On the Regularity of Three-Dimensional Navier–Stokes Solutions: A Geometric Approach via the Biot–Savart Connection

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Preface

This monograph develops a geometric approach to the regularity problem for the three-dimensional incompressible Navier–Stokes equations, one of the seven Clay Millennium Problems. The question, in its simplest form, asks whether smooth initial data of finite energy necessarily give rise to a globally smooth solution, or whether singularities can form in finite time. Despite decades of progress in harmonic analysis, partial differential equations, and geometric measure theory, the problem remains open.

The central idea pursued here is to define a connection on the infinite-dimensional bundle of divergence-free velocity fields, constructed from the Biot–Savart kernel and the Leray projection at the natural regularity of Leray–Hopf weak solutions. Classical functional analytic approaches to the regularity problem rely on energy methods that, at a critical juncture, require estimates beyond what the energy class $L^\infty_t(L^2_\sigma) \cap L^2_t(H^1_0)$ provides. The connection formulation offers a different path: by encoding the geometric content of the flow into the curvature of a well-defined connection on the divergence-free bundle, one remains entirely within the Leray–Hopf class throughout. The programme then asks whether incompressibility, combined with the curvature constraints imposed by this connection, obstructs finite-time blowup.

A distinguishing feature of this work is its three-track architecture. Every chapter is developed simultaneously along three parallel lines:

- **Formal verification (Lean 4 and Mathlib).** All definitions, theorem statements, and (where completed) proofs are formalised in the Lean 4 proof assistant, building on the Mathlib library. Lean serves as the final arbiter of correctness: a claim is accepted only when the type-checker confirms it. The current formalisation contains a catalogue of `sorry` obligations corresponding to known theorems whose Lean proofs are deferred; these are tracked explicitly and do not represent gaps in the mathematical argument.
- **Theory (LaTeX and SymPy).** The mathematical narrative is written as a self-contained book, with complete proofs for all results. Symbolic computation in SymPy independently verifies key identities, ex-

ponents, and estimates, providing a computational cross-check on the analytic arguments.

- **Numerics.** Numerical simulations (planned for later chapters) will test the geometric predictions of the theory against direct computation of the Navier–Stokes equations, using spectral and finite-element solvers. The numerical track is not a substitute for proof, but a source of intuition and a check against false conjectures.

Status of the programme. This document records an ongoing research programme. As of the current writing, Chapters 1 and 2 are complete across all three tracks. Chapter 1 (functional analytic foundations) establishes the Sobolev space framework, embedding theorems, and the Helmholtz decomposition. Chapter 2 (Leray–Hopf weak solutions) develops the weak formulation on $\mathbb{R}^3$, constructs solutions via Galerkin approximation with Gram–Schmidt bases on the whole space, proves the energy inequality, and connects to Fefferman's Clay formulation. For both chapters the Lean formalisation compiles, the LaTeX proofs are self-contained, and the SymPy verifications pass. The remaining chapters are in various stages of development. *We do not claim to have resolved the Millennium Problem.* The approach may ultimately succeed or fail; this text is an honest record of the attempt, written so that the mathematical community can evaluate, critique, and build upon the ideas presented here.

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Chapter 1

Functional Analytic Foundations

This chapter collects the functional analytic prerequisites for the Leray–Hopf theory developed in Chapter 2. We build the framework in six steps: distributions and test functions (Section 1.1), weak derivatives and Sobolev spaces (Section 1.2), Sobolev embedding theorems (Section 1.3), the Rellich–Kondrachov compact embedding theorem (Section 1.4), the Poincaré inequality (Section 1.5), and the Helmholtz decomposition together with the Leray projector (Section 1.6). The standard references are Evans [2] and Adams–Fournier [1]; for the Navier–Stokes setting see Temam [7] and Robinson et al. [6].

Throughout, $\Omega \subset \mathbb{R}^n$ denotes an open set, and $n \geq 1$ unless otherwise stated. For the fluid applications in Chapter 2 we set $n = 3$.

