrt{1 + f'(x-h)^2}\,\mathrm{d}x = \int_a^b \sqrt{1 + f'(u)^2}\,\mathrm{d}u.$$

The left side is the length of the graph of g and the right side is that of f. p. 284

8.6.6 By the chain rule $g'(x) = \lambda f'(x/\lambda) \cdot \dfrac{1}{\lambda} = f'(x/\lambda)$, continuous on $[\lambda a, \lambda b]$. Put $\varphi(x) = \sqrt{1 + f'(x/\lambda)^2}$. Theorem 8.5.2 with $h(u) = \lambda u$, $h'(u) = \lambda$, $h(a) = \lambda a$ and $h(b) = \lambda b$ gives

$$\int_{\lambda a}^{\lambda b} \varphi(x)\,\mathrm{d}x = \int_a^b \varphi(\lambda u)\,\lambda\,\mathrm{d}u = \lambda \int_a^b \sqrt{1 + f'(u)^2}\,\mathrm{d}u.$$

The left side is the length of the graph of g and the integral on the right is the length of the graph of f. p. 284

8.6.7 By Example 8.6.3 with slopes 1 and 2 the lengths are $\sqrt{1+1} = \sqrt{2}$ and $\sqrt{1+4} = \sqrt{5}$. A linear map L would give $L(2f) = 2L(f) = 2\sqrt{2} = \sqrt{8}$. Since $\sqrt{5} \neq \sqrt{8}$, the map is not linear. A linear map also sends 0 to 0, whereas the zero function has length $L(0) = 1$. p. 284

8.6.8 Since $|\sin| \leq 1$, $\|f_n\|_\infty \leq \frac{1}{n}$. The derivative is $f_n'(x) = n\cos(n^2 x)$. From $\sqrt{1+t^2} \geq |t|$ and the order property,

$$L(f_n) = \int_0^\pi \sqrt{1 + n^2\cos^2(n^2 x)}\,\mathrm{d}x \geq n \int_0^\pi |\cos(n^2 x)|\,\mathrm{d}x.$$

The substitution $u = n^2 x$ turns the last integral into $\dfrac{1}{n^2}\int_0^{n^2\pi} |\cos u|\,\mathrm{d}u$. The function $|\cos u|$ has period π. Splitting at $\pi/2$, where cos changes sign, $\int_0^\pi |\cos u|\,\mathrm{d}u = [\sin u]_0^{\pi/2} - [\sin u]_{\pi/2}^{\pi} = 1 + 1 = 2$. By additivity and Problem 8.5.6 the integral over $[0, n^2\pi]$ is $2n^2$. Hence $L(f_n) \geq 2n$. If L were continuous at the zero function for the sup norm, then $\|f_n - 0\|_\infty \to 0$ would force $L(f_n) \to L(0) = \pi$. The lengths grow without bound. A graph can be uniformly close to a segment and still be arbitrarily long. p. 284

9.1 Strategy for a General Integrand

9.1.1 On $(0, \infty)$ the integrand equals $x^{3/2-2} + 2x^{1/2-2} = x^{-1/2} + 2x^{-3/2}$. By Propositions 8.1.5 and 8.1.6,

$$\int (x^{-1/2} + 2x^{-3/2})\,\mathrm{d}x = 2x^{1/2} - 4x^{-1/2} + C = 2\sqrt{x} - \frac{4}{\sqrt{x}} + C.$$

p. 290

9.1.2 Since $2x^2 + 4x + 10 = 2((x+1)^2 + 4)$, Eq. (9.1.3) with $u = x + 1$ and $a = 2$ gives

$$\int \frac{\mathrm{d}x}{2x^2 + 4x + 10} = \frac{1}{2}\int \frac{\mathrm{d}x}{(x+1)^2 + 2^2} = \frac{1}{2}\cdot\frac{1}{2}\arctan\frac{x+1}{2} + C = \frac{1}{4}\arctan\frac{x+1}{2} + C.$$

p. 290

9.1.3 With $\varphi(t) = t^2$ from $(0,\infty)$ onto $(0,\infty)$ the pullback is $\dfrac{t}{1+t^2}\cdot 2t\,\mathrm{d}t = 2\Big(1 - \dfrac{1}{1+t^2}\Big)\mathrm{d}t$, with antiderivative $G(t) = 2t - 2\arctan t$. By Theorem 9.1.4 with $\varphi^{-1}(x) = \sqrt{x}$,

$$\int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x = 2\sqrt{x} - 2\arctan\sqrt{x} + C.$$

p. 290

9.1.4 By Eq. (9.1.5), $x^2 + bx + c = (x + \frac{b}{2})^2 + a^2$ with $a = \frac{d}{2} > 0$. The denominator is positive and the integrand is continuous on $\mathbb{R}$. With $u = x + \frac{b}{2}$ and $\mathrm{d}u = \mathrm{d}x$, Eq. (9.1.3) gives

$$\int \frac{\mathrm{d}x}{x^2+bx+c} = \frac{1}{a}\arctan\frac{x+b/2}{a} + C = \frac{2}{d}\arctan\frac{2x+b}{d} + C.$$

p. 290

9.1.5 For $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$ the number $s = \pi - t$ lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\sin t = \sin(\pi - s) = \sin s$ (Theorem 4.5.9). The sine is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $(-1,1)$. The map $t \mapsto \pi - t$ is a bijection between the two intervals. Their composite φ is a bijection onto $(-1,1)$ with inverse $\pi - \arcsin x$. On the interval $\cos t < 0$. Hence $\sqrt{1-\sin^2 t} = |\cos t| = -\cos t$ and the pullback is $\cos t\,\mathrm{d}t/(-\cos t) = -\mathrm{d}t$. With $G(t) = -t$, Theorem 9.1.4 gives the antiderivative $G(\varphi^{-1}(x)) = \arcsin x - \pi$. It differs from the antiderivative $\arcsin x$ of Example 9.1.5 by a constant, as Theorem 8.1.3 requires. Ignoring the sign of $\cos t$ would give $\pi - \arcsin x$. Its derivative has the wrong sign. p. 291

9.1.6 On each of the two intervals the integrand equals $x + 4$, with the antiderivatives $\frac{1}{2}x^2 + 4x + C$ on that interval (Theorem 8.1.3). The constants on the two intervals are independent (Theorem 7.3.8). The function $\frac{1}{2}x^2 + 4x$ is an antiderivative of $x + 4$ on $\mathbb{R}$. The quotient is a different function and is undefined at 4. The condition $F'(4) = f(4)$ has no meaning when $f(4)$ does not exist. p. 291

9.1.7 The function e^{x^2} is continuous on $\mathbb{R}$. By Corollary 8.3.8 the area-so-far function $F(x) = \int_0^x e^{t^2}\,\mathrm{d}t$ is an antiderivative, with $F'(x) = e^{x^2}$ by Theorem 8.3.4. Hence $F(0) = 0$ and $F'(0) = e^0 = 1$. The integrand is no pullback of a tabulated 1-form (the factor $2x$ is missing), no product with a factor that differentiation simplifies, and neither rational nor trigonometric nor a root of a quadratic. The strategy assigns it to the area-so-far function and to numerical integration. That the methods do not apply is no contradiction: existence of an antiderivative and the existence of a formula for it are different statements. p. 291

9.1.8 The map $\varphi(t) = -t$ is a differentiable bijection of $(-a,a)$ onto itself. The pullback of $f(x)\,\mathrm{d}x$ is $f(-t)\cdot(-1)\,\mathrm{d}t = f(t)\,\mathrm{d}t$ because f is odd. By Theorem 8.4.2, $F\circ\varphi$ is an antiderivative of this pullback, that is, of f. By Theorem 8.1.3 there is a constant C with $F(-t) = F(t) + C$ for every $t \in (-a,a)$. At $t = 0$ this reads $F(0) = F(0) + C$. Hence $C = 0$ and $F(-t) = F(t)$. p. 291

9.2 Integration by Parts

9.2.1 The derivatives of $u = x^2$ are $x^2, 2x, 2, 0$ and every V_k is e^x. By Theorem 9.2.9 with $n = 2$,

$$\int x^2 e^x\,\mathrm{d}x = x^2e^x - 2xe^x + 2e^x + C = (x^2 - 2x + 2)e^x + C.$$

p. 296

9.2.2 Write I for one antiderivative of $e^x\cos x$. With $u=\cos x$ and $v=e^x$, Theorem 9.2.1(i) shows that $e^x\cos x+G$ is an antiderivative of $e^x\cos x$ whenever G is one of $e^x\sin x$. With $u=\sin x$ and $v=e^x$, the function $G=e^x\sin x-I$ is one. Hence $e^x\cos x+e^x\sin x-I$ is an antiderivative of $e^x\cos x$ and equals $I+C_0$ by Theorem 8.1.3. Then $2I=e^x(\sin x+\cos x)-C_0$. Restoring the constant,

$$\int e^x\cos x\,\mathrm{d}x=\frac{e^x}{2}(\sin x+\cos x)+C.$$

p. 296

9.2.3 With $u=\arctan x$ and $\mathrm{d}v=\mathrm{d}x$ we have $\mathrm{d}u=\mathrm{d}x/(1+x^2)$ and $v=x$. Then

$$\int\arctan x\,\mathrm{d}x=x\arctan x-\int\frac{x}{1+x^2}\,\mathrm{d}x=x\arctan x-\frac{1}{2}\ln(1+x^2)+C.$$

The last integral is the pullback by $w=1+x^2$. Its logarithm needs no absolute value because $w>0$ (Problem 8.4.7). p. 296

9.2.4 With $u=x$ and $v=-e^{-x}$, Eq. (9.2.2) gives

$$\int_0^1 xe^{-x}\,\mathrm{d}x=\left[-xe^{-x}\right]_0^1+\int_0^1 e^{-x}\,\mathrm{d}x=-\frac{1}{e}+\left(1-\frac{1}{e}\right)=1-\frac{2}{e}.$$

p. 296

9.2.5 For $n=0$ the formula reads $f(x)=f(a)+\int_a^x f'(t)\,\mathrm{d}t$. That identity is Theorem 8.3.5 for the continuous function f'. Let $n\geq 1$ and assume the formula for $n-1$. Its remainder is $R=\dfrac{1}{(n-1)!}\int_a^x(x-t)^{n-1}f^{(n)}(t)\,\mathrm{d}t$. In the variable t take $u=f^{(n)}(t)$ and $\mathrm{d}v=(x-t)^{n-1}\,\mathrm{d}t$, with $\mathrm{d}u=f^{(n+1)}(t)\,\mathrm{d}t$ and $v=-(x-t)^n/n$. Both have continuous derivatives on I and Eq. (9.2.2) gives

$$\int_a^x(x-t)^{n-1}f^{(n)}(t)\,\mathrm{d}t=\Big[-\frac{(x-t)^n}{n}f^{(n)}(t)\Big]_{t=a}^{t=x}+\frac{1}{n}\int_a^x(x-t)^nf^{(n+1)}(t)\,\mathrm{d}t.$$

The bracket equals $\dfrac{(x-a)^n}{n}f^{(n)}(a)$. Dividing by $(n-1)!$,

$$R=\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{1}{n!}\int_a^x(x-t)^nf^{(n+1)}(t)\,\mathrm{d}t,$$

and the formula for n follows. For the bound, the order property applied on the interval between a and x gives

$$\Big|\frac{1}{n!}\int_a^x(x-t)^nf^{(n+1)}(t)\,\mathrm{d}t\Big|\leq\frac{M}{n!}\Big|\int_a^x|x-t|^n\,\mathrm{d}t\Big|=\frac{M}{n!}\cdot\frac{|x-a|^{n+1}}{n+1}=\frac{M|x-a|^{n+1}}{(n+1)!}.$$

p. 297

9.2.6 With $u=(\ln x)^n$ and $v=x$, both in $C^1((0,\infty))$, $\mathrm{d}u=n(\ln x)^{n-1}x^{-1}\,\mathrm{d}x$ and $v\,\mathrm{d}u=n(\ln x)^{n-1}\,\mathrm{d}x$. Equation (9.2.1) is the stated formula. Applying it with $n=3,2,1$ and $\int\mathrm{d}x=x$,

$$\begin{aligned}\int(\ln x)^3\,\mathrm{d}x&=x(\ln x)^3-3\Big(x(\ln x)^2-2(x\ln x-x)\Big)+C\\&=x\big((\ln x)^3-3(\ln x)^2+6\ln x-6\big)+C.\end{aligned}$$

p. 297

9.2.7 The symbol $\int \mathrm{d}x/x$ denotes the set of functions $\ln x + C$ with C ranging over $\mathbb{R}$ (Definition 8.1.4). The right side $1 + \int \mathrm{d}x/x$ is the set of functions $1 + \ln x + C$. As C ranges over $\mathbb{R}$, so does $1 + C$. The two sets coincide. The equation is therefore true. Cancelling the integral on both sides treats a family as a single number. Theorem 9.2.1 in the precise form (i) states that if G is one antiderivative of $v\,u' = -1/x$, then $uv - G = 1 - G$ is one antiderivative of $1/x$. With $G = -\ln x$ the function $1 + \ln x$ is indeed an antiderivative of $1/x$. p. 297

9.2.8 The factor $x^2 e^{x^3}$ has the antiderivative $\frac{1}{3}e^{x^3}$ by the substitution $w = x^3$. Take $u = x^3$, $\mathrm{d}v = x^2 e^{x^3}\,\mathrm{d}x$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and $v = \frac{1}{3}e^{x^3}$. Then

$$\int x^5 e^{x^3}\,\mathrm{d}x = \frac{1}{3}x^3 e^{x^3} - \int x^2 e^{x^3}\,\mathrm{d}x = \frac{1}{3}(x^3 - 1)e^{x^3} + C.$$

p. 297

9.2.9 With $u = f$ and $v = -\cos(\lambda x)/\lambda$, Eq. (9.2.2) gives

$$\int_a^b f(x)\sin(\lambda x)\,\mathrm{d}x = \Big[-\frac{f(x)\cos(\lambda x)}{\lambda}\Big]_a^b + \frac{1}{\lambda}\int_a^b f'(x)\cos(\lambda x)\,\mathrm{d}x.$$

Since $|\cos| \le 1$, the bracket is at most $(|f(a)| + |f(b)|)/\lambda$ in absolute value. The function $f'\cos(\lambda x)$ is continuous with sup norm at most $\|f'\|_\infty$. By Eq. (8.2.6) its integral is at most $(b-a)\|f'\|_\infty$. The triangle inequality gives the estimate. Its right side is a constant divided by λ and tends to 0 as $\lambda \to \infty$. p. 297

9.2.10 With $u = \sin x$, $u' = \cos x$, $u'' = -\sin x$ and $V_1 = V_2 = e^x$, Eq. (9.2.5) with $n = 1$ reads

$$\int e^x \sin x\,\mathrm{d}x = e^x \sin x - e^x \cos x + \int (-\sin x)e^x\,\mathrm{d}x.$$

The last integral is the negative of the original. As in Worked Example 9.2.4, for one antiderivative I the function $e^x(\sin x - \cos x) - I$ is another and equals $I + C_0$ for a constant C_0. The general answer is $\frac{1}{2}e^x(\sin x - \cos x) + C$. The polynomial case of Theorem 9.2.9 terminates because $u^{(n+1)} = 0$. The derivatives of $\sin x$ cycle through $\pm\sin x$ and $\pm\cos x$, none of them 0. The remainder in Eq. (9.2.5) is therefore never zero. Only the cyclic equation closes the computation. p. 297

9.3 Trigonometric Integrals

9.3.1 By Eq. (9.3.3), $\sin 4x \cos 2x = \frac{1}{2}[\sin 2x + \sin 6x]$. Integrating term by term,

$$\int \sin 4x \cos 2x\,\mathrm{d}x = -\frac{\cos 2x}{4} - \frac{\cos 6x}{12} + C.$$

p. 303

9.3.2 By Eq. (9.3.10) with $k = 1$ and $n = 3$, the integrand is the pullback by $u = \sec x$ of $(u^2 - 1)u^2\,\mathrm{d}u$. Hence

$$\int \tan^3 x \sec^3 x\,\mathrm{d}x = \int (u^4 - u^2)\,\mathrm{d}u = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$$

p. 303

9.3.3 By Eq. (9.3.12) with $n = 5$, $\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\int \sec^3 x\,dx$. Substituting Worked Example 9.3.4,

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.$$

p. 303

9.3.4 The integrand is continuous on $(-\pi, \pi)$. By Eq. (9.3.13), $2 + \cos\varphi(t) = \dfrac{3+t^2}{1+t^2}$ and the pullback is $\dfrac{1+t^2}{3+t^2}\cdot\dfrac{2\,dt}{1+t^2} = \dfrac{2\,dt}{3+t^2}$. By Eq. (9.1.3) with $a = \sqrt{3}$ an antiderivative is $G(t) = \frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}}$. By Theorem 9.3.15,

$$\int \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}}\arctan\Big(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\Big) + C \qquad (-\pi < x < \pi).$$

p. 303

9.3.5 By Eq. (4.5.4) and Eq. (9.3.2), $\sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{8} - \frac{1}{8}\cos 4x$. Hence

$$\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx = \Big[\frac{x}{8} - \frac{\sin 4x}{32}\Big]_0^{\pi/2} = \frac{\pi}{16}.$$

p. 303

9.3.6 For $0 \le x \le \frac{\pi}{2}$ we have $0 \le \sin x \le 1$ and therefore $\sin^{2m+1} x \le \sin^{2m} x \le \sin^{2m-1} x$. The order property Theorem 8.2.15(iv) gives $W_{2m+1} \le W_{2m} \le W_{2m-1}$. All three are positive. Dividing by W_{2m+1} and using the recursion $W_{2m+1} = \frac{2m}{2m+1}W_{2m-1}$,

$$1 \le \frac{W_{2m}}{W_{2m+1}} \le \frac{W_{2m-1}}{W_{2m+1}} = \frac{2m+1}{2m}.$$

The right side tends to 1. The squeeze theorem (Theorem 5.1.11) gives $W_{2m}/W_{2m+1} \to 1$. By Eq. (9.3.8),

$$\frac{W_{2m}}{W_{2m+1}} = \frac{\pi}{2}\prod_{j=1}^{m}\frac{(2j-1)(2j+1)}{(2j)^2} = \frac{\pi}{2P_m}, \qquad P_m = \prod_{j=1}^{m}\frac{(2j)^2}{(2j-1)(2j+1)}.$$

Hence $\pi/(2P_m) \to 1$ and $P_m \to \pi/2$ by Theorem 5.1.9. p. 304

9.3.7 For an integer $k \neq 0$, $\int_{-\pi}^{\pi}\cos kx\,dx = \big[\sin kx/k\big]_{-\pi}^{\pi} = 0$. For every integer k the integrand $\sin kx$ is odd and has integral 0 over $[-\pi,\pi]$ (Corollary 8.5.4). By Eq. (9.3.3), $\sin mx\sin nx = \frac{1}{2}\big[\cos(m-n)x - \cos(m+n)x\big]$. Since $m+n \neq 0$, the second term integrates to 0. The first integrates to 0 when $m \neq n$ and to $\frac{1}{2}\cdot 2\pi = \pi$ when $m = n$. The product of cosines is the same with a plus sign. The product $\sin mx\cos nx$ is odd and has integral 0. p. 304

9.3.8 Since $\tan^2 x = \sec^2 x - 1$, the integrand is $\tan^{n-2} x\sec^2 x - \tan^{n-2} x$. The first term is the pullback by $u = \tan x$ of $u^{n-2}\,du$, with antiderivative $\tan^{n-1} x/(n-1)$. Linearity gives the formula. With $n = 4$ and then $n = 2$,

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \Big(\tan x - \int dx\Big) = \frac{\tan^3 x}{3} - \tan x + x + C.$$

p. 304

9.3.9 The integrand is positive and continuous on $[0, 2\pi]$. Its integral is positive by Theorem 8.2.15(iv). Theorem 8.3.5 needs an antiderivative continuous on all of $[0, 2\pi]$. On $(\pi, 3\pi)$ the function $\tan(x/2)$ is again defined and F is again an antiderivative there (Theorem 9.3.15 for $x - 2\pi$). At π the function F is undefined and has the one-sided limits $F(\pi^-) = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$ and $F(\pi^+) = -\frac{\pi}{\sqrt{3}}$. Let $\Phi(x) = \int_0^x \mathrm{d}t/(2 + \cos t)$, continuous on $[0, 2\pi]$ by Theorem 8.3.4. On $[0, \pi)$ it equals $F(x) - F(0)$. Letting $x \to \pi^-$ gives $\int_0^\pi = \frac{\pi}{\sqrt{3}}$. On $(\pi, 2\pi]$ the two functions $\Phi(x) - \Phi(\pi)$ and $F(x) - F(\pi^+)$ are antiderivatives of the integrand that tend to 0 at π^+. They differ by a constant (Theorem 8.1.3) whose value is the difference of these limits, 0. Hence $\int_\pi^{2\pi} = F(2\pi) - F(\pi^+) = 0 + \frac{\pi}{\sqrt{3}}$. The total is $\frac{2\pi}{\sqrt{3}}$. p. 304

9.3.10 For $x = \arctan t$ the proof of Theorem 9.3.15 gives $\sin^2 x = t^2 \cos^2 x = t^2/(1 + t^2)$. The 1-form $\mathrm{d}x$ pulls back to $\mathrm{d}t/(1 + t^2)$. Hence $1 + \sin^2 x$ pulls back to $(1 + 2t^2)/(1 + t^2)$ and the integrand to $\mathrm{d}t/(1 + 2t^2) = \frac{1}{2}\,\mathrm{d}t/(\frac{1}{2} + t^2)$. By Eq. (9.1.3) with $a = 1/\sqrt{2}$ an antiderivative is $\frac{1}{\sqrt{2}} \arctan(\sqrt{2}\,t)$. Theorem 9.1.4 with the inverse $t = \tan x$ gives the answer. The Weierstrass substitution would produce $\sin^2 x = 4t^2/(1+t^2)^2$ and a denominator of degree 4. Here $1+\sin^2 x = (1 + 2\tan^2 x)/(1 + \tan^2 x)$ is a rational function of $\tan x$ alone. The substitution by $\tan x$ keeps the degree at 2. p. 304

9.4 Trigonometric Substitutions

9.4.1 With $x = 2\sec\theta$ on $(0, \frac{\pi}{2})$, $\sqrt{x^2 - 4} = 2\tan\theta$ and the pullback is $\dfrac{2\tan\theta \cdot 2\sec\theta\tan\theta\,\mathrm{d}\theta}{2\sec\theta} = 2\tan^2\theta\,\mathrm{d}\theta = 2(\sec^2\theta - 1)\,\mathrm{d}\theta$, with antiderivative $2\tan\theta - 2\theta$. With $\theta = \arccos(2/x)$ and $\tan\theta = \frac{1}{2}\sqrt{x^2 - 4}$ (Lemma 9.4.2),

$$\int \frac{\sqrt{x^2 - 4}}{x}\,\mathrm{d}x = \sqrt{x^2 - 4} - 2\arccos\frac{2}{x} + C.$$

p. 309

9.4.2 With $u = x + 2$ the radicand is $u^2 + 9$. The substitution $u = 3\tan\theta$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ gives $\sqrt{u^2 + 9} = 3\sec\theta$ and the pullback $\sec\theta\,\mathrm{d}\theta$, with antiderivative $\ln(\sec\theta + \tan\theta)$. The argument is positive because $\sec\theta = \frac{1}{3}\sqrt{u^2 + 9} > \frac{1}{3}|u| \geq -\tan\theta$. By Lemma 9.4.2,

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 + 4x + 13}} = \ln\frac{u + \sqrt{u^2 + 9}}{3} + C = \ln\Big(x + 2 + \sqrt{x^2 + 4x + 13}\Big) + C.$$

p. 309

9.4.3 With $x = 2\sin\theta$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\sqrt{4 - x^2} = 2\cos\theta$ and the pullback is $4\sin^2\theta \cdot 2\cos\theta \cdot 2\cos\theta\,\mathrm{d}\theta = 16\sin^2\theta\cos^2\theta\,\mathrm{d}\theta = (2 - 2\cos 4\theta)\,\mathrm{d}\theta$, as in Problem 9.3.5. An antiderivative is $2\theta - \frac{1}{2}\sin 4\theta$. By the double-angle formulas $\sin 4\theta = 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)$, and Lemma 9.4.2 gives $\sin\theta = x/2$ and $\cos\theta = \frac{1}{2}\sqrt{4 - x^2}$. Hence $\sin 4\theta = \frac{1}{2}x(2 - x^2)\sqrt{4 - x^2}$ and

$$\int x^2\sqrt{4 - x^2}\,\mathrm{d}x = 2\arcsin\frac{x}{2} + \frac{x(x^2 - 2)\sqrt{4 - x^2}}{4} + C.$$

p. 309

9.4.4 By Proposition 9.4.1 the map $x = 5\sec\theta$ is a bijection from $(\frac{\pi}{2}, \pi)$ onto $(-\infty, -5)$ with $\sqrt{x^2 - 25} = -5\tan\theta$. The pullback is $5\sec\theta\tan\theta\,\mathrm{d}\theta/(-5\tan\theta) = -\sec\theta\,\mathrm{d}\theta$, with antiderivative

$-\ln|\sec\theta+\tan\theta|$ (Proposition 9.3.12). By Lemma 9.4.2, $\sec\theta = x/5$ and $\tan\theta = -\sqrt{x^2-25}/5$. The sum $\sec\theta+\tan\theta$ is negative, with absolute value $(\sqrt{x^2-25}-x)/5$. Hence

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-25}} = -\ln\Big(\sqrt{x^2-25}-x\Big)+C \qquad (x<-5),$$

with $\ln 5$ absorbed into the constant. Writing $5\tan\theta$ for the radical as on the other interval gives an antiderivative of the negative of the integrand. p. 309

9.4.5 For $|x|>1$ the number $1/x$ lies in $(-1,1)$. By the chain rule and Theorem 6.4.10,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos\frac{1}{x} = -\frac{1}{\sqrt{1-x^{-2}}}\cdot\Big(-\frac{1}{x^2}\Big) = \frac{1}{x^2\sqrt{(x^2-1)/x^2}} = \frac{1}{|x|\sqrt{x^2-1}}.$$

On $(1,\infty)$ this is $1/(x\sqrt{x^2-1})$. On $(-\infty,-1)$ it is the negative of it. There the antiderivative is $-\arccos(1/x)$. With $x=\sec\theta$ from $(0,\frac{\pi}{2})$ onto $(1,\infty)$ the pullback is $\sec\theta\tan\theta\,\mathrm{d}\theta/(\sec\theta\tan\theta) = \mathrm{d}\theta$, and Theorem 9.1.4 with the inverse $\theta=\arccos(1/x)$ gives $\arccos(1/x)+C$ again. p. 309

9.4.6 With $g(\theta)=\sin\theta$, $g(\pi)=0$ and $g(\frac{\pi}{2})=1$, and $g'=\cos$ continuous, Theorem 8.5.2 gives

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_\pi^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,\mathrm{d}\theta = \int_\pi^{\pi/2} (-\cos\theta)\cos\theta\,\mathrm{d}\theta = \int_{\pi/2}^{\pi} \cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{4},$$

by Eq. (9.3.2). The value is the area of a quarter of the unit disc (Example 9.4.3). With $\cos\theta$ in place of the radical the integrand becomes $\cos^2\theta$ on the reversed interval. The result is then $-\frac{\pi}{4}$. The sign of the radical is not part of the substitution formula and must be read off the interval. p. 309

9.4.7 By Proposition 9.4.1 the map $x=a\tan\theta$ is a bijection from $(-\frac{\pi}{2},\frac{\pi}{2})$ onto $\mathbb{R}$ with $\sqrt{a^2+x^2}=a\sec\theta$. The pullback is $a\sec\theta\cdot a\sec^2\theta\,\mathrm{d}\theta = a^2\sec^3\theta\,\mathrm{d}\theta$. By Worked Example 9.3.4 an antiderivative is $\frac{a^2}{2}(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta))$, with $\sec\theta+\tan\theta>0$ on this interval. By Lemma 9.4.2, $\sec\theta=\sqrt{a^2+x^2}/a$ and $\tan\theta=x/a$. Substituting,

$$\frac{a^2}{2}\Big(\frac{x\sqrt{a^2+x^2}}{a^2}+\ln\frac{x+\sqrt{a^2+x^2}}{a}\Big) = \frac{1}{2}x\sqrt{a^2+x^2}+\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})-\frac{a^2}{2}\ln a,$$

and Theorem 9.1.4 gives the formula. p. 309

9.4.8 Since $f'=2x$, Eq. (8.6.1) gives $L=\int_0^1\sqrt{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^2\sqrt{1+u^2}\,\mathrm{d}u$ by Theorem 8.5.2 with $u=2x$. The antiderivative of Problem 9.4.7 with $a=1$ gives

$$\begin{aligned} L &= \frac{1}{2}\Big[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2})\Big]_0^2 \\ &= \frac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5}) \approx 1.4789. \end{aligned}$$

p. 309

9.4.9 For $|x|\le a$ the ellipse contains the points with $|y|\le b\sqrt{1-x^2/a^2}$. The area is the integral of the length $2b\sqrt{1-x^2/a^2}$ of this segment (Definition 8.5.6). With $g(\theta)=a\sin\theta$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$, $g'(\theta)=a\cos\theta$ is continuous and $\cos\theta\ge 0$. Hence $\sqrt{1-g(\theta)^2/a^2}=\cos\theta$ and

$$2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x = 2ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \pi ab.$$

Theorem 8.5.2 transfers the integral to the image interval and never inverts g. Its hypotheses are a continuous g' on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and a continuous integrand on $[-a, a]$. Both are satisfied here. Table 9.4.1 lists open intervals because it serves indefinite integrals of every integrand with the radical. One of them is $1/\sqrt{a^2 - x^2}$, undefined at $\pm a$. p. 309

9.4.10 The region under the graph of the integrand is the upper half of a disc of radius 2. By Example 9.4.3 the disc has area $\pi r^2 = 4\pi$ and the half disc has area 2π. Hence $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\pi$. p. 310

9.4.11 $\operatorname{avg}(f) = \dfrac{1}{2-(-2)} \displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}$. p. 310

9.5 Integration of Rational Functions by Partial Fractions

9.5.1 Division gives $x^2 + 3x - 1 = (x^2 - 1) \cdot 1 + 3x$ and $\dfrac{x^2 + 3x - 1}{x^2 - 1} = 1 + \dfrac{3x}{(x-1)(x+1)}$. Cover-up gives $\frac{3}{2}$ at $x = 1$ and $\dfrac{-3}{-2} = \frac{3}{2}$ at $x = -1$. Hence

$$\int \frac{x^2 + 3x - 1}{x^2 - 1}\,dx = x + \frac{3}{2}\ln|x-1| + \frac{3}{2}\ln|x+1| + C = x + \frac{3}{2}\ln|x^2 - 1| + C$$

on each of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. p. 315

9.5.2 By Proposition 9.5.4, $A = \dfrac{1+4+1}{(1+2)^2} = \frac{2}{3}$ and $C = \dfrac{4-8+1}{-2-1} = 1$. The coefficient of x^2 in the cleared identity $x^2 + 4x + 1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$ gives $A + B = 1$ and $B = \frac{1}{3}$. Hence

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+2)^2}\,dx = \frac{2}{3}\ln|x-1| + \frac{1}{3}\ln|x+2| - \frac{1}{x+2} + C.$$

p. 315

9.5.3 Cover-up gives $\dfrac{1}{x^2 - 9} = \dfrac{1/6}{x-3} - \dfrac{1/6}{x+3}$. By Eq. (9.5.7), $\displaystyle\int \frac{dx}{x^2 - 9} = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$. On $(-3, 3)$, $|x-3| = 3 - x$ and $|x+3| = x + 3$. Hence

$$\int \frac{dx}{x^2 - 9} = \frac{1}{6}\ln\frac{3-x}{3+x} + C \qquad (-3 < x < 3).$$

The unbarred $\frac{1}{6}\ln(x-3)$ is the logarithm of a negative number at every point of the interval. p. 315

9.5.4 The discriminant is $1 + 80 = 81 > 0$. Hence $x^2 - x - 20 = (x-5)(x+4)$. Cover-up gives $A = \dfrac{16}{9}$ at $x = 5$ and $B = \dfrac{-11}{-9} = \dfrac{11}{9}$ at $x = -4$. Hence

$$\int \frac{3x + 1}{x^2 - x - 20}\,dx = \frac{16}{9}\ln|x-5| + \frac{11}{9}\ln|x+4| + C.$$

Completing the square gives $(x - \frac{1}{2})^2 - \frac{81}{4}$. The constant is negative. The hypothesis $b^2 < 4c$ of Eq. (9.5.3) fails. The form $u^2 + k^2$ of Proposition 9.5.6 never arises. p. 315

9.5.5 Cover-up gives $B = \dfrac{2-2+4}{1+1} = 2$. In the cleared identity

$$2x^3 - 2x^2 + 4x = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

the coefficient of x^3 gives $A + C = 2$, the constant term gives $-A + B + D = 0$, and the coefficient of x gives $A + C - 2D = 4$. Hence $D = -1$, $A = 1$ and $C = 1$. The decomposition is $\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{x}{x^2+1} - \dfrac{1}{x^2+1}$, and

$$\int \frac{2x^3 - 2x^2 + 4x}{(x-1)^2(x^2+1)}\,dx = \ln|x-1| - \frac{2}{x-1} + \frac{1}{2}\ln(x^2+1) - \arctan x + C.$$

p. 315

9.5.6 Cover-up gives $A = \dfrac{-5}{1} = -5$ at $x = -1$ and $B = \dfrac{-9}{-1} = 9$ at $x = -2$. The antiderivative $-5\ln(x+1) + 9\ln(x+2)$ is continuous on $[0,1]$, and Theorem 8.3.5 gives

$$\int_0^1 \frac{4x-1}{(x+1)(x+2)}\,dx = (-5\ln 2 + 9\ln 3) - 9\ln 2 = 9\ln 3 - 14\ln 2.$$

p. 315

9.5.7 Multiplying the decomposition by $(x-r)^m$ gives, near r,

$$g(x) = \sum_{k=1}^{m} A_k(x-r)^{m-k} + (x-r)^m R(x) = \sum_{j=0}^{m-1} A_{m-j}(x-r)^j + (x-r)^m R(x),$$

where R is a rational function with no pole at r, infinitely often differentiable near r. By the Leibniz rule (Theorem 7.3.21) every derivative of order $j < m$ of $(x-r)^m R(x)$ is a sum of terms with a factor $(x-r)^{m-i}$ for some $i \le j$. It vanishes at r. The jth derivative of $(x-r)^i$ at r is $j!$ when $i = j$ and 0 otherwise. Hence $g^{(j)}(r) = j!\,A_{m-j}$. For Worked Example 9.5.2, $m = 3$, $q_1 = 1$ and $g(x) = 2x - 1$. Then $C = A_3 = g(-1) = -3$, $B = A_2 = g'(-1) = 2$ and $A = A_1 = \frac{1}{2}g''(-1) = 0$. p. 315

9.5.8 The function $f = p/q$ is even because p and q are. For every x with $q(x) \neq 0$,

$$f(x) = f(-x) = \sum_i \Big(\frac{A_i}{-x - r_i} + \frac{B_i}{-x + r_i}\Big) = \sum_i \Big(\frac{-B_i}{x - r_i} + \frac{-A_i}{x + r_i}\Big).$$

The right side is a decomposition of f of the form of Theorem 9.5.2. By uniqueness its coefficient of $1/(x-r_i)$ equals that of the original decomposition: $-B_i = A_i$. p. 315

9.5.9 Write $q_i(x) = q(x)/(x-r_i) = \prod_{j\neq i}(x-r_j)$. Then $q = (x-r_i)q_i$. The product rule gives $q'(r_i) = q_i(r_i) + 0 = \prod_{j\neq i}(r_i - r_j) \neq 0$. By Proposition 9.5.4 the coefficient of $1/(x-r_i)$ in the decomposition of $1/q$ is $1/q_i(r_i) = 1/q'(r_i)$. The cleared identity is $1 = \sum_i q_i(x)/q'(r_i)$ for every real x (Corollary 4.2.8). Each q_i has degree $n-1$ and leading coefficient 1. Since $n - 1 \ge 1$, the coefficient of x^{n-1} on the left is 0. On the right it is $\sum_i 1/q'(r_i)$. For $q = (x-1)(x-2)(x-4)$, $q'(1) = (-1)(-3) = 3$, $q'(2) = (1)(-2) = -2$ and $q'(4) = (3)(2) = 6$, and $\frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$. p. 316

9.5.10 By Eq. (9.5.6) with $k = 1$ and $n = 2$, $\int \dfrac{\mathrm{d}x}{(1+x^2)^2} = \dfrac{x}{2(1+x^2)} + \dfrac{1}{2}\arctan x + C$. By Theorem 8.3.5,

$$\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{4} + \frac{1}{2}\cdot\frac{\pi}{4} = \frac{1}{4} + \frac{\pi}{8}.$$

p. 316

9.6 Integral Tables and Computer Algebra Systems

9.6.1 With $n = 2$ and $a = -1$, $\int x^2 e^{-x}\,\mathrm{d}x = -x^2e^{-x} + 2\int xe^{-x}\,\mathrm{d}x$. With $n = 1$, $\int xe^{-x}\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C$. Substituting,

$$\int x^2e^{-x}\,\mathrm{d}x = -(x^2 + 2x + 2)e^{-x} + C.$$

p. 319

9.6.2 With $u = \sin x$ the integrand is the pullback of $u\,\mathrm{d}u$ and an antiderivative is $\frac{1}{2}\sin^2 x$. With $u = \cos x$ it is the pullback of $-u\,\mathrm{d}u$, with antiderivative $-\frac{1}{2}\cos^2 x$. Their difference is $\frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$ on $\mathbb{R}$. By Proposition 9.6.4 the two answers describe the same family.

p. 320

9.6.3 Both arguments are positive exactly when $x > 3$. On $(-3, 3)$ the first logarithm has a negative argument and on $(-\infty, -3)$ both do. With the real logarithm the expression is therefore an antiderivative on $(3, \infty)$ only. With the complex logarithm each negative argument y contributes $\ln|y| + i\pi$. On $(-\infty, -3)$ the two terms $\pm\frac{i\pi}{6}$ cancel and the expression is the real function $\frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3|$. On $(-3, 3)$ only the first term contributes $\frac{i\pi}{6}$. The expression then differs from the same real function by a constant and is again an antiderivative there. By partial fractions, $\dfrac{1}{x^2-9} = \dfrac{1}{6}\left(\dfrac{1}{x-3} - \dfrac{1}{x+3}\right)$, and Eq. (9.5.7) gives the antiderivative $\frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3|$ on each of the three intervals. By Proposition 9.6.4 the general antiderivative adds a locally constant function, one constant per interval.

p. 320

9.6.4 For $|x| < |a|$ the number x/a lies in $(-1, 1)$. By the chain rule and Theorem 6.4.10,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\frac{x}{a} = \frac{1}{\sqrt{1 - x^2/a^2}}\cdot\frac{1}{a} = \frac{|a|}{a\sqrt{a^2 - x^2}},$$

using $\sqrt{1 - x^2/a^2} = \sqrt{a^2 - x^2}/|a|$. For $a < 0$ the factor $|a|/a$ is -1. Since the integrand depends on a only through a^2, the entry $\arcsin(x/|a|) + C$ is correct for every $a \neq 0$ by Proposition 9.1.7 with the positive constant $|a|$.

p. 320

9.6.5 The quotient $c\sin x/(1 + c\cos x)$ and its arctangent have period 2π. The term $x/\sqrt{3}$ increases by $2\pi/\sqrt{3}$ when x increases by 2π. Hence $F(x + 2\pi) = F(x) + 2\pi/\sqrt{3}$. Since F is an antiderivative on $\mathbb{R}$, Theorem 8.3.5 gives $\displaystyle\int_0^{4\pi} \frac{\mathrm{d}x}{2 + \cos x} = F(4\pi) - F(0) = 2\cdot\frac{2\pi}{\sqrt{3}} = \frac{4\pi}{\sqrt{3}}$.