1.1 Distributions and test functions

1.1.1 Test functions and the LF topology

Definition 1.1 (Test functions)**.** Let $K \subset \Omega$ be compact. Define

$$\mathcal{D}_K(\Omega) \;=\; \big\{\, \varphi \in C^\infty(\Omega) : \operatorname{supp}\varphi \subseteq K \,\big\}$$

with the Fréchet topology induced by the family of seminorms

$$p_m(\varphi) \;=\; \max_{|\alpha|\leq m} \sup_{x\in\Omega} |\partial^\alpha \varphi(x)|\,, \qquad m \in \mathbb{N}.$$

The *space of test functions* is

$$\mathcal{D}(\Omega) \;=\; C_c^\infty(\Omega) \;=\; \bigcup_{K \Subset \Omega} \mathcal{D}_K(\Omega),$$

equipped with the *LF (strict inductive limit) topology*: a sequence (φ_j) converges in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $K \Subset \Omega$ such that $\operatorname{supp}\varphi_j \subseteq K$ for all j and $p_m(\varphi_j - \varphi) \to 0$ for every $m \in \mathbb{N}$.

Remark 1.2. The LF topology makes $\mathcal{D}(\Omega)$ a complete locally convex space, though it is not metrisable. Sequential continuity nevertheless suffices for the theory of distributions.

1.1.2 Distributions

Definition 1.3 (Distributions)**.** A *distribution* on Ω is a sequentially continuous linear functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$. The space of all distributions is the topological dual

$$\mathcal{D}'(\Omega),$$

equipped with the weak-$*$ topology. We write $\langle T, \varphi \rangle$ for the duality pairing.

Example 1.4 (Locally integrable functions)**.** Every $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f, \varphi \rangle \;=\; \int_\Omega f(x)\, \varphi(x)\, \mathrm{d}x, \qquad \varphi \in \mathcal{D}(\Omega).$$

The assignment $f \mapsto T_f$ is injective, so $L^1_{\mathrm{loc}}(\Omega)$ embeds continuously into $\mathcal{D}'(\Omega)$.

1.1.3 Distributional derivatives

Definition 1.5 (Distributional derivative)**.** For a multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the *distributional (partial) derivative* of $T \in \mathcal{D}'(\Omega)$ is the distribution $\partial^\alpha T$ defined by

$$\langle \partial^\alpha T, \varphi \rangle \;=\; (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle, \qquad \varphi \in \mathcal{D}(\Omega).$$

The sign is chosen so that, for smooth f, the distributional derivative $\partial^\alpha T_f$ coincides with $T_{\partial^\alpha f}$ (integration by parts).

1.2 Weak derivatives and Sobolev spaces

1.2.1 Weak partial derivatives

Definition 1.6 (Weak partial derivative[1])**.** Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}^n$. A function $v \in L^1_{\mathrm{loc}}(\Omega)$ is a *weak α-th partial derivative* of u, written $v = D^\alpha u$, if

$$\int_\Omega v(x)\, \varphi(x)\, \mathrm{d}x \;=\; (-1)^{|\alpha|} \int_\Omega u(x)\, \partial^\alpha \varphi(x)\, \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.1}$$

Theorem 1.7 (Uniqueness of weak derivatives[2])**.** *If $v_1, v_2 \in L^1_{\mathrm{loc}}(\Omega)$ are both weak α-th partial derivatives of u, then $v_1 = v_2$ almost everywhere in Ω.*

[1] Lean 4: `NavierStokes.IsWeakPartialDeriv`
[2] Lean 4: `NavierStokes.weakPartialDeriv_unique`

Proof. Let $w = v_1 - v_2$. By definition,

$$\int_\Omega w(x)\,\varphi(x)\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

This is precisely the content of the *fundamental lemma of the calculus of variations*: if $w \in L^1_{\mathrm{loc}}(\Omega)$ satisfies $\int_\Omega w\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $w = 0$ almost everywhere. Indeed, for any Lebesgue point x_0 of w and any sequence of mollifiers $\varphi_\varepsilon = \rho_\varepsilon(\cdot - x_0)$ supported in Ω, one obtains $w(x_0) = \lim_{\varepsilon\to 0} \int w\varphi_\varepsilon = 0$. □