p. 320

9.6.6 By the chain rule,

$$F_2'(x) = \frac{1}{1 + \left(\dfrac{1+x}{1-x}\right)^2}\cdot\frac{(1-x) + (1+x)}{(1-x)^2} = \frac{2}{(1-x)^2 + (1+x)^2} = \frac{1}{1+x^2}.$$

By Proposition 9.6.4 the difference is constant on each interval. At $x = 0$ it is $\arctan 1 - 0 = \frac{\pi}{4}$. As $x \to \infty$, $\dfrac{1+x}{1-x} \to -1$ and $F_2 - F_1 \to -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$, the constant value on $(1, \infty)$.

p. 320

9.6.7 Let $h \neq 0$ with $c+h \in I$. The function F is continuous on the closed interval between c and $c+h$ and differentiable inside it. By Theorem 7.1.9 there is ξ_h strictly between c and $c+h$ with $F(c+h)-F(c) = F'(\xi_h)h = f(\xi_h)h$. As $h \to 0$, $\xi_h \to c$ and $f(\xi_h) \to f(c)$ by continuity. Hence the difference quotient tends to $f(c)$. The function F_0 has different one-sided limits $\pm\pi/\sqrt{3}$ at π (Problem 9.3.9). No choice of $F_0(\pi)$ makes it continuous there. The hypothesis therefore fails. p. 320

9.6.8 By Eq. (9.2.4) with $n = 3$ and $a = 1$, $\int x^3 e^x \, dx = e^x(x^3 - 3x^2 + 6x - 6) + C$. By Theorem 8.3.5,

$$\int_0^1 x^3 e^x \, dx = e\,(1 - 3 + 6 - 6) - (-6) = 6 - 2e.$$

p. 320

9.7 Numerical Integration

9.7.1 With $M = 4$, $b - a = \frac{\pi}{2}$ and $n = 6$, Eq. (9.7.3) gives $|E_T| \leq \dfrac{4(\pi/2)^3}{12 \cdot 36} = \dfrac{\pi^3}{864} \approx 0.035887$. With $h = \frac{\pi}{12}$ the values $\sin(2x_k)$ are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0$, and Eq. (9.7.1) gives $T_6 = \frac{\pi}{24} \cdot 2(1+\sqrt{3}+1) = \frac{\pi(\sqrt{3}+2)}{12} \approx 0.977049$. The integral is $\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = 1$, and $|E_T| \approx 0.022951$ lies within the bound. p. 325

9.7.2 With $h = 1$ the values are $1, 14, 51$ and $S_2 = \frac{1}{3}(1 + 56 + 51) = 36$. The integral is $\left[\frac{x^4}{2} - \frac{x^2}{2}\right]_1^3 = 36$. For a polynomial of degree at most 3 the fourth derivative is 0, and Eq. (9.7.7) gives $E_S = 0$ for every even n. For x^4 the fourth derivative is 24 and the error is not 0 (Worked Example 9.7.2). p. 326

9.7.3 With $y_k = e^{k/4}$, Eqs. (9.7.1) and (9.7.6) give $T_4 \approx 1.727222$ and $S_4 \approx 1.718319$. Against $e - 1 \approx 1.718282$ the errors are $E_T \approx 0.008940$ and $E_S \approx 0.0000370$. Since every derivative of e^x is e^x, Proposition 9.7.10 predicts $E_T \approx \frac{1}{12n^2}(e-1)$ and $E_S \approx \frac{1}{180n^4}(e-1)$. At $n = 4$ these are 0.008949 and 0.0000373, close to the observed errors. p. 326

9.7.4 The largest value of $24/x^5$ on $[1, 2]$ is 24 at $x = 1$. With $M = 24$, Eq. (9.7.8) gives $|E_S| \leq \dfrac{24}{180 \cdot 256} = \dfrac{1}{1920}$. With $h = \frac{1}{4}$ the values $1/x_k$ at $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$ give $S_4 = \frac{1}{12}(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}) = \frac{1747}{2520}$. The error $\frac{1747}{2520} - \ln 2 \approx 0.000107$ lies within the bound. p. 326

9.7.5 For $0 \leq t \leq h$ put $F(t) = 2t\,f(\mu) - \int_{\mu-t}^{\mu+t} f(x)\,dx$. By Theorem 8.3.4 and the chain rule, $F'(t) = 2f(\mu) - f(\mu+t) - f(\mu-t)$ and $F''(t) = -f'(\mu+t) + f'(\mu-t)$. In particular $F(0) = F'(0) = 0$. Let $K = F(h)/h^3$ and $G(t) = F(t) - Kt^3$. Then $G(0) = G(h) = 0$. Rolle's theorem (Theorem 7.1.7) gives $t_1 \in (0, h)$ with $G'(t_1) = 0$. Since $G'(0) = 0$ as well, it gives $t_2 \in (0, t_1)$ with

$$0 = G''(t_2) = -\big(f'(\mu + t_2) - f'(\mu - t_2)\big) - 6Kt_2.$$

By the Mean Value Theorem (Theorem 7.1.9), $f'(\mu+t_2) - f'(\mu-t_2) = 2t_2 f''(c)$ for some c. Hence $K = -f''(c)/3$ and $F(h) = -\frac{h^3}{3}f''(c)$. This is the stated identity. On $[a, b]$ apply it to each subinterval of width $\Delta = 2h$ with midpoint μ_k. Adding, $\int_a^b f - M_n = \dfrac{\Delta^3}{24}\sum_{k=1}^n f''(c_k) = \dfrac{(b-a)\Delta^2}{24} \cdot \dfrac{1}{n}\sum_{k=1}^n f''(c_k)$. The Intermediate Value Theorem replaces the average by $f''(\xi)$. p. 326

9.7.6 Let $h = (b-a)/(2m)$ and $y_k = f(a+kh)$ for $k = 0, \dots, 2m$. Then $T_{2m} = \frac{h}{2}(y_0 + 2\sum_{k=1}^{2m-1} y_k + y_{2m})$, and T_m uses only the even points with width $2h$: $T_m = h(y_0 + 2\sum_{j=1}^{m-1} y_{2j} + y_{2m})$. In $\frac{1}{3}(4T_{2m} - T_m)$ the endpoints receive $\frac{1}{3}(2h - h) = \frac{h}{3}$, the odd points $\frac{1}{3} \cdot 4h = \frac{4h}{3}$ and the interior even points $\frac{1}{3}(4h - 2h) = \frac{2h}{3}$. These are the weights of S_{2m} (Eq. (9.7.6)). For $f \in C^2$, Proposition 9.7.10 gives errors approximately 4κ for T_m and κ for T_{2m} with $\kappa = \frac{(b-a)^2}{48m^2}(f'(b) - f'(a))$. The error of the combination is $\frac{1}{3}$ of four times the second error minus the first. In it the two leading terms 4κ cancel. p. 326

9.7.7 By Eq. (9.7.1), $T_1 = \frac{1}{2}(0+1) = 0.5$, $T_2 = \frac{1}{4}(0 + 2\sqrt{1/2} + 1) \approx 0.603553$ and $T_4 = \frac{1}{8}(0 + 2(\frac{1}{2} + \sqrt{1/2} + \sqrt{3/4}) + 1) \approx 0.643283$. The errors against $\frac{2}{3}$ are about -0.166667, -0.063113 and -0.023384, with ratios 2.64 and 2.70. Theorem 9.7.3 requires f'' continuous on the closed interval $[0, 1]$. Here $f''(x) = -\frac{1}{4}x^{-3/2}$ is unbounded near 0 and no M exists. The observed ratios stay well below the value 4 that order 2 would give. p. 326

9.7.8 Let $L_n = \sum_{k=1}^{n} f(x_{k-1})h$ and $R_n = \sum_{k=1}^{n} f(x_k)h$. By Eq. (9.7.1), $T_n = \sum_k \frac{h}{2}(y_{k-1} + y_k) = \frac{1}{2}(L_n + R_n)$. Both are Riemann sums of the continuous f on partitions of mesh $h = (b-a)/n \to 0$. By Theorem 8.2.13 and Definition 8.2.4 they tend to $\int_a^b f$. Hence so does their average. p. 326

9.8 Improper Integrals

9.8.1 By Eq. (9.8.6) the integral converges only if $\int_0^1$ and $\int_1^2$ both do. By Eq. (9.8.4), $\int_0^1 \frac{\mathrm{d}x}{(x-1)^2} = \lim_{c \to 1^-}(-\frac{1}{c-1} - 1) = \infty$. Hence the integral diverges. The value -2 applies Theorem 8.3.5 with a function undefined at 1, outside its hypotheses. p. 332

9.8.2 By Eq. (9.8.2), $\int_{-\infty}^0 e^x \,\mathrm{d}x = \lim_{a \to -\infty}(1 - e^a) = 1$. By Eq. (9.8.1), $\int_0^\infty e^x \,\mathrm{d}x = \lim_{b \to \infty}(e^b - 1) = \infty$. One piece diverges. By Definition 9.8.2 the integral diverges. p. 332

9.8.3 Both $f(x) = \dfrac{x+1}{x^3+3}$ and $g(x) = \dfrac{1}{x^2}$ are positive and continuous on $[1, \infty)$, and $\dfrac{f(x)}{g(x)} = \dfrac{x^3+x^2}{x^3+3} = \dfrac{1+1/x}{1+3/x^3} \to 1$. The integral of g converges by Theorem 9.8.7. By Theorem 9.8.10 so does that of f. p. 332

9.8.4 Theorem 9.8.9 infers convergence of the smaller integral from that of the larger. It infers divergence of the larger from that of the smaller. The given inequality makes the integrand the larger function. The smaller function $\frac{1}{\sqrt{2}}x^{-2}$ has a convergent integral. Neither conclusion applies. Since $x^4 + x \geq x^4$ for $x \geq 1$, $0 < \dfrac{1}{\sqrt{x^4+x}} \leq \dfrac{1}{x^2}$. The integral of x^{-2} converges (Theorem 9.8.7). So does the given integral. p. 332

9.8.5 The function $h(x) = x - \ln(x+1)$ has $h(0) = 0$ and $h'(x) = \dfrac{x}{x+1} > 0$ for $x > 0$. By Theorem 7.2.4 it is positive on $(0, \infty)$. Hence $0 < \ln(x+1) < x$ and $\dfrac{1}{\ln(x+1)} > \dfrac{1}{x}$ for $x \geq 1$. The integral of $1/x$ over $[1, \infty)$ diverges (Theorem 9.8.7), and Theorem 9.8.9 shows that the given integral diverges. p. 332

9.8.6 With $u = \ln x$ and $v = -\dfrac{1}{2x^2}$, Eq. (9.2.2) gives

$$\int_1^b \frac{\ln x}{x^3}\,\mathrm{d}x = \Big[-\frac{\ln x}{2x^2}\Big]_1^b + \int_1^b \frac{\mathrm{d}x}{2x^3} = -\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4}.$$

By Theorem 5.2.26 the first two terms tend to 0. The integral is $\frac{1}{4}$. p. 332

9.8.7 For $n = 0$ the integral is $1 = 0!$ by Worked Example 9.8.1. Let $n \geq 1$ and assume the result for $n-1$. With $u = x^n$ and $v = -e^{-x}$, Eq. (9.2.2) on $[0, b]$ gives

$$\int_0^b x^n e^{-x}\,\mathrm{d}x = -b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}\,\mathrm{d}x.$$

By Theorem 5.2.26, $b^n e^{-b} \to 0$. By the inductive hypothesis the last integral tends to $(n-1)!$. Hence the integral converges to $n\,(n-1)! = n!$ (Theorem 1.4.2). p. 332

9.8.8 With $u = x^{-p}$ and $v = -\cos x$, Eq. (9.2.2) gives

$$\int_1^b \frac{\sin x}{x^p}\,\mathrm{d}x = \cos 1 - \frac{\cos b}{b^p} - p\int_1^b \frac{\cos x}{x^{p+1}}\,\mathrm{d}x.$$

Since $p > 0$, $|\cos b|/b^p \leq b^{-p} \to 0$. Since $p + 1 > 1$, the last integral converges absolutely by comparison with $x^{-(p+1)}$ (Theorems 9.8.7 and 9.8.9) and converges by Theorem 9.8.13. Hence the integral converges. For $p > 1$, $|\sin x|/x^p \leq x^{-p}$ and the convergence is absolute. For $0 < p \leq 1$ and $x \geq 1$, $x^p \leq x$ and $|\sin x|/x^p \geq |\sin x|/x \geq 0$. The integral of $|\sin x|/x$ over $[1, \infty)$ diverges by Proposition 9.8.14, and Theorem 9.8.9 shows that the convergence is conditional. p. 332

9.8.9 The function f_n is continuous, equal to $\frac{1}{n}$ on $[0, n]$ and between 0 and $\frac{1}{n}$ afterwards. Hence $\|f_n\|_\infty = \frac{1}{n}$. Its integral over $[0, b]$ is constant for $b \geq n+1$ and equals $\int_0^n \frac{1}{n}\,\mathrm{d}x + \int_n^{n+1} \frac{1}{n}(n+1-x)\,\mathrm{d}x = 1 + \frac{1}{2n}$. A constant K as stated would give $1 + \frac{1}{2n} \leq \frac{K}{n}$, false for $n > K$. On a bounded interval the integral is a bounded functional with norm equal to the length (Theorem 8.2.17). On $[0, \infty)$ the length is infinite. The sup norm no longer controls the integral. p. 332

9.8.10 Where $\sin x \neq 0$, $\dfrac{f(x)}{g(x)} = 1 + \dfrac{\sin^2 x/x}{\sin x/\sqrt{x}} = 1 + \dfrac{\sin x}{\sqrt{x}}$. Here $\left|\dfrac{\sin x}{\sqrt{x}}\right| \leq x^{-1/2} \to 0$. By Problem 9.8.8 with $p = \frac{1}{2}$, $\int_1^\infty g$ converges. If $\int_1^\infty f$ converged, then by Theorem 5.2.13 so would $\int_1^\infty (f - g) = \int_1^\infty \dfrac{\sin^2 x}{x}\,\mathrm{d}x$, which diverges by the proof of Proposition 9.8.14. Hence $\int f$ diverges. The hypothesis that fails is positivity. The functions f and g change sign. The quotient f/g is also undefined at the zeros $k\pi$ of g. The limit Eq. (9.8.9) exists only along the points where $g \neq 0$. The bounds $\frac{L}{2}g < f < \frac{3L}{2}g$ of the proof then fall outside the hypothesis $0 \leq f$ of Theorem 9.8.9. p. 332

9.8.11 For $n \geq 2$ the triangle $T_n(x) = \max(0, 1 - 2^n|x - n|)$ is continuous, vanishes outside $[n-2^{-n}, n+2^{-n}]$, has value 1 at n and has integral $\frac{1}{2}\cdot 2\cdot 2^{-n}\cdot 1 = 2^{-n}$. Since $2^{-n} \leq \frac{1}{4}$, the supports of different triangles are disjoint and lie in $[1, \infty)$. Let f be the sum of the triangles. Near every point at most one triangle is nonzero. It is continuous and non-negative with $f(n) = 1$. For $b \geq 1$ choose an integer $N > b$. The integral of f over $[1, b]$ is at most $\sum_{n=2}^{N} 2^{-n} < \frac{1}{2}$ by Eq. (4.6.11). By Lemma 9.8.8 the integral converges, although f takes the value 1 at arbitrarily large x. p. 333

10.1 Sequences and Complete Spaces

10.1.1 By (10.1.1), $a_{10} = 2 + 9 \cdot 3 = 29$. By (10.1.3), $s_{10} = 5(4 + 27) = 155$. By (10.1.2), $b_n = 2\left(\frac{1}{3}\right)^{n-1}$ and so tends to 0 by Theorem 10.1.4 since $\frac{1}{3} < 1$. p. 341

10.1.2 The even terms $1+\frac{1}{2k}$ decrease from $\frac{3}{2}$ to the limit 1. The odd terms $-1-\frac{1}{2k-1}$ increase from -2 to the limit -1. Hence $\sup a_n = a_2 = \frac{3}{2}$ and $\inf a_n = a_1 = -2$. Since every $U_n = \sup_{k\geq n} a_k$ is an even term $1+\frac{1}{k}$ with $k \geq n$, $U_n \to 1$ and $\limsup a_n = 1$. The same argument for $-a_n$ gives $\liminf a_n = -1$. p. 341

10.1.3 The first is the fifth limit of Theorem 10.1.4 at $x = 3$. The second is the square of $(1-1/n)^n \to e^{-1}$ and tends to e^{-2} by Theorem 5.1.9. p. 341

10.1.4 For $n \geq 4$,

$$0 \leq \frac{2^n}{n!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\prod_{j=5}^{n}\frac{2}{j} \leq \frac{2}{3}\Big(\frac{2}{5}\Big)^{n-4}.$$

The right-hand side tends to 0. Theorem 5.1.11 gives $2^n/n! \to 0$.

L'Hôpital's rule needs a differentiable real extension of numerator and denominator. The factorial is defined only at integers and the rule applies only after an extension of $n!$ has been chosen and shown differentiable. No earlier chapter provides such an extension. Limits with factorials are settled by bounds, by Theorem 10.1.4 or by the ratio test. p. 341

10.1.5

1. Bounds. Since $a_1 = 1 < 2$ and $a_n < 2$ implies $a_{n+1} < \sqrt{4} = 2$, induction gives $a_n < 2$. Every term is a positive square root.

2. Monotonicity. For $0 < a_n < 2$, $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2-a_n)(1+a_n) > 0$. Since both terms are positive, $a_{n+1} > a_n$.

By Theorem 5.1.18 the sequence converges to some $L \in [1,2]$. Since $t \mapsto \sqrt{2+t}$ is continuous, Theorem 5.3.5 gives $L = \sqrt{2+L}$ and hence $L^2 - L - 2 = (L-2)(L+1) = 0$. The root -1 is excluded and $a_n \to 2$. p. 341

10.1.6 Since $a_k + b_k \leq \sup_{j\geq n} a_j + \sup_{j\geq n} b_j$ for $k \geq n$, the right-hand side is an upper bound for $\{a_k + b_k : k \geq n\}$ and

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k.$$

Letting $n \to \infty$ and using Theorems 5.1.9 and 5.1.10 gives the inequality. For $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ the sum is 0 for every n while each upper limit is 1. Hence $0 < 2$. p. 342

10.1.7 Since $\|(-1)^n\mathbf{v}\| = \|\mathbf{v}\|$ for every n, the norms are constant. If $\mathbf{v}_n \to \mathbf{w}$, then $2\|\mathbf{v}\| = \|\mathbf{v}_{n+1} - \mathbf{v}_n\| \leq \|\mathbf{v}_{n+1} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}_n\| \to 0$, contradicting $\mathbf{v} \neq \mathbf{0}$. The norm of a limit is the limit of the norms. The converse fails. p. 342

10.1.8 Each f_n is nonnegative and tends to 0 at every $x \in [0,1]$, at $x = 0$ trivially and for $x > 0$ by the growth ranking. Since f_n increases on $[0,1/n]$ and decreases on $[1/n,\infty)$, its maximum on $[0,1]$ is $f_n(1/n) = e^{-1}$ and does not tend to 0. Since for $n \geq 2$ the point $1/n$ lies at or left of $\frac{1}{2}$, on $[\frac{1}{2},1]$ the maximum is $f_n(\frac{1}{2}) = \frac{n}{2}e^{-n/2} \to 0$ by Theorem 5.2.26. The convergence is uniform on $[\frac{1}{2},1]$ and not on $[0,1]$. p. 342

10.1.9 Homogeneity and the triangle inequality follow from Theorem 8.2.15 since $|cf| = |c||f|$ and $|f+g| \leq |f| + |g|$. If $f \neq 0$, some x_0 has $|f(x_0)| = \eta > 0$ and $|f| \geq \eta/2$ on an interval of positive length ℓ in $[0,1]$ by continuity. The order property then gives $\|f\|_1 \geq \eta\ell/2 > 0$.

Since $x^n \geq 0$, $\|x^n\|_1 = \int_0^1 x^n\,dx = 1/(n+1)$ while the maximum of x^n on $[0,1]$ is 1. A constant K with $\|f\|_\infty \leq K\|f\|_1$ would give $1 \leq K/(n+1)$ for all n. That inequality fails for $n \geq K$. The two norms are therefore not equivalent, in contrast with Theorem 3.6.15 whose proof uses the finite dimension of $\mathbb{R}^m$. p. 342

10.1.10 With $\varepsilon = 1$ there is an N with $\|\mathbf{v}_n - \mathbf{v}_{N+1}\| < 1$ for $n > N$ and hence $\|\mathbf{v}_n\| < 1 + \|\mathbf{v}_{N+1}\|$ for those n. The largest of this bound and $\|\mathbf{v}_1\|, \dots, \|\mathbf{v}_N\|$ bounds every term.

Let $\mathbf{v}_{n_k} \to \mathbf{v}$ and $\varepsilon > 0$. Choose N with $\|\mathbf{v}_m - \mathbf{v}_n\| < \varepsilon/2$ for $m, n > N$ and then k with $n_k > N$ and $\|\mathbf{v}_{n_k} - \mathbf{v}\| < \varepsilon/2$. For $n > N$, $\|\mathbf{v}_n - \mathbf{v}\| \le \|\mathbf{v}_n - \mathbf{v}_{n_k}\| + \|\mathbf{v}_{n_k} - \mathbf{v}\| < \varepsilon$. p. 342

10.1.11 By the hint the difference is positive and its denominator is smallest at $x = 0$ where the difference equals $(1/n)/\sqrt{1/n} = 1/\sqrt{n}$. Hence $\|f_n - |x|\|_\infty = 1/\sqrt{n} \to 0$. Each f_n is differentiable on $[-1, 1]$ by the chain rule since $x^2 + 1/n > 0$. The uniform limit $|x|$ has no derivative at 0. Uniform convergence preserves continuity (Lemma 10.1.16) and does not preserve differentiability. p. 342

10.1.12 For $m > n$ both functions vanish on $[0, \frac{1}{2}]$, equal 1 on $[\frac{1}{2} + \frac{1}{n}, 1]$ and lie in $[0, 1]$. Hence $\|g_n - g_m\|_1 \le \int_{1/2}^{1/2+1/n} 1 \, \mathrm{d}x = 1/n$. The sequence is therefore Cauchy.

Suppose $\|g_n - g\|_1 \to 0$ with g continuous. Since $g_n = 0$ on $[0, \frac{1}{2}]$, $\int_0^{1/2} |g| \le \|g - g_n\|_1 \to 0$. Hence $\int_0^{1/2} |g| = 0$ and the argument of Problem 10.1.9 gives $g = 0$ on $[0, \frac{1}{2}]$. Fix $\delta > 0$. Since $g_n = 1$ on $[\frac{1}{2} + \delta, 1]$ for $n > 1/\delta$, $\int_{1/2+\delta}^1 |g - 1| \le \|g - g_n\|_1 \to 0$ and $g = 1$ on $[\frac{1}{2} + \delta, 1]$. As δ is arbitrary, $g = 1$ on $(\frac{1}{2}, 1]$. Then $g(\frac{1}{2}) = 0$ while $g(x) = 1$ for every $x > \frac{1}{2}$, contradicting continuity at $\frac{1}{2}$. No limit exists in $C([0, 1])$. The space is therefore not complete in this norm. The same sequence is not Cauchy in the sup norm, as Theorem 10.1.17 requires. p. 342

10.2 Infinite Series

10.2.1 The first term is 3 and the ratio is $\frac{2}{5}$, with $|r| < 1$. By Theorem 10.2.7 the sum is $3/(1 - \frac{2}{5}) = 5$. p. 347

10.2.2 Since $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, (10.2.4) with $b_n = 1/n$ gives the sum $b_1 - \lim b_n = 1$. p. 347

10.2.3 Both pieces are convergent geometric series: $\sum 3/2^n = \frac{3}{2}/(1 - \frac{1}{2}) = 3$ and $\sum 2/3^n = \frac{2}{3}/(1 - \frac{1}{3}) = 1$. By Theorem 10.2.10 the series converges to $3 - 1 = 2$. p. 348

10.2.4 The series is geometric with first term 1 and ratio $(x - 2)/3$. By Theorem 10.2.7 it converges exactly when $|x - 2| < 3$ (that is, $-1 < x < 5$) with sum $1/(1 - \frac{x-2}{3}) = 3/(5 - x)$. At $x = -1$ and $x = 5$ the terms have absolute value 1 and the series diverges by Theorem 10.2.4. Section 11.1 treats series of this kind in general. p. 348

10.2.5 Since $|x^n/2^n| \le 2^{-n} = M_n$ on $[-1, 1]$ and $\sum M_n = 2$, Theorem 10.2.18 gives uniform convergence. Since the series is geometric with ratio $x/2$, $f(x) = 1/(1 - x/2) = 2/(2 - x)$. The tail is

$$f(x) - s_N(x) = \sum_{n>N} \left(\frac{x}{2}\right)^n = \frac{(x/2)^{N+1}}{1 - x/2},$$

whose absolute value is largest at $x = 1$ where it equals 2^{-N}. Hence $\|f - s_N\|_\infty = 2^{-N}$. Since $2^{-9} \approx 0.00195$ and $2^{-10} \approx 0.00098$, the least N is 10. p. 348

10.2.6 Since each of the n terms $1/k$ with $n < k \le 2n$ is at least $1/(2n)$, the block is at least $n/(2n) = \frac{1}{2}$. With $\varepsilon = \frac{1}{2}$ no N satisfies (10.2.6) because $m = 2n$ with $n > N$ violates it. By Theorem 10.2.12 the series diverges. p. 348

10.2.7 For fixed n the partial sums of the tail are $\mathbf{s}_m - \mathbf{s}_n$ with $m > n$ and converge to $\mathbf{S} - \mathbf{s}_n$ by Theorem 5.1.31. Hence the tail converges and $\mathbf{r}_n = \mathbf{S} - \mathbf{s}_n$. Since $\mathbf{s}_n \to \mathbf{S}$, $\|\mathbf{r}_n\| = \|\mathbf{S} - \mathbf{s}_n\| \to 0$. p. 348

10.2.8 By the proof of Theorem 10.1.4, for $n > K$, $|z|^n/n! \le (|z|^K/K!)2^{K-n} = C\,2^{-n}$. Since the partial sums of $\sum |z|^n/n!$ increase and are bounded by the finitely many terms with $n \le K$ plus $C\sum 2^{-n}$, the series converges by Theorem 5.1.18. Since $\mathbb{C}$ is complete, Theorem 10.2.15 gives convergence. Definition 11.3.7 uses this series to define the complex exponential. p. 348

10.2.9 Pointwise convergence on $(-1,1)$ is Theorem 10.2.7. Since $|x^n| \le r^n$ on $[-r,r]$ and $\sum r^n$ converges, Theorem 10.2.18 applies. On $(-1,1)$, $|s_N(x)| \le N+1$ while $1/(1-x) \to \infty$ as $x \to 1^-$. The difference is therefore unbounded, no s_N is within 1 of the sum everywhere, and the convergence is not uniform. The M-test fails because $\sup_{(-1,1)}|x^n| = 1$ and $\sum 1$ diverges. p. 348

10.2.10 In the series of the hint the block of k terms equal to $(-1)^{k+1}/k$ sums to $(-1)^{k+1}$. The partial sums at the ends of the blocks are therefore $1, 0, 1, 0, \ldots$ and inside each block move monotonically between 0 and 1. Every partial sum lies in $[0,1]$ and the terms tend to 0. Since the partial sums equal 1 and 0 infinitely often, they have two subsequences with different limits and the series diverges (Theorem 5.1.23). p. 348

10.2.11 Let $\{\mathbf{x}_n\}$ be Cauchy. Choose n_1 with $\|\mathbf{x}_m - \mathbf{x}_n\| < \frac{1}{2}$ for $m, n \ge n_1$ and inductively $n_{k+1} > n_k$ with $\|\mathbf{x}_m - \mathbf{x}_n\| < 2^{-k-1}$ for $m, n \ge n_{k+1}$. Put $\mathbf{v}_1 = \mathbf{x}_{n_1}$ and $\mathbf{v}_{k+1} = \mathbf{x}_{n_{k+1}} - \mathbf{x}_{n_k}$. Then $\|\mathbf{v}_{k+1}\| < 2^{-k}$, the series $\sum \|\mathbf{v}_k\|$ converges by comparison with a geometric series and Theorem 5.1.18, and by hypothesis $\sum \mathbf{v}_k$ converges. Its partial sums are $\mathbf{v}_1 + \cdots + \mathbf{v}_k = \mathbf{x}_{n_k}$ by telescoping. The subsequence $\{\mathbf{x}_{n_k}\}$ therefore converges and by Problem 10.1.10 the whole sequence converges to the same limit. Hence V is complete. p. 348

10.3 Integral Test

10.3.1 (a) $p = \frac{3}{2} > 1$ and Theorem 10.3.3 gives convergence. (b) $\sqrt[3]{n^2} = n^{2/3}$ gives $p = \frac{2}{3} \le 1$ and the series diverges. p. 352

10.3.2 The function is continuous, positive and decreasing on $[1,\infty)$ and satisfies $\int_1^b dx/(x^2+1) = \arctan b - \pi/4 \to \pi/4$. The series converges. Letting $n \to \infty$ in (10.3.3) gives $S - a_1 \le \pi/4 \le S$, that is $\pi/4 \le S \le \frac{1}{2} + \pi/4$. The sum is about 1.0767. p. 352

10.3.3 For $p > 0$ the function $1/(x(\ln x)^p)$ is continuous, positive and decreasing on $[2,\infty)$. The substitution $u = \ln x$ turns its integral into $\int_{\ln 2}^{\infty} u^{-p}\,du$. That integral converges exactly when $p > 1$ (Eq. (9.8.7)). Theorem 10.3.2 settles $p > 0$. Since $(\ln n)^p \le 1$ for $p \le 0$ and $n \ge 3$, the terms are at least $1/n$ and the partial sums exceed those of the divergent harmonic series from $n = 3$ onwards (Theorem 10.3.1). The series diverges. p. 352

10.3.4 For $c > 0$, $\int_c^{\infty} x^{-3}\,dx = 1/(2c^2)$. By Eq. (10.3.5) at $n = 4$, $\frac{1}{2\cdot 25} \le R_4 \le \frac{1}{2\cdot 16}$, that is $\frac{1}{50} \le R_4 \le \frac{1}{32}$. The true tail is about 0.02439. p. 352

10.3.5 The upper bound is $\int_n^{\infty} x^{-2}\,dx = \lim_{b\to\infty}(1/n - 1/b) = 1/n$. Requiring $1/n \le 0.001$ gives $n \ge 1000$. For 10^{-6} the same inequality gives $n \ge 10^6$. The lower bound $R_n \ge 1/(n+1)$ shows that the estimate is sharp: 999 terms leave an error of at least 0.001. The error bounds of Section 10.6 and of the Taylor series of Section 11.3 apply to other series and fall much faster: Problem 10.6.4 reaches 0.001 for $1/e$ with seven terms. p. 352

10.3.6 Since $\ln(n+1) - \ln n = \int_n^{n+1} dx/x$ and $1/x \ge 1/(n+1)$ on $[n, n+1]$, the order property gives $d_n - d_{n+1} = \int_n^{n+1} dx/x - 1/(n+1) \ge 0$. By (10.3.3) with $f(x) = 1/x$, $H_n - 1 \le \ln n \le H_n - \frac{1}{n}$.

Hence $\frac{1}{n} \le d_n \le 1$. A nonincreasing sequence bounded below converges by Theorem 5.1.18. p. 352

10.3.7 If $f \ge 0$ on $[1, \infty)$, Theorem 10.3.2 applies. Otherwise some x_0 has $f(x_0) = -c < 0$ and $f(x) \le -c$ for $x \ge x_0$. Then $f(n) \le -c$ for $n \ge x_0$, the terms do not tend to 0 and the series diverges by Theorem 10.2.4. Also $\int_1^b f \le \int_1^{x_0} f - c(b - x_0) \to -\infty$. Hence the integral diverges. Both diverge and the conclusion of the integral test holds. p. 352

10.3.8 Let T_n be the tent with peak $1/n$ at $x = n$ and base $[n - 1/n^2, n + 1/n^2]$ and let $f = e^{-x} + \sum_{n \ge 2} T_n$. Since the bases are disjoint, f is continuous and positive. Since the area of T_n is $1/n^3$, for each b $\int_1^b f \le \int_1^\infty e^{-x}\,dx + \sum_{n \ge 2} 1/n^3$ and the integral converges by Proposition 5.2.23. Since $f(n) \ge 1/n$, the series diverges by comparison of partial sums with the harmonic series. Since the function rises to each peak, it is not nonincreasing. p. 352

10.3.9 Letting $n \to \infty$ in (10.3.3) gives $S - a_1 \le \int_1^\infty f \le S$, that is $1 \le S \le 2$. The sum is $\pi^2/6 \approx 1.645$. For any nonincreasing f the sum exceeds the integral by the total area of the parts of the right-hand rectangles in Fig. 10.3.1 above the curve. That area is positive whenever f is strictly decreasing. p. 352

10.3.10 The block $2^k \le n < 2^{k+1}$ has 2^k terms, each at most a_{2^k}. Hence $\sum_{n=1}^{2^{K+1}-1} a_n \le \sum_{k=0}^{K} 2^k a_{2^k}$. The block $2^{k-1} < n \le 2^k$ has 2^{k-1} terms, each at least a_{2^k}. Hence $\sum_{n=2}^{2^K} a_n \ge \sum_{k=1}^{K} 2^{k-1} a_{2^k}$. Since both series have nonnegative terms, each inequality turns by Theorem 10.3.1 bounded partial sums of one series into bounded partial sums of the other.

For $a_n = n^{-p}$ with $p > 0$ the condensed series is $\sum 2^k 2^{-kp} = \sum (2^{1-p})^k$. That geometric series with ratio 2^{1-p} converges exactly when $p > 1$ (Theorem 10.2.7). For $p \le 0$ the terms do not tend to 0. p. 352

10.4 Comparison Tests

10.4.1 For $n \ge 1$, $0 < 1/(n^2 + n) < 1/n^2$ and $\sum 1/n^2$ converges by Theorem 10.3.3. Part (1) of Theorem 10.4.1 gives convergence. The series also telescopes to 1 (Problem 10.2.2). p. 356

10.4.2 For $n \ge 1$, $(n + 1)/n^2 > 1/n$ and $\sum 1/n$ diverges by Theorem 10.2.5. Part (2) of Theorem 10.4.1 gives divergence. p. 356

10.4.3 With $b_n = 1/n$, $a_n/b_n = n/\sqrt{n^2 + 1} = 1/\sqrt{1 + 1/n^2} \to 1$ and the limit is finite and nonzero. Since $\sum 1/n$ diverges, part (1) of Theorem 10.4.2 gives divergence. p. 356

10.4.4 Since $(\ln n)/n^{1/4} \to 0$ (Theorem 5.2.26), there is an N with $\ln n \le n^{1/4}$ for $n \ge N$. For those n, $0 \le (\ln n)/n^{3/2} \le 1/n^{5/4}$ and $\sum 1/n^{5/4}$ converges by Theorem 10.3.3. Part (1) of Theorem 10.4.1 from N onwards gives convergence. The exponent $\frac{3}{2}$ exceeds 1 by $\frac{1}{2}$ and the logarithm absorbs $\frac{1}{4}$ of that margin. The series $\sum (\ln n)/n$ has no margin left and diverges. p. 356

10.4.5 The geometric series $\sum (3/4)^n$ converges. With $b_n = (3/4)^n$,

$$\frac{a_n}{b_n} = \frac{1 + n^2/3^n}{1 + n^5/4^n} \xrightarrow[n \to \infty]{} 1$$

because $n^2/3^n \to 0$ and $n^5/4^n \to 0$ (Lemma 5.2.25). Part (1) of Theorem 10.4.2 gives convergence. Replacing n^2 and n^5 by any powers changes neither limit. Replacing 4^n by 3^n sends the terms to 1 and gives divergence. p. 356

10.4.6 From $(|a|-|b|)^2 \geq 0$, $|a_n b_n| \leq \frac{1}{2}(a_n^2 + b_n^2)$. The right-hand side is the general term of a convergent series by Theorem 10.2.10. Hence Theorem 10.4.1 gives convergence of $\sum |a_n b_n|$. With $b_n = 1/n$, $\sum b_n^2$ converges by Theorem 10.3.3. p. 356

10.4.7 Since $a_n/(1+a_n) \leq a_n$, convergence of $\sum a_n$ gives convergence of the second series by Theorem 10.4.1. Conversely, if $\sum a_n/(1+a_n)$ converges, its terms tend to 0. Since $a_n = t/(1-t)$ with $t = a_n/(1+a_n)$, also $a_n \to 0$. Then $1 + a_n \leq 2$ for large n and hence $a_n \leq 2a_n/(1+a_n)$. Theorem 10.4.1 gives convergence of $\sum a_n$. p. 356

10.4.8 The ratios are $1/n \to 0$ and $1/\ln n \to 0$. The first series converges by Theorem 10.3.3 and the second diverges by Worked Example 10.3.2. Since a limit $c = 0$ against a divergent comparison series is consistent with either behaviour, part (2) requires $\sum b_n$ to converge. p. 356

10.4.9 The tails r_n are positive, decrease to 0 (Problem 10.2.7) and satisfy $b_n = r_{n-1} - r_n$. Hence

$$a_n = \frac{r_{n-1} - r_n}{\sqrt{r_{n-1}}} = \frac{(\sqrt{r_{n-1}} - \sqrt{r_n})(\sqrt{r_{n-1}} + \sqrt{r_n})}{\sqrt{r_{n-1}}} \leq 2(\sqrt{r_{n-1}} - \sqrt{r_n}).$$

Since the right-hand side telescopes, the partial sums of $\sum a_n$ are at most $2\sqrt{r_0}$ and the series converges by Theorem 10.3.1. Since $r_{n-1} \to 0$, $a_n/b_n = 1/\sqrt{r_{n-1}} \to \infty$. If one convergent series $\sum b_n$ were a comparison series for all others, every convergent $\sum a_n$ would have a_n/b_n bounded, contradicting this construction. p. 356

10.5 Ratio and Root Tests

10.5.1 Since $a_{n+1}/a_n = 2/(n+1) \to 0 < 1$, Theorem 10.5.2 gives convergence. Theorem 11.3.2 shows that the sum is $e^2 - 1$. p. 359

10.5.2 $a_{n+1}/a_n = \frac{1}{2}(1+1/n)^2 \to \frac{1}{2}$. The same computation shows that $\sum n^p/r^n$ converges for every p whenever $r > 1$. p. 360

10.5.3 The term is $(n/3^n)^n$ with $a_n^{1/n} = n/3^n \to 0$ by Lemma 5.2.25. Theorem 10.5.1 gives convergence. p. 360

10.5.4 For even n, $a_n = 2^{1-n}$ and $a_{n+1} = 2^{-n-2}$, with ratio $\frac{1}{8}$. For odd n, $a_n = 2^{-n-1}$ and $a_{n+1} = 2^{-n}$, with ratio 2. Neither part of Theorem 10.5.2 applies. The roots $a_n^{1/n} = 2^{-1+(-1)^n/n}$ tend to $\frac{1}{2}$. Theorem 10.5.1 gives convergence. Since the series has nonnegative terms, its odd-index and even-index parts are the geometric series $\sum_{k\geq 1} 4^{-k} = \frac{1}{3}$ and $\sum_{k\geq 1} 2 \cdot 4^{-k} = \frac{2}{3}$. By Problem 10.5.9 the sum is $\frac{1}{3} + \frac{2}{3} = 1$. p. 360

10.5.5 With $a_n = n!/n^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \to \frac{1}{e}$$

by Theorems 5.1.9 and 5.1.20. By Lemma 10.5.4, $a_n^{1/n} = (n!)^{1/n}/n \to 1/e$. p. 360