1.2.2 Sobolev spaces

Definition 1.8 ($W^{1,p}(\Omega)$[3])**.** For $1 \le p \le \infty$, the *Sobolev space* $W^{1,p}(\Omega)$ consists of all $u \in L^p(\Omega)$ whose weak partial derivatives $D_i u$ exist in $L^p(\Omega)$ for each $i = 1, \dots, n$. It is a Banach space under

$$\|u\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|u\|_{L^p}^p + \sum_{i=1}^n \|D_i u\|_{L^p}^p\right)^{1/p}, & 1 \le p < \infty, \\ \|u\|_{L^\infty} + \sum_{i=1}^n \|D_i u\|_{L^\infty}, & p = \infty. \end{cases} \tag{1.2}$$

Definition 1.9 ($H^1(\Omega) = W^{1,2}(\Omega)$[4])**.** The space $H^1(\Omega) = W^{1,2}(\Omega)$ is a *Hilbert space* with inner product

$$\langle u, v\rangle_{H^1} = \int_\Omega u(x)\,v(x)\,\mathrm{d}x + \sum_{i=1}^n \int_\Omega D_i u(x)\,D_i v(x)\,\mathrm{d}x. \tag{1.3}$$

Definition 1.10 ($H^1_0(\Omega)$[5])**.** The space $H^1_0(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$:

$$H^1_0(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}.$$

Functions in $H^1_0(\Omega)$ satisfy a homogeneous Dirichlet boundary condition in the trace sense: $u\big|_{\partial\Omega} = 0$ whenever $\partial\Omega$ is sufficiently regular.

1.3 Sobolev embedding theorems

The embedding theorems quantify how membership in a Sobolev space controls pointwise or L^q regularity. The critical exponent for $W^{1,p}(\mathbb{R}^n)$ is

$$p^* = \frac{np}{n-p}, \qquad 1 \le p < n.$$

[3]Lean 4: `NavierStokes.SobolevW1p`
[4]Lean 4: `NavierStokes.SobolevH1`
[5]Lean 4: `NavierStokes.SobolevH1Zero`

1.3.1 Subcritical (Gagliardo–Nirenberg–Sobolev) embedding

Theorem 1.11 (Subcritical Sobolev embedding[6]). *Let $1 \le p < n$. There exists a constant $C = C(n,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \;\le\; C\,\|\nabla u\|_{L^p(\mathbb{R}^n)}\,. \tag{1.4}$$

Consequently, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ continuously.

Proof. We first establish the case $p = 1$. For $u \in C_c^\infty(\mathbb{R}^n)$ and each $i = 1, \ldots, n$, the fundamental theorem of calculus gives

$$|u(x)| \;\le\; \int_{-\infty}^{\infty} |\partial_i u(x_1, \ldots, t, \ldots, x_n)|\, \mathrm{d}t,$$

so

$$|u(x)|^{n/(n-1)} \;\le\; \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |\partial_i u|\, \mathrm{d}x_i\Big)^{1/(n-1)}.$$

Integrating over $\mathbb{R}^n$ and applying the generalised Hölder inequality (with $n-1$ factors, each in L^1 of the remaining variable) yields $\|u\|_{L^{n/(n-1)}} \le \prod_i \|\partial_i u\|_{L^1}^{1/n} \le \|\nabla u\|_{L^1}$, which is the Gagliardo–Nirenberg inequality (the case $p = 1$, $p^* = n/(n-1)$).

For $1 < p < n$, set $\gamma = p(n-1)/(n-p) > 1$ and apply the $p = 1$ result to $v = |u|^\gamma$. Since $|\nabla v| = \gamma\,|u|^{\gamma-1}\,|\nabla u|$, Hölder's inequality with exponents p and $p/(p-1)$ gives $\|\nabla v\|_{L^1} \le \gamma\,\|u\|_{L^{\gamma p/(p-1)}}^{\gamma-1}\,\|\nabla u\|_{L^p}$. The exponents are arranged so that $\gamma n/(n-1) = \gamma p/(p-1) = p^*$, and rearranging produces the desired inequality $\|u\|_{L^{p^*}} \le C(n,p)\,\|\nabla u\|_{L^p}$.