10.5.6 By the computation of Problem 10.5.5, $a_{n+1}/a_n = x(1+1/n)^{-n} \to x/e$. Theorem 10.5.2 gives convergence for $x < e$ and divergence for $x > e$. At $x = e$ the limit is 1 and part (2) of the theorem applies. Since $(1+1/n)^n < e$ for every n by Theorem 5.1.20, the ratio $e(1+1/n)^{-n}$ exceeds 1 for every n and the terms increase from $a_1 = e$. The series therefore diverges. p. 360

10.5.7 The ratio $(n^2+1)/(n^2+2n+2)$ tends to 1 and so does the root $(n^2+1)^{-1/n}$ by Lemma 10.5.4. Since $0 < 1/(n^2+1) < 1/n^2$ and $\sum 1/n^2$ converges by Theorem 10.3.3, Theorem 10.4.1 gives convergence. p. 360

10.5.8 Induction gives $a_n \le a_1 r^{n-1}$. Comparison with the geometric series (Theorem 10.4.1) gives convergence. For $a_n = 1/n$, $a_{n+1}/a_n = n/(n+1) < 1$ for every n and the series diverges by Theorem 10.2.5. Since the ratios approach 1, no fixed $r < 1$ bounds them. p. 360

10.5.9 The partial sums satisfy $s_{2m} = \sum_{k\le m} a_{2k-1} + \sum_{k\le m} a_{2k}$ with limit $A+B$. Since the terms are nonnegative, $\{s_n\}$ is nondecreasing with $s_n \le s_{2n} \le A+B$ and converges by Theorem 10.3.1. Its subsequence s_{2m} tends to $A+B$. By Theorem 5.1.23 so does the whole sequence. p. 360

10.5.10 Multiplying the hypothesis by a_n gives $na_n - na_{n+1} \ge pa_{n+1}$, that is $na_n - (n+1)a_{n+1} \ge (p-1)a_{n+1} > 0$. Summing over $n = N, \dots, m$, the left side telescopes:

$$(p-1)\sum_{n=N}^{m} a_{n+1} \le Na_N - (m+1)a_{m+1} \le Na_N.$$

The partial sums of $\sum a_n$ are bounded and the series converges by Theorem 10.3.1. For $a_n = 1/n^2$, $n(1-n^2/(n+1)^2) = n(2n+1)/(n+1)^2 \to 2$. The hypothesis therefore holds with $p = \frac{3}{2}$ for large n although $a_{n+1}/a_n \to 1$. p. 360

10.6 Alternating Series and Rearrangement

10.6.1 Since the sizes $1/(2n+1)$ decrease to 0, the series converges. Since $\dfrac{1/(2n+1)}{1/n} \to \frac{1}{2}$, Theorem 10.4.2 and the divergence of $\sum 1/n$ give divergence of the absolute series. p. 364

10.6.2 By Problem 10.5.1, $\sum 2^n/n!$ converges. The series therefore converges absolutely and converges by Corollary 10.2.16 without the alternating series test. p. 364

10.6.3 The absolute series is the p-series, convergent exactly for $p > 1$ (Theorem 10.3.3). For $0 < p \le 1$ the sizes n^{-p} decrease to 0 and Theorem 10.6.2 gives conditional convergence. For $p \le 0$ the terms have absolute value at least 1 and Theorem 10.2.4 gives divergence. p. 364

10.6.4 Since the sizes $1/n!$ are nonincreasing and tend to 0, (10.6.4) applies. Since the first factorial past 1000 is $7! = 5040$, stopping after index 6 leaves an error of at most $1/5040$. The sum of the seven terms is $\frac{265}{720} \approx 0.368056$, against $1/e \approx 0.367879$. Seven terms replace the thousand that Problem 10.3.5 needed for $\sum 1/n^2$. p. 364

10.6.5 Since the running sum is positive after 1, negative terms follow: $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} = \frac{1}{12} > 0$ and then $\frac{1}{12} - \frac{1}{8} = -\frac{1}{24} < 0$. The next positive term gives $-\frac{1}{24} + \frac{1}{3} = \frac{7}{24} > 0$. Negative terms then resume with $\frac{7}{24} - \frac{1}{10} = \frac{23}{120}$ and $\frac{23}{120} - \frac{1}{12} = \frac{13}{120}$. The construction continues with $-\frac{1}{14}$ and $-\frac{1}{16}$ before the next positive term. p. 364

10.6.6 If $\sum |a_n|$ converges, so do $\sum a_n^{\pm}$ by Theorem 10.4.1. Conversely, if both converge, so does their sum $\sum |a_n|$ by Theorem 10.2.10. Suppose $\sum a_n$ converges conditionally. Both parts cannot converge, by the first statement. If $\sum a_n^-$ converged, then $\sum a_n^+ = \sum (a_n + a_n^-)$ would converge. The same argument applies with the roles exchanged. Both therefore diverge. p. 364

10.6.7 If only finitely many terms are negative, $a_n = |a_n|$ from some index on. By Theorem 10.2.10 the convergence of $\sum a_n$ then gives convergence of $\sum |a_n|$ and hence absolute convergence. The same argument applies to $\sum (-a_n)$ if only finitely many terms are positive. p. 364

10.6.8 The positive part $\sum a_n^+$ of $\sum(-1)^{n+1}/n^2$ has the term $1/n^2$ at odd n and 0 at even n. The negative part $\sum a_n^-$ has 0 at odd n and $1/n^2$ at even n. Both converge by Problem 10.6.6. By Problem 10.5.9 applied to a_n^+, whose even-index series is 0, the sum of the positive part is $\sum_k 1/(2k-1)^2$. In the same way the negative part sums to $\sum_k 1/(2k)^2$. The even part is $\frac{1}{4}\sum 1/k^2 = \pi^2/24$ by Theorem 10.2.10. Since by Problem 10.5.9 the odd and even parts of $\sum 1/n^2$ add up to $\pi^2/6$, the odd part is $\pi^2/6 - \pi^2/24 = \pi^2/8$. Since both parts converge, the series is the difference of its positive and negative parts by Theorem 10.2.10 and equals $\pi^2/8 - \pi^2/24 = \pi^2/12$. For the alternating harmonic series both parts diverge and the difference is undefined. Theorem 10.6.8 shows that no value follows from the split. p. 365

10.6.9 Since $a_n \to 0$ by Theorem 10.2.4, $a_n \le 1$ for n beyond some N and then $0 \le a_n^2 \le a_n$. Part (1) of Theorem 10.4.1 gives convergence. For $a_n = (-1)^n/\sqrt{n}$ the series converges by Theorem 10.6.2 while $\sum a_n^2 = \sum 1/n$ diverges. p. 365

10.6.10 At the end of each block the partial sum is 0. Inside block k it is $2c_k$ or c_k. Every partial sum therefore lies in $[0, 2c_k]$ with $c_k \to 0$ and the series converges to 0. The cubes of block k sum to $8c_k^3 - c_k^3 - c_k^3 = 6/k$ and the series of cubes therefore diverges with the harmonic series. Without a sign condition, convergence of $\sum a_n$ does not determine the behaviour of $\sum a_n^3$, in contrast with Problem 10.6.9. p. 365

10.6.11 Writing $b_k = B_k - B_{k-1}$ and collecting the coefficient of each B_k gives the identity of the hint. Since $a_k - a_{k+1} \ge 0$ and $|B_k| \le K$,

$$\Big|\sum_{k=n+1}^{m} a_k b_k\Big| \le K\Big(a_m + a_{n+1} + \sum_{k=n+1}^{m-1}(a_k - a_{k+1})\Big) = 2Ka_{n+1}.$$

Since $a_{n+1} \to 0$, the block sums satisfy (10.2.6). Theorem 10.2.12 in the complete space $\mathbb{R}$ then gives convergence. With $b_k = (-1)^{k+1}$ the partial sums are 1 and 0. Hence $\sum(-1)^{k+1}a_k$ converges whenever a_k decreases to 0. p. 365

11.1 Power Series

11.1.1 Since $|c_n/c_{n+1}| = (n+1)!/n! = n+1 \to \infty$, $R = \infty$ and the series converges absolutely for every real x. Section 11.2 identifies the sum as e^x. p. 372

11.1.2 Since the roots are $\frac{1}{2}$ for even n and $\frac{1}{3}$ for odd n, $\limsup|c_n|^{1/n} = \frac{1}{2}$ and $R = 2$ by Theorem 11.1.5. For even n, $c_{n+1}/c_n = 2^n/3^{n+1} \to 0$. For odd n, $c_{n+1}/c_n = 3^n/2^{n+1} \to \infty$. The ratios have no limit. p. 372

11.1.3 Since $|c_n/c_{n+1}| = \sqrt{(n+1)/n} \to 1$, $R = 1$. At $x = 1$ the series $\sum(-1)^n/\sqrt{n}$ converges by Theorem 10.6.2. At $x = -1$ it is $\sum 1/\sqrt{n}$ and diverges by Theorem 10.3.3. p. 372

11.1.4 By Theorem 11.1.15 on $(-1,1)$, $1/(1-x)^2 = \sum_{n\ge1} nx^{n-1}$. Multiplying by x gives $x/(1-x)^2 = \sum_{n\ge1} nx^n$. At $x = \frac{1}{2}$ the sum is $\frac{1}{2}/\frac{1}{4} = 2$. p. 372

11.1.5 For $|x| < 1$, $1/(1+x^4) = \sum(-1)^n x^{4n}$. Since $[0, \frac{1}{2}]$ lies inside $(-1,1)$, Theorem 11.1.15 gives

$$\int_0^{1/2} \frac{\mathrm{d}x}{1+x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)2^{4n+1}}.$$

The sizes $u_0 = 0.5$, $u_1 = 0.00625$, $u_2 \approx 0.00021701$, $u_3 \approx 0.00000939$ and $u_4 \approx 0.00000045$ decrease to 0. By Theorem 10.6.3, four terms leave an error of at most u_4. The sum of the four

terms is $0.5 - 0.00625 + 0.00021701 - 0.00000939 = 0.49395762$ and rounds to 0.49396 at five places. p. 372

11.1.6 Put $t = x^2$. By Theorem 11.1.5 the series $\sum c_n t^n$ converges absolutely for $|t| < R$ and diverges for $|t| > R$. Hence $\sum c_n x^{2n}$ converges absolutely for $|x| < \sqrt{R}$ and diverges for $|x| > \sqrt{R}$. A series with both properties has radius $\sqrt{R}$ by the uniqueness of the dividing number in Theorem 11.1.5. p. 372

11.1.7 By Theorem 10.2.10 the series $g(x) = \sum_m 2c_{2m+1}x^{2m+1}$ has sum $f(x) - f(-x) = 0$ on $(-R, R)$. Applying (11.1.9) k times and setting $x = 0$ leaves $k!$ times the coefficient of x^k. Every coefficient of g is therefore $g^{(k)}(0)/k! = 0$. Hence $c_{2m+1} = 0$. p. 372

11.1.8 On $[-b, b]$, $|\sin(nx)/n^3| \leq 1/n^3$ and $|\cos(nx)/n^2| \leq 1/n^2$, both majorants converging by Theorem 10.3.3. By Theorem 10.2.18 the series and the derived series converge uniformly on $[-b, b]$ and the derived series has a continuous sum. Part (3) of Theorem 11.1.2, with $\sum \sin(-nb)/n^3$ convergent, gives $f' = \sum \cos(nx)/n^2$ on $[-b, b]$. Every real x lies in the interior of some such interval. p. 372

11.1.9 On $[-1, 1]$, $|x^n/n^2| \leq 1/n^2$ and Theorem 10.2.18 gives uniform convergence. The derived series $\sum x^{n-1}/n$ has radius 1 (Lemma 11.1.14). At $x = -1$ it is a convergent alternating series and at $x = 1$ it is the harmonic series. Since differentiation preserves the radius and may lose an endpoint, Theorem 11.1.15 makes no claim at $x = \pm R$. p. 372

11.1.10 For $x > 0$, $n^2 x e^{-nx} \to 0$ by Theorem 5.2.26 and $f_n(0) = 0$. Integration by parts gives $\int_0^1 n^2 x e^{-nx}\,\mathrm{d}x = 1 - (n+1)e^{-n}$ with limit 1. The integrals of the partial sums f_n of $\sum u_n$ do not tend to the integral of the sum 0. By part (2) of Theorem 11.1.2 the convergence cannot be uniform. Indeed $\|f_n\|_\infty = f_n(1/n) = n/e \to \infty$. p. 372

11.2 Taylor and Maclaurin Series

11.2.1 Substituting $-2x$ gives $\sum(-2x)^n/n!$ with first terms $1 - 2x + 2x^2 - \frac{4}{3}x^3$. Since the exponential row holds for every real x, the new series does too. p. 376

11.2.2 From (11.2.6), $\cos(x^2) = \sum(-1)^n x^{4n}/(2n)!$, and multiplying by x gives the series. The powers advance in fours and the factorials in twos. p. 376

11.2.3 The sizes $u_n = 2^{-2n-1}/(2n+1)$ are 0.5, 0.0416667, 0.00625, 0.0011161, 0.0002170, 0.0000444 and $u_6 \approx 0.0000094$. Since they decrease to 0, six terms leave an error below $u_6 < 10^{-5}$ by Theorem 10.6.3. Since the six terms sum to 0.4636398, $\arctan\frac{1}{2} \approx 0.4636$. The value is 0.46364761. p. 376

11.2.4 Both series converge absolutely for every x. Hence Theorem 11.1.17 applies. With $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ and $\sin x = x - \frac{x^3}{6} + \cdots$, the coefficients are 1 for x, 1 for x^2, $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ for x^3 and $\frac{1}{6} - \frac{1}{6} = 0$ for x^4. p. 376

11.2.5 By (11.2.6), $\cos 2x = \sum(-1)^n 2^{2n} x^{2n}/(2n)!$. Since the $n = 0$ term is 1, halving $1 - \cos 2x = \sum_{n\geq 1}(-1)^{n+1}2^{2n}x^{2n}/(2n)!$ gives the series. Squaring $x - x^3/6 + \cdots$ by (11.1.13) gives the same $x^2 - x^4/3 + \cdots$. p. 376

11.2.6 Differentiating $f(-x) = -f(x)$ k times gives $(-1)^k f^{(k)}(-x) = -f^{(k)}(x)$. Since for even k and $x = 0$ the identity reads $f^{(k)}(0) = -f^{(k)}(0)$, $f^{(k)}(0) = 0$ and every even coefficient vanishes. p. 376

11.2.7 The series has $R = \infty$. By (11.1.9), $E'(x) = \sum_{n\geq 1} nx^{n-1}/n! = \sum_{m\geq 0} x^m/m! = E(x)$, with $m = n-1$. Also $E(0) = 1$. By Theorem 7.3.18 with $\lambda = 1$, $E(x) = E(0)e^x = e^x$ for every real x. p. 376

11.2.8 Both series converge absolutely. The Cauchy coefficient is $\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = (x+y)^n/n!$. Therefore Theorem 11.1.17 gives $E(x)E(y) = \sum(x+y)^n/n! = E(x+y)$. The same computation for complex x and y is Proposition 11.3.8. p. 376

11.2.9 By Theorem 11.2.1 the series $\sum(-1)^n x^{2n}$ for $|x| < 1$ is the Maclaurin series. Since its coefficients have $\limsup|c_m|^{1/m} = 1$, $R = 1$. At $x = 1$ the terms $(-1)^n$ do not tend to 0. Smoothness on the real line does not determine the radius. Let a power series about 0 have sum $1/(1+x^2)$ on some interval $(-r, r)$. By Theorem 11.2.1 its coefficients are the Maclaurin coefficients of $1/(1+x^2)$. The series is therefore $\sum(-1)^n x^{2n}$ with radius 1, and no power series about 0 represents the function on $(-r, r)$ with $r > 1$. Over the complex numbers $1+z^2$ vanishes at $z = \pm i$ at distance 1 from the centre. p. 376

11.2.10 The series has radius 1 and represents $\ln(1+x)$ only for $-1 < x \leq 1$. Since at $x = 2$ its terms $(-1)^{n+1}2^n/n$ grow without bound, it diverges by Theorem 10.2.4 and the equation asserts nothing. The value $\ln 3$ is $-\ln\frac{1}{3} = -\ln(1+(-\frac{2}{3}))$ with the logarithm on the right given by the series at $x = -\frac{2}{3}$. p. 376

11.3 Convergence of Taylor Series

11.3.1 $P_3(0.3) = 0.3 - 0.027/6 = 0.2955$. The fourth derivative of sin is sin, bounded by 1. Hence $|R_3| \leq 0.0081/24 = 0.0003375$. The value is 0.29552021. Since $P_4 = P_3$, applying (11.3.2) at $n = 4$ gives the sharper bound $0.3^5/5! \approx 2.0 \times 10^{-5}$. p. 380

11.3.2 The third derivative of cos is sin, bounded by 1. For $x \neq 0$, (11.3.2) at $n = 2$ gives the bound $|x|^3/3!$ and at $x = 0$ both sides vanish. p. 380

11.3.3 Since $9! = 362880$ and $10! = 3628800$, the bound $1/(n+1)!$ falls below 10^{-6} first at $n = 9$. The cosine series has only even powers, so $P_9 = P_8$ has five nonzero terms, summing to $1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40320} \approx 0.5403026$. The value is $\cos 1 \approx 0.5403023$. p. 380

11.3.4 The sizes 0.2, 0.02, 0.0026667, 0.0004, 0.000064 decrease to 0. Four terms give 0.1822667 with error at most $u_5 = 0.000064$. Every number in $[0.1822667, 0.1823307]$ rounds to 0.1823. The value is 0.1823216. On $[0, 0.2]$ the derivative bound is $M_n = n!$ and (11.3.2) gives $0.2^{n+1}/(n+1)$. That is the bound of Theorem 10.6.3. For $-1 < x < 0$ the bound on $[x, 0]$ is $M_n = n!/(1+x)^{n+1}$ and (11.3.2) gives $\left(|x|/(1+x)\right)^{n+1}/(n+1)$. That bound tends to 0 only for $x \geq -\frac{1}{2}$. On $(-1, -\frac{1}{2})$ Theorem 11.3.1 proves nothing. The series converges to $\ln(1+x)$ there by Theorem 11.1.15. p. 380

11.3.5 By Theorem 11.3.1, $2^{13}/13! \approx 1.3 \times 10^{-6}$ and $2^{14}/14! \approx 1.9 \times 10^{-7}$. Hence $n+1 = 14$ and $n = 13$. By Theorem 10.6.3, $u_6 = 2^{13}/13! \approx 1.3 \times 10^{-6}$ and $u_7 = 2^{15}/15! \approx 2.5 \times 10^{-8}$. Therefore the terms of index 0 to 6 suffice. The polynomial through degree 13 has seven nonzero terms. The two bounds agree once the zero coefficient of x^{14} is used. Since $P_{14} = P_{13}$ for the sine, Theorem 11.3.1 at $n = 14$ bounds the same error by $2^{15}/15! = u_7$. p. 380

11.3.6 By Proposition 11.3.8 and induction on n, $\exp(i\theta)^n = \exp(in\theta)$ with real part $\cos n\theta$ by Theorem 11.3.9. The series $\sum\exp(in\theta)/n!$ is the exponential series at $w = \exp(i\theta)$ and so converges to $\exp w$. Real parts of a convergent complex series converge to the real part of the sum. With $w = \cos\theta + i\sin\theta$, Proposition 11.3.8 gives $\exp w = e^{\cos\theta}\exp(i\sin\theta) = e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$ with real part $e^{\cos\theta}\cos(\sin\theta)$. p. 381

11.3.7 Since every t between a and x has $|t| \le |a| + |x-a|$, $M_n = e^{|a|+|x-a|}$ serves for every n and the bound tends to 0 by Theorem 10.1.4. Since every derivative at a is e^a, the series at $x = a + h$ is $\sum e^a h^k/k!$. p. 381

11.3.8 By Theorem 11.3.9 the left side is $\cos(\alpha+\beta)+i\sin(\alpha+\beta)$ and the right side is $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$. Two complex numbers are equal exactly when their real parts and their imaginary parts are equal. p. 381

11.3.9 Since every derivative of ϕ at 0 is 0, the functions e^x and $e^x + \phi(x)$ have the same derivatives at 0 and the same Maclaurin series $\sum x^n/n!$ with sum e^x. Since the second function differs from that sum at every $x \neq 0$, it is not analytic at 0. A Taylor series records the derivatives at one point and cannot detect ϕ. p. 381

11.3.10 For $x > 0$, induction gives $g^{(k)}(x) = p_k(1/x)e^{-1/x}$ with $p_{k+1}(u) = u^2(p_k(u) - p_k'(u))$. Since Theorem 5.2.26 gives $t^m e^{-t} \to 0$ for $t = 1/x$, every such expression tends to 0 as $x \to 0^+$. On $x < 0$ every derivative is 0. The difference-quotient induction of Proposition 11.3.5 gives $g^{(k)}(0) = 0$ for every k and continuity of each $g^{(k)}$. The product ψ is smooth by the product and chain rules. For $0 < x < 1$ both factors are positive and outside $(0, 1)$ one factor vanishes.

Let f be analytic on $\mathbb{R}$ and zero on an interval containing x_0 in its interior and let s be the supremum of the $x \ge x_0$ with $f = 0$ on $[x_0, x]$. If $s < \infty$, every derivative of f vanishes on (x_0, s) and hence at s by continuity. Since the Taylor series at s is then 0, analyticity at s gives $f = 0$ on an interval around s, contrary to the choice of s. Hence $f = 0$ on $[x_0, \infty)$ and the same argument to the left gives $f = 0$ on $\mathbb{R}$. The function ψ is therefore smooth and not analytic. p. 381

11.4 Binomial Series and Applications of Taylor Series

11.4.1 $\binom{1/3}{1} = \frac{1}{3}$ and $\binom{1/3}{2} = \frac{1}{3}(-\frac{2}{3})/2 = -\frac{1}{9}$. Since the exponent is not a nonnegative integer, the series holds for $|x| < 1$. p. 384

11.4.2 The sizes $u_n = 2^{-2n-1}/(n!(2n+1))$ are 0.5, 0.0416667, 0.003125, 0.0001860, $u_4 \approx$ 0.0000090 and $u_5 \approx 0.00000037$. Four terms give 0.4612723 with error below u_4. That bound leaves the fifth decimal place open. Five terms give 0.4612814 with error below u_5. The integral is 0.46128 to five places. p. 384

11.4.3 The sizes are 1, $\frac{1}{3}$, $\frac{1}{10}$, $\frac{1}{42}$, $\frac{1}{216}$, $\frac{1}{1320}$, $\frac{1}{9360}$ and $u_7 = \frac{1}{75600} \approx 1.3 \times 10^{-5}$. The terms through $n = 6$ sum to 0.746836 with error below u_7. The integral is therefore 0.7468. The value is 0.7468241. p. 384

11.4.4 With $m = -\frac{1}{2}$, $\binom{-1/2}{1} = -\frac{1}{2}$ and $\binom{-1/2}{2} = \frac{3}{8}$. Hence $(1-t^2)^{-1/2} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots$ for $|t| < 1$. Integrating from 0 to x by Theorem 11.1.15 gives $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$. p. 385

11.4.5 The numerator of $\binom{-1/2}{k}$ is $\prod_{j=1}^{k}(-\frac{2j-1}{2}) = (-1)^k \frac{1\cdot 3\cdots(2k-1)}{2^k}$. Here $1 \cdot 3 \cdots (2k-1) = (2k)!/(2^k k!)$. Hence $\binom{-1/2}{k} = (-1)^k(2k)!/(4^k(k!)^2)$ and $\binom{-1/2}{k}(-4)^k = \binom{2k}{k}$. The binomial series at $-4x$ gives the sum for $|4x| < 1$. p. 385

11.4.6 By Eq. (4.6.18), $\binom{m}{k+1} = \binom{m}{k} \cdot \dfrac{m-k}{k+1}$. Hence $(k+1)\binom{m}{k+1} = (m-k)\binom{m}{k}$. Adding $k\binom{m}{k}$ gives $m\binom{m}{k}$. p. 385

11.4.7 By (11.2.6), $\cos x - 1 + x^2/2 = x^4/24 - x^6/720 + \cdots$ with leading coefficient $\frac{1}{24}$ at $k = 4$. The denominator is x^4 with $d_4 = 1$. The limit is $\frac{1}{24}$. If g starts at a higher power than f, as with $f(x) = x$ and $g(x) = x^2$, then $d_k = 0$ and $f/g = 1/x$ is unbounded near 0. p. 385

11.4.8 Let $u_k = \left|\binom{1/2}{k}\right|$. The ratio $u_{k+1}/u_k = (k-\frac{1}{2})/(k+1) = 1 - \frac{3/2}{k+1}$ is below 1 and the sizes therefore decrease. The signs alternate from $k = 1$ on. Since $1 - t \le e^{-t}$, $u_k \le u_1 \exp(-\frac{3}{2}\sum_{j=2}^{k} 1/j)$ and the bound tends to 0 because the harmonic series diverges. By Theorem 10.6.2 the series converges at $x = 1$. By Theorem 11.1.12 its sum is the limit of $\sqrt{1+x}$ as $x \to 1^-$, equal to $\sqrt{2}$ by continuity. p. 385

11.4.9

(i) $m > 0$. For integers $k > m$ the identity of the hint gives $ku_k - (k+1)u_{k+1} = mu_k > 0$. The positive numbers $v_k = ku_k$ therefore decrease for $k > m$. Fix an integer $K > m$. For $N > K$,

$$m\sum_{k=K}^{N} u_k = \sum_{k=K}^{N}(v_k - v_{k+1}) = v_K - v_{N+1} \le v_K.$$

By this bound and Theorem 10.3.1 the series $\sum u_k$ converges. At $x = \pm 1$ the terms have size u_k and the series converges absolutely.

(ii) $-1 < m < 0$. Every factor $m - j$ of the numerator in Eq. (4.6.18) is negative. Hence $\binom{m}{k}$ has the sign of $(-1)^k$. At $x = 1$ the series alternates. Its sizes decrease because $u_{k+1}/u_k = (k-m)/(k+1) < 1$. Since $1 - t \le e^{-t}$, $u_k = \prod_{j=0}^{k-1}(1 - \frac{m+1}{j+1}) \le \exp(-(m+1)\sum_{j=1}^{k} 1/j)$. The bound tends to 0 because the harmonic series diverges. By Theorem 10.6.2 the series converges at $x = 1$. At $x = -1$ the terms $\binom{m}{k}(-1)^k = u_k$ are positive. Since $-m > 0$, the identity of the hint gives $(k+1)u_{k+1} = (k-m)u_k \ge ku_k$. Hence $ku_k \ge u_1$ and $u_k \ge u_1/k$. The series diverges by comparison with the harmonic series (Theorems 10.3.3 and 10.4.1).

(iii) $m \le -1$. Here $u_{k+1}/u_k = (k-m)/(k+1) \ge 1$ and $u_k \ge u_0 = 1$. The terms do not tend to 0 at $x = \pm 1$ and the series diverges by Theorem 10.2.4. p. 385

11.4.10 Since $e^x - 1 - x = x^2/2 + x^3/6 + \cdots$, the limit is $\frac{1}{2}$ and the quotient is $\frac{1}{2} + x/6 + \cdots$ near 0. With L'Hôpital's rule, $(e^x - 1)/(2x)$ is again of the form 0/0. A second application gives $e^x/2 \to \frac{1}{2}$. p. 385

11.5 Analytic Functions and the Shift Operator

11.5.1 Since $1 - x = \frac{3}{2}(1 - \frac{2}{3}(x + \frac{1}{2}))$, the geometric series gives $\sum (2/3)^{k+1}(x+\frac{1}{2})^k$ for $|x + \frac{1}{2}| < \frac{3}{2}$. The theorem guarantees only $1 - \frac{1}{2} = \frac{1}{2}$, the distance from b to the edge of $(-1, 1)$. The new series reaches the point $x = 1$ where f is undefined and stops there. p. 389

11.5.2 The terms are $h^k f^{(k)}(0)/k! = (-1)^{k+1}h^k/k$ for $k \ge 1$ and 0 for $k = 0$. The series is (11.1.15) at h with sum $\ln(1 + h)$, the value $(T_h f)(0)$. The condition $|0 - 0| + |h| < 1$ is that of Theorem 11.5.7. p. 389

11.5.3 The terms are nonnegative. For fixed $m \ge 2$, $\sum_{n\ge 2} m^{-n} = m^{-2}/(1 - m^{-1}) = 1/(m(m-1))$. Then $\sum_{m\ge 2} 1/(m(m-1)) = 1$ by telescoping. The iterated sum by rows m is therefore finite, Theorem 11.5.2 applies, and the iterated sum in the other order equals 1. p. 389

11.5.4 Every derivative of exp at x is e^x and $R = \infty$. By Theorem 11.5.7, $e^{x+y} = \sum_k y^k e^x/k! = e^x \sum_k y^k/k! = e^x e^y$, using Theorem 11.3.2 for the last sum. Problem 11.2.8 reached the same identity through the Cauchy product. p. 389

11.5.5 Let $x_0 \in (\alpha, \beta)$ and let s be the supremum of the $x < R$ with $f = 0$ on $[x_0, x]$. If $s < R$, all derivatives of f vanish on (x_0, s) and at s by continuity. By Theorem 11.5.5, f equals its Taylor series at s near s. Since that series is 0, $f = 0$ on an interval around s, contrary to the

choice of s. Hence $f = 0$ on $[x_0, R)$ and the same argument to the left gives $f = 0$ on $(-R, R)$. By Theorem 11.2.1 every c_n is then 0. p. 389

11.5.6 Row 0 is $(1, 0, 0, ...)$ and row $m \geq 1$ has -1 in column $m-1$ and 1 in column m. Column n has 1 in row n and -1 in row $n+1$. The square $m, n \leq N$ contains the $N+1$ diagonal entries and the N entries $a_{m+1,m}$ with $m+1 \leq N$. Its sum is therefore 1. Each column sums to 0. Since the absolute row sums are 2 for $m \geq 1$, the hypothesis $B < \infty$ fails. In the proof the tail of rows cannot be discarded because no M makes $\sum_{m>M} b_m$ small. p. 389

11.5.7 For complex $w \neq 0$ and $|x - b| < |w|$, $\dfrac{1}{(x-b)-w} = -\sum_{k\geq 0}(x-b)^k w^{-k-1}$ by the geometric series in $\mathbb{C}$. With $w = i - b$ and $\bar{w} = -i - b$, $\dfrac{1}{x-i} - \dfrac{1}{x+i} = -\sum_k (w^{-k-1} - \bar{w}^{-k-1})(x-b)^k = -\sum_k 2i\,\mathrm{Im}(w^{-k-1})(x-b)^k$. Hence $g(x) = -\sum_k \mathrm{Im}(w^{-k-1})(x-b)^k$ for $|x-b| < |w| = \sqrt{1+b^2}$. By Theorem 11.2.1 these are the Taylor coefficients. Writing $w = |w|\exp(i\theta)$ with $\theta \in (0, \pi)$, Theorem 11.3.9 and Problem 11.2.8 give $|c_k| = |w|^{-k-1}|\sin((k+1)\theta)|$. Hence $\limsup |c_k|^{1/k} \leq |w|^{-1}$. For a lower bound let $\eta = \frac{1}{4}\sin\theta > 0$. If $|\sin j\theta| < \eta$, then $|\cos j\theta| > \frac{9}{10}$ and the addition formula gives $|\sin((j+1)\theta)| \geq |\cos j\theta|\sin\theta - |\sin j\theta| > \frac{1}{2}\sin\theta > \eta$. Of any two consecutive integers j, one therefore has $|\sin j\theta| \geq \eta$. For the infinitely many such $k+1$, $|c_k|^{1/k} \geq \eta^{1/k}|w|^{-(k+1)/k} \to |w|^{-1}$. The upper limit is $|w|^{-1}$ and the radius is $\sqrt{1+b^2}$ by Theorem 11.1.5. For $0 < |b| < 1$ it exceeds the distance $1 - |b|$ guaranteed by Theorem 11.5.5 from the Maclaurin series. For $|b| \geq 1$ that theorem guarantees nothing. p. 389

Notes

Preface

1. **First Edition (14 October 2026)** Retitles the book *Single-Variable Calculus* and rebuilds it as an honours text in four parts: Foundations of Analysis, Differential Calculus, Integral Calculus and Infinite Series. Every result is proved in the text or in Appendix A. Limits, continuity, the derivative, differentiation as an operator and the convergence of sequences and series are stated in normed spaces. Hints, answers and solutions are in Appendices B, C and D. The section numbers given in the entries below are those of the edition they describe.

   Edition 0.4 (10 October 2026) Revises the writing throughout.

   Edition 0.3 (26 July 2026) Adds Chapter 10 on infinite sequences and series. All ten sections of the twelfth edition's chapter appear, from sequences and the geometric series through the integral, comparison, ratio, root and alternating series tests to power series, Taylor and Maclaurin series, and the binomial series with its applications, followed by an optional eleventh section on the analysis that justifies these tests. Chapter 0 gains the discrete methods the new chapter uses: sigma notation and reindexing, the algebra of sums, the standard closed forms, factorials and the growth ordering, binomial coefficients with a non-integer upper index, and the bookkeeping of signs. Section 10.9 derives Euler's formula a second time from the exponential series and so completes the cross-check proposed in Section 8.2. Four new figures: the rectangles of the integral test, the partial sums of an alternating series approaching their limit, an interval of convergence, and the Taylor polynomials of the sine.

   Edition 0.2 (18 July 2026) Adds an optional enrichment to Section 8.2 that derives Euler's formula from the imaginary unit as a quarter-turn of the plane, with a three-dimensional figure in which the unit circle and the graphs of sine and cosine appear as three projections of one helix. The enrichment also recovers the product-to-sum identities from Euler's formula. Adds a title page and a copyright page, sets the index of problems by topic in three columns, and breaks the widest displayed equation across aligned lines so that its number sits beside the last line.

   Edition 0.1 (17 July 2026) First consolidated draft. Chapter 0 collects the foundations used in later chapters: sets and completeness, limits and continuity, the rules of differentiation and integration that Calculus II assumes, and a strategy for a general integrand. Chapter 8 follows with all seven sections on techniques of integration: integration by parts, trigonometric integrals, trigonometric substitutions, partial fractions, integral tables and computer algebra systems, numerical integration, and improper integrals. Chapters 4 to 6 contain the material on antiderivatives, definite integrals and arc length that Chapter 8 builds upon. Numbering every result by its section in the twelfth edition of Thomas gives headings, theorems and equations the numbers the course text uses. Each section opens with a study guide and worked examples ahead of problems whose answers are all checked against a computer algebra system.

Index of Notation

$\neg P$	not P (negation)
$P \wedge Q$	P and Q (conjunction)
$P \vee Q$	P or Q or both (disjunction)
$P \implies Q$	if P then Q (implication)
$P \iff Q$	P if and only if Q (biconditional)
$\equiv$	is logically equivalent to
$\mathrm{T},\ \mathrm{F}$	the truth values true and false
$\forall x \in S$	for every x in the set S
$\exists x \in S$	there exists at least one x in S
$\exists!\, x \in S$	there exists exactly one x in S
$\mathbb{N},\ \mathbb{Z}$	natural numbers $1, 2, 3, \ldots$; all integers
$\mathbb{Q},\ \mathbb{R}$	rational numbers; real numbers
$a \mid b$	a divides b
$P(n)$	a statement depending on the integer n
$\displaystyle\sum_{k=1}^{n} a_k$	the sum $a_1 + a_2 + \cdots + a_n$
$n!$	$1 \cdot 2 \cdots n$ for $n \in \mathbb{N}$, and $0! = 1$
$\gcd(a, b)$	the greatest common divisor of a and b
$x \in A,\ x \notin A$	x is (is not) an element of A
$\{x \in A : P(x)\}$	the elements of A that satisfy P
$\emptyset$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B,\ A \cap B,\ A \setminus B$	union, intersection, difference
$A^c,\ A \times B$	complement, Cartesian product
$f : A \to B$	a function from A to B
$f(S),\ f^{-1}(T)$	image of S, preimage of T
$g \circ f$	composition, $x \mapsto g(f(x))$
f^{-1}	inverse function of a bijection f

$f\|_S$	restriction of f to S
$\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R}$	the natural numbers, integers, rationals and reals
$x \in S$	x is an element of the set S
$S \subseteq T$	every element of S is an element of T
$\emptyset$	the empty set
$[a,b],\ (a,b)$	closed and open intervals (Definition 2.1.8)
$0.1\overline{36}$	a decimal whose block 36 repeats for ever
$-a,\ a^{-1}$	the additive and multiplicative inverses of a
$a-b,\ a/b$	$a+(-b)$ and $a\,b^{-1}$
$a \leq b$	$a < b$ or $a = b$
$a > b$	$b < a$
$\|a\|$	the absolute value of a (Definition 2.3.5)
$\|a-b\|$	the distance between a and b
$\delta,\ \varepsilon$	positive numbers used as tolerances
$\sup S,\ \inf S$	supremum and infimum of S (Definition 2.4.3)
$\max S,\ \min S$	greatest and least element of S (Definition 2.4.2)
$\lfloor x \rfloor$	the integer part of x (Lemma 2.4.11)
$\sqrt{a}$	the nonnegative real number whose square is $a \geq 0$ (Theorem 2.4.8)
$\mathbf{u}, \mathbf{v}, \mathbf{w}$	vectors (bold); numbers are written in italic
$\mathbb{R}^2$	the plane: ordered pairs (x, y) of real numbers
$\mathbf{v} = (v_1, v_2)$	a vector of the plane and its components
$\mathbf{0}$	the zero vector
$\overrightarrow{PQ}$	the displacement from the point P to the point Q
$T_p\mathbb{R}$	the displacements based at the point p of the line (Definition 3.1.8)
$V,\ W$	real vector spaces
$\mathbb{R}^n$	the n-tuples $(x_1, \ldots, x_n)$ of real numbers
$\mathcal{P}_n$	the real polynomials of degree at most n together with 0
$\mathcal{F}(I)$, $C(I)$	all real functions on the interval I; the continuous ones
$\mathbb{R}^{\mathbb{N}}$	the real sequences $(a_1, a_2, \ldots)$
span(...)	the set of linear combinations of a list (Definition 3.2.16)
$\dim V$	the dimension of V (Definition 3.2.26)
$\mathbf{e}_1, \ldots, \mathbf{e}_n$	the standard basis of $\mathbb{R}^n$

$[\mathbf{v}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{v}$ in the basis $\mathcal{B}$
$T\colon V \to W$	a linear map from V to W
$\ker T$, $\operatorname{im} T$	kernel and image of T (Definitions 3.3.7 and 3.3.8)
$A = (a_{ij})$	a matrix with entry a_{ij} in row i and column j
$A\mathbf{x}$, AB	matrix times vector, matrix product
I, id	the identity matrix, the identity map
A^{-1}, T^{-1}	inverse matrix, inverse map
$R_{c,s}$, $F_{c,s}$	rotation and reflection of the plane (Definitions 3.3.21 and 3.3.22)
M_m	multiplication by the function m
λ, μ	linear functionals
V^*	the dual space of V (Definition 3.4.7)
$\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$	the dual basis of a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ (Theorem 3.4.8)
δ_{ij}	1 if $i = j$ and 0 otherwise
E_a	evaluation at a, $E_a f = f(a)$ (Definition 3.4.10)
$\mathbf{u} \cdot \mathbf{v}$	the dot product (Definition 3.5.2)
$\lvert\mathbf{v}\rvert$	the length of $\mathbf{v}$
$\mathbf{u} \perp \mathbf{v}$	$\mathbf{u}$ and $\mathbf{v}$ are orthogonal
$\operatorname{proj}_{\mathbf{v}} \mathbf{u}$	the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$
$\mathbf{v}^{\perp}$	the quarter turn $R_{0,1}\mathbf{v} = (-v_2, v_1)$
$\det A$	the determinant of a 2×2 matrix (Definition 3.5.13)
$\det(\mathbf{v}, \mathbf{w})$	the determinant of the matrix with columns $\mathbf{v}, \mathbf{w}$
$\lVert\mathbf{v}\rVert$	a norm of $\mathbf{v}$ (Definition 3.6.2)
$\lVert\mathbf{x}\rVert_1$, $\lVert\mathbf{x}\rVert_2$, $\lVert\mathbf{x}\rVert_\infty$	the 1-, 2- and ∞-norms on $\mathbb{R}^n$ (Definition 3.6.6)
$\mathcal{B}(I)$	the bounded real functions on the interval I
$\lVert f\rVert_\infty$	the sup norm of a bounded function (Definition 3.6.9)
$d(\mathbf{u}, \mathbf{v})$, $B_r(\mathbf{v})$	distance and open ball (Definition 3.6.12)
$\mathcal{L}(V, W)$	the bounded linear maps from V to W
$\lVert T\rVert$	the operator norm of a bounded linear map (Definition 3.6.19)
$\operatorname{dom} f$	the domain of the function f
$\Gamma(f)$	the graph $\{(x, f(x)) : x \in \operatorname{dom} f\}$ of f
$\lfloor x \rfloor$	the floor of x, the largest integer not exceeding x
$\deg p$	the degree of a nonzero polynomial p