The extension from $C_c^\infty(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ follows by density: approximate $u \in W^{1,p}$ by a sequence $(u_k) \subset C_c^\infty$ in the $W^{1,p}$ norm, apply the inequality to each u_k, and pass to the limit using completeness of L^{p^*}. For bounded domains with Lipschitz boundary, the Stein extension operator provides a bounded map $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$, reducing to the whole-space case. See [2], Section 5.6. □

Remark 1.12 (The case $n = 3$, $p = 2$). For the Navier–Stokes problem with $n = 3$ and $p = 2$, one has

$$p^* = \frac{3 \cdot 2}{3 - 2} = 6,$$

so $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ with the explicit bound

$$\|u\|_{L^6(\mathbb{R}^3)} \;\le\; C\,\|\nabla u\|_{L^2(\mathbb{R}^3)}\,. \tag{1.5}$$

This embedding is used repeatedly in the energy estimates of Chapter 2.

[6] Lean 4: `NavierStokes.sobolev_embedding_subcritical`

1.3.2 Supercritical (Morrey) embedding

Theorem 1.13 (Morrey embedding[7]). *Let $p > n$ and set $\alpha = 1 - n/p \in (0,1)$. Then there exists a constant $C = C(n,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$,*

$$|u(x) - u(y)| \;\le\; C\,|x-y|^{\alpha}\,\|u\|_{W^{1,p}(\mathbb{R}^n)}, \qquad a.e.\ x, y \in \mathbb{R}^n. \tag{1.6}$$

In particular, $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$ continuously.

Proof. By density it suffices to prove the estimate for $u \in C_c^\infty(\mathbb{R}^n)$. Fix $x, y \in \mathbb{R}^n$ and set $r = |x-y|$. For any $z \in B_r(x)$, the fundamental theorem of calculus along the segment from z to x gives

$$u(x) - u(z) \;=\; \int_0^1 \nabla u\big(z + t(x-z)\big)\cdot(x-z)\,\mathrm{d}t.$$

Averaging over $z \in B_r(x)$ and denoting $u_{B_r} =_{B_r(x)} u\,\mathrm{d}z$, we obtain

$$\begin{aligned} |u(x) - u_{B_r}| \;&\le\; {}_{B_r(x)}\int_0^1 \big|\nabla u\big(z + t(x-z)\big)\big|\;|x-z|\;\mathrm{d}t\,\mathrm{d}z \\ &\le\; \frac{1}{|B_r|}\int_{B_r(x)} \frac{|\nabla u(w)|}{|x-w|^{n-1}}\,\mathrm{d}w, \end{aligned}$$

where the last inequality follows from the change of variables $w = z + t(x-z)$ and the bound $|x-z| \le r$. Applying Hölder's inequality with exponents p and $p' = p/(p-1)$ to the integral on the right yields

$$|u(x) - u_{B_r}| \;\le\; \frac{1}{|B_r|}\,\|\nabla u\|_{L^p(B_r)}\left(\int_{B_r(x)} |x-w|^{-(n-1)p'}\;\mathrm{d}w\right)^{1/p'}.$$

Since $p > n$, one has $(n-1)p' < n$, so the integral converges and equals $C(n,p)\,r^{n-(n-1)p'}$. Combining this with $|B_r| = \omega_n\,r^n$ gives $|u(x) - u_{B_r}| \le C\,r^{1-n/p}\,\|\nabla u\|_{L^p(B_r)}$. The same bound holds with x replaced by y (since $B_r(y) \subset B_{2r}(x)$), so the triangle inequality yields

$$|u(x) - u(y)| \;\le\; |u(x) - u_{B_r}| + |u(y) - u_{B_r}| \;\le\; C\,|x-y|^{1-n/p}\,\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Setting $\alpha = 1 - n/p$ and absorbing the full $W^{1,p}$ norm via the Sobolev inequality completes the proof. The extension from C_c^∞ to $W^{1,p}$ follows by approximation: the Hölder estimate passes to the limit, and the representative obtained is continuous after modification on a set of measure zero. ∎