$\sqrt[n]{a} = a^{1/n}$	the positive nth root of $a > 0$
a^x	the power with base $a > 0$ and real exponent x
e, $\exp x = e^x$	Euler's number and the natural exponential
$\log_a x$	the logarithm of $x > 0$ to the base $a > 0$, $a \neq 1$
$\ln x$	the natural logarithm $\log_e x$
$P(\theta)$	the point of the unit circle at arc length θ from $(1, 0)$
$\sin, \cos, \tan$	sine, cosine, tangent; $\sec, \csc, \cot$ their reciprocals
$\arcsin, \arccos, \arctan$	the inverse trigonometric functions
$\sum_{k=m}^{n} a_k$	the sum $a_m + a_{m+1} + \cdots + a_n$
$n!$	the factorial $n(n-1)\cdots 2 \cdot 1$, with $0! = 1$
$\binom{n}{k}$	the binomial coefficient
$\{a_n\}$	the sequence whose nth term is a_n
$a_n \to L$	the sequence $\{a_n\}$ converges to L (Definition 5.1.3)
$\lim_{n\to\infty} a_n$	the limit of a convergent sequence
$\{a_{n_k}\}$	a subsequence of $\{a_n\}$, with $n_1 < n_2 < \cdots$
e	the limit of $(1 + \frac{1}{n})^n$ (Theorem 5.1.20)
$\mathbf{v}_n \to \mathbf{v}$	convergence in a normed space (Definition 5.1.30)
$\lim_{x\to a} f(x) = L$	$f(x)$ tends to L as x tends to a (Definition 5.2.1)
$\lim_{x\to a^+} f(x)$	the limit from the right, through $x > a$ (Definition 5.2.10)
$\lim_{x\to a^-} f(x)$	the limit from the left, through $x < a$
$\lim_{x\to\infty} f(x)$	the limit at infinity (Definition 5.2.20)
$f(x) \ll g(x)$	$f(x)/g(x) \to 0$ as $x \to \infty$ (Theorem 5.2.26)
$Q(h)$	the difference quotient of f at a (Definition 6.1.2)
$f'(a)$	the derivative of f at a
f'	the derivative function $x \mapsto f'(x)$
$o(h)$	a quantity $r(h)$ with $r(h)/h \to 0$ as $h \to 0$ (Definition 6.1.7)
$f'_+(a)$, $f'_-(a)$	the right-hand and left-hand derivatives (Definition 6.1.13)
$T_p\mathbb{R}$, $\mathbf{v}$	displacements at the point p, and one of them
$\omega_p(\mathbf{v})$	the value of the 1-form ω at p on $\mathbf{v}$
d	the differential operator $f \mapsto \mathrm{d}f$ (Definition 6.2.5)
$\mathrm{d}x$	the 1-form $\mathbf{v} \mapsto \mathbf{v}$ (Eq. (6.2.4))
$f(x)\,\mathrm{d}x$	the 1-form $\mathbf{v} \mapsto f(x)\,\mathbf{v}$ (Eq. (6.2.5))

$\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}}{\mathrm{d}x}f$	the quotient of 1-forms $\mathrm{d}y$ by $\mathrm{d}x$, equal to the derivative (Eq. (6.2.7))
$g^*\omega$	the pullback of ω by g (Definition 6.2.11)
$\mathrm{d}F_{\mathbf{a}}$	the differential at $\mathbf{a}$ of a map between normed spaces (Definition 6.2.14)
$C(I)$	the continuous real functions on the interval I
$C^k(I)$	functions on I with k derivatives, the kth continuous
$C^\infty(I)$	functions on I with derivatives of every order
D, D^n	differentiation $f \mapsto f'$ and its nth power $f \mapsto f^{(n)}$ (this chapter only)
$\ker T$, $\operatorname{im} T$	kernel and image of a linear map T (Definitions 3.3.7 and 3.3.8)
$\|f\|_{C^1}$	the C^1 norm $\|f\|_\infty + \|f'\|_\infty$ (Eq. (7.3.7))
$\|f\|_{C^k}$	the C^k norm $\sum_{j=0}^{k} \|f^{(j)}\|_\infty$
id	the identity map $f \mapsto f$
$\mathcal{P}_n$	the real polynomials of degree at most n
J	the integral operator, a right inverse of D
$P_n(x)$	the Taylor polynomial of order n of f at a
$R_n(x)$	the remainder $f(x) - P_n(x)$
$o((x-a)^n)$	a function r with $r(x)/(x-a)^n \to 0$ as $x \to a$
e^{hD}	the operator $\sum_{k\geq 0} h^k D^k/k!$, a finite sum on polynomials
$F'(x) = f(x)$	F is an antiderivative of f (Definition 8.1.1)
$\int f(x)\,\mathrm{d}x$	all antiderivatives of the 1-form $f(x)\,\mathrm{d}x$ (Definition 8.1.4)
$+C$	constant of integration, one per interval
P, $\|P\|$	a partition $a = x_0 < \cdots < x_n = b$ and its mesh $\max_k \Delta x_k$
Δx_k, c_k	displacement of and sample point in the kth step
$\int_{[a,b]} \omega$	integral of the 1-form ω over $[a,b]$ oriented from a to b
$\int_a^b f(x)\,\mathrm{d}x$	definite integral of $f(x)\,\mathrm{d}x$ (Definition 8.2.4)
$U(f,P)$, $L(f,P)$	upper and lower sums of f on P (Definition 8.2.8)
$\|f\|_\infty$	sup norm $\max_{a\leq x\leq b}\|f(x)\|$ of $f \in C([a,b])$ (Definition 3.6.9)
$\operatorname{avg}(f)$	average value $\frac{1}{b-a}\int_a^b f\,\mathrm{d}x$
$\big[F(x)\big]_a^b$	shorthand for $F(b) - F(a)$
$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f$	derivative of an integral with variable upper limit
$\int_{u(x)}^{v(x)} f$	both limits functions of x (chain rule applies)

J	the integration map $f \mapsto \int_a^x f(t)\,\mathrm{d}t$ (Theorem 8.3.10)
$\|J\|$	operator norm of J for the sup norm (Proposition 8.3.11)
$u = g(x)$	the inner function chosen as the new variable
$\mathrm{d}u = g'(x)\,\mathrm{d}x$	the pullback of $\mathrm{d}u$, matched against the integrand
$g^*\omega$	pullback of the 1-form ω by g (Eq. (6.2.8))
$g(a)$, $g(b)$	the transformed limits after $u = g(x)$
$\int_{-a}^{a}$	symmetric interval (test the integrand for parity)
top $-$ bottom	the integrand for an area between curves
L	arc length of $y = f(x)$ over $[a, b]$
$\mathrm{d}s$	arc-length density $\sqrt{1 + (f')^2}\,\lvert\mathrm{d}x\rvert$ (Eq. (8.6.3))
$1 + (f')^2$	the radicand (write it as a perfect square when possible)
I, J	intervals of the old variable x and of the new variable t
$\varphi : J \to I$	a substitution $x = \varphi(t)$ that is a bijection (Theorem 9.1.4)
$\varphi^*\omega$	pullback of the 1-form ω by φ (Definition 6.2.11)
a	the positive scale in the templates of Proposition 9.1.7
u, $\mathrm{d}v$	a function and a 1-form whose product is the integrand
$\mathrm{d}u$, v	$\mathrm{d}u = u'(x)\,\mathrm{d}x$ (Eq. (6.2.6)) and v an antiderivative of $\mathrm{d}v$
$C^1(I)$	functions with a continuous derivative on the interval I (Definition 7.3.2)
V_k	iterated antiderivatives: $V_k' = V_{k-1}$ (Theorem 9.2.9)
LIATE	a heuristic order of kinds for the choice of u
m, n	non-negative integer exponents in $\sin^m x \cos^n x$ or $\tan^m x \sec^n x$
p, q	frequencies in a product $\sin px \cos qx$
W_n	the integral $\int_0^{\pi/2} \sin^n x\,\mathrm{d}x$ (Proposition 9.3.10)
t	the Weierstrass variable, $x = 2\arctan t$ and $t = \tan(x/2)$
$R(s, c)$	a rational function of two variables
a	the positive constant fixed by the shape of the radical
θ	the new variable, with the interval of Table 9.4.1
φ	the substitution $x = \varphi(\theta)$, a bijection
u	the shifted variable $x + \beta/(2\alpha)$ after completing the square
p, q	numerator and denominator of $f = p/q$
r, m	a real root of q and its multiplicity
b, c, n	an irreducible factor $x^2 + bx + c$ ($b^2 < 4c$) and its multiplicity

$A_k,\ B_k,\ C_k$	the coefficients of the decomposition
$u,\ k$	the shifted variable $x + \frac{b}{2}$ and $k = \sqrt{c - b^2/4}$
U	an open set, here a union of finitely many disjoint open intervals
$F_1,\ F_2$	two antiderivatives of the same integrand
log	the logarithm as printed by a computer algebra system
$n,\ h$	the number of subintervals and their common width $(b-a)/n$
$x_k,\ y_k$	the kth partition point $a + kh$ and the value $f(x_k)$
$T_n,\ S_n$	the Trapezoidal and Simpson's Rule estimates
$E_T,\ E_S$	the signed errors $T_n - \int_a^b f$ and $S_n - \int_a^b f$
M	an upper bound for $\lvert f'' \rvert$ or $\lvert f^{(4)} \rvert$ on $[a, b]$
β	the improper endpoint: ∞ or a finite point near which f is unbounded
$F(b)$	the proper integral $\int_a^b f(x)\,\mathrm{d}x$ for $a \le b < \beta$
$\int_a^\beta f$	the improper integral $\lim_{b \to \beta^-} F(b)$
p	the exponent in the benchmark integrand x^{-p}
$\limsup a_n$	the upper limit of a real sequence (Definition 10.1.6)
$\liminf a_n$	the lower limit, $-\limsup(-a_n)$
$\lVert v \rVert$	the norm of a vector v in a normed space V
$C([a, b])$	the continuous real functions on $[a, b]$
$\lVert f \rVert_\infty$	the sup norm $\sup_x \lvert f(x) \rvert$ (Definition 3.6.9)
$\mathbf{s}_n$	the nth partial sum, $\mathbf{v}_1 + \cdots + \mathbf{v}_n$
$\sum \mathbf{v}_n$	the series and its sum when it converges
H_n	the nth partial sum of the harmonic series
M_n	a constant bounding $\lVert u_n \rVert_\infty$ in the M-test
f	a continuous nonincreasing function with $f(n) = a_n$
p	the exponent in the term $1/n^p$
R_n	the tail $S - s_n$, the error in stopping at n terms
a_n	the general term of the series under test
b_n	the general term of the comparison series
c	the finite positive limit of a_n/b_n
N	the index beyond which the comparison inequality is satisfied
ρ	the upper limit of $\lvert a_n \rvert^{1/n}$, or the limit of $\lvert a_{n+1}/a_n \rvert$

r	a number strictly between ρ and 1, used in both proofs
u_n	the size of the nth term, with $a_n = (-1)^{n+1}u_n$
L	the sum of a convergent alternating series
σ	a rearrangement, a bijection of the positive integers
a	the centre of the power series
c_n	the coefficient of $(x-a)^n$
R	the radius of convergence, in $[0,\infty]$
f	the sum of the series on its interval of convergence
$f^{(k)}(a)$	the kth derivative of f at a, with $f^{(0)} = f$
$P_n(x)$	the Taylor polynomial of order n
$R_n(x)$	the remainder $f(x) - P_n(x)$
c	the intermediate point supplied by Theorem 7.4.8
M	a bound on $\lvert f^{(n+1)}\rvert$ between a and x
$\exp z$	the complex exponential $\sum z^n/n!$ (Definition 11.3.7)
m	the exponent of the binomial, any real number
$\binom{m}{k}$	the generalised binomial coefficient of Eq. (4.6.18)
u_n	the size of the nth term of an alternating series
a_{mn}	a double sequence, indexed by $m, n \geq 0$
T_h	the shift $(T_h f)(x) = f(x+h)$
$E_h f$	the Taylor shift series $\sum_{k\geq 0} h^k f^{(k)}/k!$

Problems by Topic

1.1	Problem 1.1.1	⋆	2.3	Problem 2.3.1	⋆	3.5	Problem 3.5.3	⋆
1.1	Problem 1.1.2	⋆	2.3	Problem 2.3.2	⋆	3.5	Problem 3.5.4	⋆⋆
1.1	Problem 1.1.3	⋆⋆	2.3	Problem 2.3.3	⋆⋆	3.5	Problem 3.5.5	⋆⋆
1.1	Problem 1.1.4	⋆⋆	2.3	Problem 2.3.4	⋆⋆	3.5	Problem 3.5.6	⋆⋆
1.1	Problem 1.1.5	⋆⋆	2.3	Problem 2.3.5	⋆⋆	3.5	Problem 3.5.7	⋆⋆⋆
1.1	Problem 1.1.6	⋆⋆⋆	2.3	Problem 2.3.6	⋆⋆	3.5	Problem 3.5.8	⋆⋆
1.2	Problem 1.2.1	⋆	2.3	Problem 2.3.7	⋆⋆⋆	3.5	Problem 3.5.9	⋆⋆
1.2	Problem 1.2.2	⋆	2.3	Problem 2.3.8	⋆⋆	3.6	Problem 3.6.1	⋆
1.2	Problem 1.2.3	⋆	2.4	Problem 2.4.1	⋆	3.6	Problem 3.6.2	⋆
1.2	Problem 1.2.4	⋆	2.4	Problem 2.4.2	⋆	3.6	Problem 3.6.3	⋆⋆
1.2	Problem 1.2.5	⋆⋆	2.4	Problem 2.4.3	⋆⋆	3.6	Problem 3.6.4	⋆⋆
1.2	Problem 1.2.6	⋆⋆⋆	2.4	Problem 2.4.4	⋆⋆	3.6	Problem 3.6.5	⋆⋆
1.3	Problem 1.3.1	⋆	2.4	Problem 2.4.5	⋆⋆⋆	3.6	Problem 3.6.6	⋆⋆
1.3	Problem 1.3.2	⋆	2.4	Problem 2.4.6	⋆⋆	3.6	Problem 3.6.7	⋆⋆
1.3	Problem 1.3.3	⋆	2.4	Problem 2.4.7	⋆⋆⋆	3.6	Problem 3.6.8	⋆⋆
1.3	Problem 1.3.4	⋆⋆	3.1	Problem 3.1.1	⋆	3.6	Problem 3.6.9	⋆⋆
1.3	Problem 1.3.5	⋆⋆	3.1	Problem 3.1.2	⋆	3.6	Problem 3.6.10	⋆⋆
1.3	Problem 1.3.6	⋆⋆	3.1	Problem 3.1.3	⋆	3.6	Problem 3.6.11	⋆⋆
1.4	Problem 1.4.1	⋆	3.1	Problem 3.1.4	⋆⋆	3.6	Problem 3.6.12	⋆⋆⋆
1.4	Problem 1.4.2	⋆⋆	3.1	Problem 3.1.5	⋆⋆	4.1	Problem 4.1.1	⋆
1.4	Problem 1.4.3	⋆⋆	3.1	Problem 3.1.6	⋆⋆	4.1	Problem 4.1.2	⋆
1.4	Problem 1.4.4	⋆⋆	3.1	Problem 3.1.7	⋆	4.1	Problem 4.1.3	⋆⋆
1.4	Problem 1.4.5	⋆⋆	3.2	Problem 3.2.1	⋆	4.1	Problem 4.1.4	⋆⋆
1.4	Problem 1.4.6	⋆⋆⋆	3.2	Problem 3.2.2	⋆	4.1	Problem 4.1.5	⋆⋆
1.4	Problem 1.4.7	⋆	3.2	Problem 3.2.3	⋆	4.1	Problem 4.1.6	⋆⋆
1.4	Problem 1.4.8	⋆⋆	3.2	Problem 3.2.4	⋆⋆	4.1	Problem 4.1.7	⋆⋆
1.4	Problem 1.4.9	⋆⋆⋆	3.2	Problem 3.2.5	⋆⋆	4.2	Problem 4.2.1	⋆
1.5	Problem 1.5.1	⋆	3.2	Problem 3.2.6	⋆⋆	4.2	Problem 4.2.2	⋆⋆
1.5	Problem 1.5.2	⋆	3.2	Problem 3.2.7	⋆⋆	4.2	Problem 4.2.3	⋆
1.5	Problem 1.5.3	⋆	3.2	Problem 3.2.8	⋆⋆	4.2	Problem 4.2.4	⋆⋆
1.5	Problem 1.5.4	⋆⋆	3.3	Problem 3.3.1	⋆	4.2	Problem 4.2.5	⋆⋆
1.5	Problem 1.5.5	⋆⋆	3.3	Problem 3.3.2	⋆	4.2	Problem 4.2.6	⋆⋆
1.5	Problem 1.5.6	⋆⋆⋆	3.3	Problem 3.3.3	⋆	4.2	Problem 4.2.7	⋆⋆
1.5	Problem 1.5.7	⋆⋆⋆	3.3	Problem 3.3.4	⋆⋆	4.2	Problem 4.2.8	⋆⋆⋆
2.1	Problem 2.1.1	⋆	3.3	Problem 3.3.5	⋆⋆	4.3	Problem 4.3.1	⋆
2.1	Problem 2.1.2	⋆	3.3	Problem 3.3.6	⋆⋆	4.3	Problem 4.3.2	⋆
2.1	Problem 2.1.3	⋆	3.3	Problem 3.3.7	⋆⋆⋆	4.3	Problem 4.3.3	⋆⋆
2.1	Problem 2.1.4	⋆⋆	3.3	Problem 3.3.8	⋆⋆	4.3	Problem 4.3.4	⋆⋆
2.1	Problem 2.1.5	⋆⋆	3.4	Problem 3.4.1	⋆	4.3	Problem 4.3.5	⋆⋆
2.1	Problem 2.1.6	⋆⋆	3.4	Problem 3.4.2	⋆	4.3	Problem 4.3.6	⋆⋆
2.1	Problem 2.1.7	⋆⋆⋆	3.4	Problem 3.4.3	⋆	4.3	Problem 4.3.7	⋆⋆⋆
2.2	Problem 2.2.1	⋆	3.4	Problem 3.4.4	⋆⋆	4.4	Problem 4.4.1	⋆
2.2	Problem 2.2.2	⋆	3.4	Problem 3.4.5	⋆⋆	4.4	Problem 4.4.2	⋆
2.2	Problem 2.2.3	⋆⋆	3.4	Problem 3.4.6	⋆	4.4	Problem 4.4.3	⋆⋆
2.2	Problem 2.2.4	⋆⋆	3.4	Problem 3.4.7	⋆⋆⋆	4.4	Problem 4.4.4	⋆⋆
2.2	Problem 2.2.5	⋆⋆	3.4	Problem 3.4.8	⋆⋆	4.4	Problem 4.4.5	⋆⋆
2.2	Problem 2.2.6	⋆⋆	3.5	Problem 3.5.1	⋆	4.4	Problem 4.4.6	⋆⋆
2.2	Problem 2.2.7	⋆⋆	3.5	Problem 3.5.2	⋆	4.4	Problem 4.4.7	⋆