[7]Lean 4: `NavierStokes.sobolev_embedding_supercritical`

1.4 Compact embeddings: Rellich–Kondrachov

The Rellich–Kondrachov theorem upgrades the continuous Sobolev embedding to a *compact* one, at the cost of passing to a lower-order space. Compactness is essential for the convergence of Galerkin approximations in Chapter 2.

Theorem 1.14 (Rellich–Kondrachov[8]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then the inclusion*

$$H^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact*: every bounded sequence in $H^1(\Omega)$ has a subsequence that converges strongly in $L^2(\Omega)$. More generally, for $1 \le p < n$ and $1 \le q < p^*$, the inclusion $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

Proof. Let (u_j) be bounded in $H^1(\Omega)$ with $\|u_j\|_{H^1} \le M$ for all j. We establish precompactness in $L^2(\Omega)$ by a mollification and diagonal argument.

Let $\tilde{u}_j$ denote the zero extension of u_j to $\mathbb{R}^n$, and let ρ_ε be a standard mollifier with $\operatorname{supp}\rho_\varepsilon \subset B_\varepsilon(0)$. Define $u_j^\varepsilon = \rho_\varepsilon * \tilde{u}_j$. By the standard mollification estimate (a consequence of Jensen's inequality and the characterisation of H^1 via difference quotients),

$$\left\|u_j^\varepsilon - u_j\right\|_{L^2(\Omega)} \;\le\; \varepsilon\,\|\nabla u_j\|_{L^2(\Omega)} \;\le\; M\varepsilon$$

for every j and every $\varepsilon > 0$.

For each fixed ε, the mollified family $(u_j^\varepsilon)_j$ is uniformly bounded and equicontinuous on $\overline{\Omega}$. Indeed, Young's convolution inequality gives

$$\left\|u_j^\varepsilon\right\|_{L^\infty} \;\le\; \|\rho_\varepsilon\|_{L^2}\,\|u_j\|_{L^2} \;\le\; C(\varepsilon)\,M,$$

and similarly $\left\|\nabla u_j^\varepsilon\right\|_{L^\infty} \le \|\nabla\rho_\varepsilon\|_{L^2}\,\|u_j\|_{L^2} \le C'(\varepsilon)\,M$, so

$$\left|u_j^\varepsilon(x) - u_j^\varepsilon(y)\right| \;\le\; C'(\varepsilon)\,M\,|x-y|$$

for all $x, y \in \overline{\Omega}$. By the Arzelà–Ascoli theorem, $(u_j^\varepsilon)_j$ is precompact in $C(\overline{\Omega})$, hence also in $L^2(\Omega)$ (since Ω is bounded).

Now apply a diagonal argument. Choose $\varepsilon_k = 1/k$ and extract successive subsequences: from (u_j) extract $(u_j^{(1)})$ such that $(u_j^{(1),\,1/1})_j$ converges in L^2; from $(u_j^{(1)})$ extract $(u_j^{(2)})$ such that $(u_j^{(2),\,1/2})_j$ converges; and so on. The diagonal sequence $v_k = u_k^{(k)}$ then satisfies the following: for any $\delta > 0$, choose ε so that $M\varepsilon < \delta/3$, then choose N so that $\left\|v_j^\varepsilon - v_k^\varepsilon\right\|_{L^2} < \delta/3$ for all $j, k \ge N$. The triangle inequality

$$\|v_j - v_k\|_{L^2} \;\le\; \left\|v_j - v_j^\varepsilon\right\|_{L^2} + \left\|v_j^\varepsilon - v_k^\varepsilon\right\|_{L^2} + \|v_k^\varepsilon - v_k\|_{L^2} \;<\; \delta$$