4.4	Problem 4.4.8	⋆⋆⋆	5.4	Problem 5.4.9	⋆⋆⋆	7.1	Problem 7.1.1	⋆
4.5	Problem 4.5.1	⋆	5.5	Problem 5.5.1	⋆	7.1	Problem 7.1.2	⋆
4.5	Problem 4.5.2	⋆⋆	5.5	Problem 5.5.2	⋆	7.1	Problem 7.1.3	⋆
4.5	Problem 4.5.3	⋆⋆	5.5	Problem 5.5.3	⋆⋆	7.1	Problem 7.1.4	⋆
4.5	Problem 4.5.4	⋆⋆	5.5	Problem 5.5.4	⋆⋆	7.1	Problem 7.1.5	⋆⋆
4.5	Problem 4.5.5	⋆⋆	5.5	Problem 5.5.5	⋆⋆	7.1	Problem 7.1.6	⋆⋆
4.5	Problem 4.5.6	⋆⋆	5.5	Problem 5.5.6	⋆⋆⋆	7.1	Problem 7.1.7	⋆⋆
4.5	Problem 4.5.7	⋆⋆⋆	5.5	Problem 5.5.7	⋆⋆⋆	7.1	Problem 7.1.8	⋆⋆
4.5	Problem 4.5.8	⋆⋆⋆	5.5	Problem 5.5.8	⋆⋆⋆	7.1	Problem 7.1.9	⋆⋆
4.5	Problem 4.5.9	⋆	6.1	Problem 6.1.1	⋆	7.1	Problem 7.1.10	⋆⋆⋆
4.5	Problem 4.5.10	⋆⋆	6.1	Problem 6.1.2	⋆	7.2	Problem 7.2.1	⋆
4.6	Problem 4.6.1	⋆	6.1	Problem 6.1.3	⋆⋆	7.2	Problem 7.2.2	⋆
4.6	Problem 4.6.2	⋆⋆	6.1	Problem 6.1.4	⋆⋆	7.2	Problem 7.2.3	⋆
4.6	Problem 4.6.3	⋆⋆	6.1	Problem 6.1.5	⋆⋆	7.2	Problem 7.2.4	⋆⋆
4.6	Problem 4.6.4	⋆⋆	6.1	Problem 6.1.6	⋆⋆	7.2	Problem 7.2.5	⋆⋆
4.6	Problem 4.6.5	⋆⋆	6.1	Problem 6.1.7	⋆⋆	7.2	Problem 7.2.6	⋆⋆
4.6	Problem 4.6.6	⋆⋆⋆	6.1	Problem 6.1.8	⋆⋆	7.2	Problem 7.2.7	⋆⋆
4.6	Problem 4.6.7	⋆⋆⋆	6.1	Problem 6.1.9	⋆⋆	7.2	Problem 7.2.8	⋆⋆
4.6	Problem 4.6.8	⋆⋆	6.1	Problem 6.1.10	⋆⋆⋆	7.2	Problem 7.2.9	⋆⋆
5.1	Problem 5.1.1	⋆	6.2	Problem 6.2.1	⋆	7.2	Problem 7.2.10	⋆⋆
5.1	Problem 5.1.2	⋆	6.2	Problem 6.2.2	⋆	7.2	Problem 7.2.11	⋆⋆
5.1	Problem 5.1.3	⋆⋆	6.2	Problem 6.2.3	⋆⋆	7.2	Problem 7.2.12	⋆⋆⋆
5.1	Problem 5.1.4	⋆⋆	6.2	Problem 6.2.4	⋆⋆	7.3	Problem 7.3.1	⋆
5.1	Problem 5.1.5	⋆⋆	6.2	Problem 6.2.5	⋆⋆	7.3	Problem 7.3.2	⋆
5.1	Problem 5.1.6	⋆⋆	6.2	Problem 6.2.6	⋆⋆	7.3	Problem 7.3.3	⋆
5.1	Problem 5.1.7	⋆⋆	6.2	Problem 6.2.7	⋆⋆⋆	7.3	Problem 7.3.4	⋆⋆
5.1	Problem 5.1.8	⋆⋆	6.2	Problem 6.2.8	⋆⋆⋆	7.3	Problem 7.3.5	⋆⋆
5.1	Problem 5.1.9	⋆⋆	6.2	Problem 6.2.9	⋆⋆	7.3	Problem 7.3.6	⋆⋆
5.1	Problem 5.1.10	⋆⋆⋆	6.2	Problem 6.2.10	⋆⋆⋆	7.3	Problem 7.3.7	⋆⋆
5.1	Problem 5.1.11	⋆⋆	6.3	Problem 6.3.1	⋆	7.3	Problem 7.3.8	⋆⋆
5.1	Problem 5.1.12	⋆⋆⋆	6.3	Problem 6.3.2	⋆	7.3	Problem 7.3.9	⋆⋆
5.2	Problem 5.2.1	⋆	6.3	Problem 6.3.3	⋆⋆	7.3	Problem 7.3.10	⋆⋆⋆
5.2	Problem 5.2.2	⋆	6.3	Problem 6.3.4	⋆⋆	7.3	Problem 7.3.11	⋆⋆
5.2	Problem 5.2.3	⋆⋆	6.3	Problem 6.3.5	⋆⋆	7.4	Problem 7.4.1	⋆
5.2	Problem 5.2.4	⋆⋆	6.3	Problem 6.3.6	⋆⋆	7.4	Problem 7.4.2	⋆
5.2	Problem 5.2.5	⋆⋆	6.3	Problem 6.3.7	⋆⋆	7.4	Problem 7.4.3	⋆⋆
5.2	Problem 5.2.6	⋆⋆	6.3	Problem 6.3.8	⋆⋆⋆	7.4	Problem 7.4.4	⋆⋆
5.2	Problem 5.2.7	⋆⋆	6.3	Problem 6.3.9	⋆⋆⋆	7.4	Problem 7.4.5	⋆⋆
5.2	Problem 5.2.8	⋆⋆⋆	6.3	Problem 6.3.10	⋆⋆⋆	7.4	Problem 7.4.6	⋆⋆
5.2	Problem 5.2.9	⋆⋆⋆	6.4	Problem 6.4.1	⋆	7.4	Problem 7.4.7	⋆⋆
5.3	Problem 5.3.1	⋆	6.4	Problem 6.4.2	⋆	7.4	Problem 7.4.8	⋆⋆
5.3	Problem 5.3.2	⋆	6.4	Problem 6.4.3	⋆⋆	7.4	Problem 7.4.9	⋆⋆⋆
5.3	Problem 5.3.3	⋆	6.4	Problem 6.4.4	⋆⋆	7.4	Problem 7.4.10	⋆⋆⋆
5.3	Problem 5.3.4	⋆⋆	6.4	Problem 6.4.5	⋆⋆	7.4	Problem 7.4.11	⋆⋆⋆
5.3	Problem 5.3.5	⋆⋆	6.4	Problem 6.4.6	⋆⋆	8.1	Problem 8.1.1	⋆
5.3	Problem 5.3.6	⋆⋆	6.4	Problem 6.4.7	⋆⋆	8.1	Problem 8.1.2	⋆
5.3	Problem 5.3.7	⋆⋆	6.4	Problem 6.4.8	⋆⋆⋆	8.1	Problem 8.1.3	⋆
5.3	Problem 5.3.8	⋆⋆⋆	6.4	Problem 6.4.9	⋆⋆⋆	8.1	Problem 8.1.4	⋆⋆
5.3	Problem 5.3.9	⋆⋆⋆	6.4	Problem 6.4.10	⋆⋆⋆	8.1	Problem 8.1.5	⋆⋆
5.3	Problem 5.3.10	⋆⋆	6.5	Problem 6.5.1	⋆	8.1	Problem 8.1.6	⋆⋆
5.4	Problem 5.4.1	⋆	6.5	Problem 6.5.2	⋆⋆	8.1	Problem 8.1.7	⋆⋆
5.4	Problem 5.4.2	⋆	6.5	Problem 6.5.3	⋆⋆	8.1	Problem 8.1.8	⋆⋆
5.4	Problem 5.4.3	⋆	6.5	Problem 6.5.4	⋆⋆	8.1	Problem 8.1.9	⋆⋆
5.4	Problem 5.4.4	⋆⋆	6.5	Problem 6.5.5	⋆⋆	8.2	Problem 8.2.1	⋆
5.4	Problem 5.4.5	⋆⋆	6.5	Problem 6.5.6	⋆⋆	8.2	Problem 8.2.2	⋆⋆
5.4	Problem 5.4.6	⋆⋆	6.5	Problem 6.5.7	⋆⋆	8.2	Problem 8.2.3	⋆⋆
5.4	Problem 5.4.7	⋆⋆	6.5	Problem 6.5.8	⋆⋆⋆	8.2	Problem 8.2.4	⋆⋆
5.4	Problem 5.4.8	⋆⋆⋆	6.5	Problem 6.5.9	⋆⋆⋆	8.2	Problem 8.2.5	⋆⋆

8.2	Problem 8.2.6	⋆⋆⋆	9.3	Problem 9.3.2	⋆	10.1	Problem 10.1.2	⋆
8.2	Problem 8.2.7	⋆⋆	9.3	Problem 9.3.3	⋆⋆	10.1	Problem 10.1.3	⋆
8.2	Problem 8.2.8	⋆⋆	9.3	Problem 9.3.4	⋆⋆	10.1	Problem 10.1.4	⋆⋆
8.3	Problem 8.3.1	⋆	9.3	Problem 9.3.5	⋆	10.1	Problem 10.1.5	⋆⋆
8.3	Problem 8.3.2	⋆⋆	9.3	Problem 9.3.6	⋆⋆⋆	10.1	Problem 10.1.6	⋆⋆
8.3	Problem 8.3.3	⋆⋆	9.3	Problem 9.3.7	⋆⋆	10.1	Problem 10.1.7	⋆
8.3	Problem 8.3.4	⋆⋆	9.3	Problem 9.3.8	⋆⋆	10.1	Problem 10.1.8	⋆⋆
8.3	Problem 8.3.5	⋆⋆	9.3	Problem 9.3.9	⋆⋆⋆	10.1	Problem 10.1.9	⋆⋆
8.3	Problem 8.3.6	⋆⋆	9.3	Problem 9.3.10	⋆⋆	10.1	Problem 10.1.10	⋆⋆
8.3	Problem 8.3.7	⋆⋆	9.4	Problem 9.4.1	⋆⋆	10.1	Problem 10.1.11	⋆⋆
8.3	Problem 8.3.8	⋆⋆	9.4	Problem 9.4.2	⋆⋆	10.1	Problem 10.1.12	⋆⋆⋆
8.3	Problem 8.3.9	⋆⋆	9.4	Problem 9.4.3	⋆⋆	10.2	Problem 10.2.1	⋆
8.3	Problem 8.3.10	⋆⋆	9.4	Problem 9.4.4	⋆⋆	10.2	Problem 10.2.2	⋆
8.4	Problem 8.4.1	⋆	9.4	Problem 9.4.5	⋆⋆	10.2	Problem 10.2.3	⋆
8.4	Problem 8.4.2	⋆⋆	9.4	Problem 9.4.6	⋆⋆	10.2	Problem 10.2.4	⋆⋆
8.4	Problem 8.4.3	⋆⋆	9.4	Problem 9.4.7	⋆⋆	10.2	Problem 10.2.5	⋆⋆
8.4	Problem 8.4.4	⋆⋆	9.4	Problem 9.4.8	⋆⋆	10.2	Problem 10.2.6	⋆
8.4	Problem 8.4.5	⋆	9.4	Problem 9.4.9	⋆⋆	10.2	Problem 10.2.7	⋆⋆
8.4	Problem 8.4.6	⋆	9.4	Problem 9.4.10	⋆	10.2	Problem 10.2.8	⋆⋆
8.4	Problem 8.4.7	⋆⋆	9.4	Problem 9.4.11	⋆	10.2	Problem 10.2.9	⋆⋆
8.4	Problem 8.4.8	⋆⋆	9.5	Problem 9.5.1	⋆⋆	10.2	Problem 10.2.10	⋆⋆
8.4	Problem 8.4.9	⋆⋆	9.5	Problem 9.5.2	⋆⋆	10.2	Problem 10.2.11	⋆⋆⋆
8.5	Problem 8.5.1	⋆	9.5	Problem 9.5.3	⋆⋆	10.3	Problem 10.3.1	⋆
8.5	Problem 8.5.2	⋆	9.5	Problem 9.5.4	⋆⋆	10.3	Problem 10.3.2	⋆
8.5	Problem 8.5.3	⋆⋆	9.5	Problem 9.5.5	⋆⋆	10.3	Problem 10.3.3	⋆⋆
8.5	Problem 8.5.4	⋆⋆	9.5	Problem 9.5.6	⋆⋆	10.3	Problem 10.3.4	⋆⋆
8.5	Problem 8.5.5	⋆⋆	9.5	Problem 9.5.7	⋆⋆⋆	10.3	Problem 10.3.5	⋆⋆
8.5	Problem 8.5.6	⋆⋆	9.5	Problem 9.5.8	⋆⋆	10.3	Problem 10.3.6	⋆⋆
8.5	Problem 8.5.7	⋆⋆⋆	9.5	Problem 9.5.9	⋆⋆	10.3	Problem 10.3.7	⋆⋆
8.5	Problem 8.5.8	⋆⋆	9.5	Problem 9.5.10	⋆	10.3	Problem 10.3.8	⋆⋆
8.5	Problem 8.5.9	⋆⋆	9.6	Problem 9.6.1	⋆	10.3	Problem 10.3.9	⋆⋆
8.6	Problem 8.6.1	⋆	9.6	Problem 9.6.2	⋆	10.3	Problem 10.3.10	⋆⋆⋆
8.6	Problem 8.6.2	⋆⋆	9.6	Problem 9.6.3	⋆⋆	10.4	Problem 10.4.1	⋆
8.6	Problem 8.6.3	⋆⋆	9.6	Problem 9.6.4	⋆⋆	10.4	Problem 10.4.2	⋆
8.6	Problem 8.6.4	⋆⋆	9.6	Problem 9.6.5	⋆⋆	10.4	Problem 10.4.3	⋆
8.6	Problem 8.6.5	⋆⋆	9.6	Problem 9.6.6	⋆⋆	10.4	Problem 10.4.4	⋆⋆
8.6	Problem 8.6.6	⋆⋆	9.6	Problem 9.6.7	⋆⋆	10.4	Problem 10.4.5	⋆⋆
8.6	Problem 8.6.7	⋆	9.6	Problem 9.6.8	⋆	10.4	Problem 10.4.6	⋆⋆
8.6	Problem 8.6.8	⋆⋆	9.7	Problem 9.7.1	⋆⋆	10.4	Problem 10.4.7	⋆⋆
9.1	Problem 9.1.1	⋆	9.7	Problem 9.7.2	⋆	10.4	Problem 10.4.8	⋆⋆
9.1	Problem 9.1.2	⋆	9.7	Problem 9.7.3	⋆⋆	10.4	Problem 10.4.9	⋆⋆⋆
9.1	Problem 9.1.3	⋆⋆	9.7	Problem 9.7.4	⋆⋆	10.5	Problem 10.5.1	⋆
9.1	Problem 9.1.4	⋆⋆	9.7	Problem 9.7.5	⋆⋆⋆	10.5	Problem 10.5.2	⋆
9.1	Problem 9.1.5	⋆⋆	9.7	Problem 9.7.6	⋆⋆	10.5	Problem 10.5.3	⋆⋆
9.1	Problem 9.1.6	⋆	9.7	Problem 9.7.7	⋆⋆	10.5	Problem 10.5.4	⋆⋆
9.1	Problem 9.1.7	⋆⋆	9.7	Problem 9.7.8	⋆⋆	10.5	Problem 10.5.5	⋆⋆
9.1	Problem 9.1.8	⋆⋆	9.8	Problem 9.8.1	⋆⋆	10.5	Problem 10.5.6	⋆⋆
9.2	Problem 9.2.1	⋆	9.8	Problem 9.8.2	⋆	10.5	Problem 10.5.7	⋆⋆
9.2	Problem 9.2.2	⋆⋆	9.8	Problem 9.8.3	⋆⋆	10.5	Problem 10.5.8	⋆
9.2	Problem 9.2.3	⋆	9.8	Problem 9.8.4	⋆⋆	10.5	Problem 10.5.9	⋆⋆
9.2	Problem 9.2.4	⋆	9.8	Problem 9.8.5	⋆⋆	10.5	Problem 10.5.10	⋆⋆⋆
9.2	Problem 9.2.5	⋆⋆⋆	9.8	Problem 9.8.6	⋆⋆	10.6	Problem 10.6.1	⋆
9.2	Problem 9.2.6	⋆⋆	9.8	Problem 9.8.7	⋆⋆	10.6	Problem 10.6.2	⋆
9.2	Problem 9.2.7	⋆⋆	9.8	Problem 9.8.8	⋆⋆	10.6	Problem 10.6.3	⋆⋆
9.2	Problem 9.2.8	⋆⋆	9.8	Problem 9.8.9	⋆⋆	10.6	Problem 10.6.4	⋆⋆
9.2	Problem 9.2.9	⋆⋆	9.8	Problem 9.8.10	⋆⋆⋆	10.6	Problem 10.6.5	⋆⋆
9.2	Problem 9.2.10	⋆⋆	9.8	Problem 9.8.11	⋆⋆⋆	10.6	Problem 10.6.6	⋆⋆
9.3	Problem 9.3.1	⋆	10.1	Problem 10.1.1	⋆	10.6	Problem 10.6.7	⋆

10.6	Problem 10.6.8	⋆⋆	11.2	Problem 11.2.4	⋆⋆	11.4	Problem 11.4.1	⋆
10.6	Problem 10.6.9	⋆⋆	11.2	Problem 11.2.5	⋆⋆	11.4	Problem 11.4.2	⋆
10.6	Problem 10.6.10	⋆⋆	11.2	Problem 11.2.6	⋆	11.4	Problem 11.4.3	⋆⋆
10.6	Problem 10.6.11	⋆⋆⋆	11.2	Problem 11.2.7	⋆⋆	11.4	Problem 11.4.4	⋆⋆
11.1	Problem 11.1.1	⋆	11.2	Problem 11.2.8	⋆⋆	11.4	Problem 11.4.5	⋆⋆
11.1	Problem 11.1.2	⋆⋆	11.2	Problem 11.2.9	⋆⋆	11.4	Problem 11.4.6	⋆
11.1	Problem 11.1.3	⋆	11.2	Problem 11.2.10	⋆	11.4	Problem 11.4.7	⋆⋆
11.1	Problem 11.1.4	⋆⋆	11.3	Problem 11.3.1	⋆	11.4	Problem 11.4.8	⋆⋆
11.1	Problem 11.1.5	⋆⋆	11.3	Problem 11.3.2	⋆	11.4	Problem 11.4.9	⋆⋆⋆
11.1	Problem 11.1.6	⋆⋆	11.3	Problem 11.3.3	⋆⋆	11.4	Problem 11.4.10	⋆⋆
11.1	Problem 11.1.7	⋆⋆	11.3	Problem 11.3.4	⋆⋆	11.5	Problem 11.5.1	⋆⋆
11.1	Problem 11.1.8	⋆⋆	11.3	Problem 11.3.5	⋆⋆	11.5	Problem 11.5.2	⋆
11.1	Problem 11.1.9	⋆⋆	11.3	Problem 11.3.6	⋆⋆	11.5	Problem 11.5.3	⋆⋆
11.1	Problem 11.1.10	⋆⋆	11.3	Problem 11.3.7	⋆⋆	11.5	Problem 11.5.4	⋆
11.2	Problem 11.2.1	⋆	11.3	Problem 11.3.8	⋆⋆	11.5	Problem 11.5.5	⋆⋆
11.2	Problem 11.2.2	⋆	11.3	Problem 11.3.9	⋆	11.5	Problem 11.5.6	⋆⋆
11.2	Problem 11.2.3	⋆⋆	11.3	Problem 11.3.10	⋆⋆⋆	11.5	Problem 11.5.7	⋆⋆⋆

References

[1] J. H. Hubbard and B. B. Hubbard. *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach*. 5th ed. Matrix Editions, 2015.

[2] G. B. Thomas, M. D. Weir, and J. Hass. *Thomas' Calculus: Early Transcendentals*. 12th ed. Pearson, 2010.