[8] Lean 4: `NavierStokes.rellich_kondrachov`

shows that (v_k) is Cauchy in $L^2(\Omega)$, and completeness gives strong convergence. The general case $W^{1,p} \hookrightarrow L^q$ for $q < p^*$ follows by interpolation between the L^{p^*} bound (from Theorem 1.11) and the L^2 compactness just established. □

Remark 1.15 (Role in the Galerkin method). In Chapter 2, the Galerkin approximations (u_m) are constructed as bounded sequences in the energy class $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. The Rellich–Kondrachov theorem (together with the Aubin–Lions lemma) provides the strong L^2 convergence needed to pass nonlinear terms to the limit.

1.5 The Poincaré inequality

Theorem 1.16 (Poincaré inequality[9])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. There exists a constant $C_\mathrm{P} = C_\mathrm{P}(\Omega) > 0$ such that*

$$\|u\|_{L^2(\Omega)} \;\le\; C_\mathrm{P}\,\|\nabla u\|_{L^2(\Omega)} \qquad \text{for all } u \in H^1_0(\Omega). \tag{1.7}$$

Proof. Suppose for contradiction that no such constant exists. Then there is a sequence $(u_j) \subset H^1_0(\Omega)$ with

$$\|u_j\|_{L^2} = 1 \qquad \text{and} \qquad \|\nabla u_j\|_{L^2} \to 0.$$

The sequence is bounded in $H^1(\Omega)$ (since $\|u_j\|^2_{H^1} = \|u_j\|^2_{L^2} + \|\nabla u_j\|^2_{L^2} \le 1 + 1 = 2$), so by Rellich–Kondrachov (Theorem 1.14), there exists a subsequence (still denoted (u_j)) with $u_j \to u$ strongly in $L^2(\Omega)$. In particular $\|u\|_{L^2} = 1$.

On the other hand, $\nabla u_j \to 0$ in $L^2(\Omega)$, and weak lower semicontinuity of the norm gives $\nabla u = 0$ almost everywhere. Since Ω is connected, u is constant almost everywhere; since $u \in H^1_0(\Omega)$, the trace of u on $\partial\Omega$ is zero, so $u \equiv 0$. This contradicts $\|u\|_{L^2} = 1$. □

Corollary 1.17 (Equivalent norm on $H^1_0(\Omega)$)**.** *On $H^1_0(\Omega)$, the seminorm $u \mapsto \|\nabla u\|_{L^2(\Omega)}$ is a norm equivalent to $\|\cdot\|_{H^1(\Omega)}$. Concretely,*

$$\|\nabla u\|_{L^2} \;\le\; \|u\|_{H^1} \;\le\; \sqrt{1 + C_\mathrm{P}^2}\,\|\nabla u\|_{L^2}\,. \tag{1.8}$$

In particular, $(H^1_0(\Omega), \langle \nabla\cdot, \nabla\cdot\rangle_{L^2})$ is a Hilbert space.

1.6 Divergence-free fields and the Helmholtz decomposition

The incompressibility constraint $\operatorname{div} u = 0$ is encoded variationally via the distributional divergence-free condition.

[9] Lean 4: `NavierStokes.poincare_inequality`

1.6.1 Distributional divergence-free fields

Definition 1.18 (Divergence-free fields). [10] A vector field $u \in L^2(\Omega; \mathbb{R}^n)$ is *distributionally divergence-free* if

$$\int_\Omega u \cdot \nabla \varphi \, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.9}$$

Equivalently, $\langle \operatorname{div} T_u, \varphi \rangle = 0$ for all $\varphi \in C_c^\infty(\Omega)$, i.e., $\operatorname{div} u = 0$ in $\mathcal{D}'(\Omega)$.

1.6.2 The space $L^2_\sigma(\Omega)$

Definition 1.19 ($L^2_\sigma(\Omega)$[11]). Define

$$L^2_\sigma(\Omega) \;=\; \overline{\left\{ u \in C_c^\infty(\Omega; \mathbb{R}^n) : \operatorname{div} u = 0 \right\}}^{\|\cdot\|_{L^2}}.$$

This is the L^2-closure of smooth, compactly supported, divergence-free vector fields.

Proposition 1.20 (Closedness of L^2_σ). [12] *The space $L^2_\sigma(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^n)$, hence itself a Hilbert space.*

Proof. $L^2_\sigma(\Omega)$ is defined as a closure, so it is closed by construction. Alternatively: the distributional divergence operator $\operatorname{div} : L^2(\Omega; \mathbb{R}^n) \to H^{-1}(\Omega)$ is continuous, so its kernel $\{u : \operatorname{div} u = 0\}$ is closed. One verifies that this kernel equals the closure of the smooth divergence-free fields when Ω has the appropriate topology (e.g., $\Omega = \mathbb{R}^n$ or Ω simply connected with Lipschitz boundary). □

1.6.3 Helmholtz decomposition

Theorem 1.21 (Helmholtz decomposition[13]). *Let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded open set with C^1 boundary, or $\Omega = \mathbb{R}^n$. Then $L^2(\Omega; \mathbb{R}^n)$ admits the orthogonal decomposition*

$$L^2(\Omega; \mathbb{R}^n) \;=\; L^2_\sigma(\Omega) \;\oplus_\perp\; G(\Omega), \tag{1.10}$$

where

$$G(\Omega) \;=\; \left\{ \nabla p : p \in H^1(\Omega) \right\}.$$

Every $u \in L^2(\Omega; \mathbb{R}^n)$ decomposes uniquely as $u = v + \nabla p$ with $v \in L^2_\sigma(\Omega)$ and $\nabla p \in G(\Omega)$.

[10] Lean 4: `NavierStokes.IsDistribDivFree`.
[11] Lean 4: `NavierStokes.L2sigma`
[12] Lean 4: `NavierStokes.l2sigma_isClosed`.
[13] Lean 4: `NavierStokes.helmholtz_decomposition`

Proof. We first establish orthogonality of $L^2_\sigma(\Omega)$ and $G(\Omega)$. Let $v \in L^2_\sigma(\Omega)$ and $\nabla p \in G(\Omega)$. Since v is the L^2-limit of smooth divergence-free fields $v_k \in C_c^\infty(\Omega;\mathbb{R}^n)$, integration by parts gives $\langle v_k, \nabla p\rangle_{L^2} = -\int_\Omega (\operatorname{div} v_k)\, p \,\mathrm{d}x = 0$ for each k, and passing to the limit yields $\langle v, \nabla p\rangle_{L^2} = 0$. Hence $L^2_\sigma(\Omega) \perp G(\Omega)$.

To construct the decomposition, let $u \in L^2(\Omega;\mathbb{R}^n)$ be given. Consider the bilinear form $a : H^1(\Omega)/\mathbb{R} \times H^1(\Omega)/\mathbb{R} \to \mathbb{R}$ defined by $a(p,q) = \int_\Omega \nabla p \cdot \nabla q \,\mathrm{d}x$ and the linear functional $\ell(q) = \int_\Omega u \cdot \nabla q \,\mathrm{d}x$. The form a is continuous and coercive on $H^1(\Omega)/\mathbb{R}$ (coercivity follows from the Poincaré–Wirtinger inequality for functions with zero mean), and ℓ is continuous since $|\ell(q)| \le \|u\|_{L^2} \|\nabla q\|_{L^2}$. By the Lax–Milgram theorem, there exists a unique $p \in H^1(\Omega)/\mathbb{R}$ such that $a(p,q) = \ell(q)$ for all q, which is the weak formulation of the Neumann problem $-\Delta p = -\operatorname{div} u$ in Ω with $\partial p/\partial \nu = u \cdot \nu$ on $\partial\Omega$.

Set $v = u - \nabla p$. For any $\varphi \in C_c^\infty(\Omega)$, the weak formulation gives $\int_\Omega v \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega u \cdot \nabla \varphi \,\mathrm{d}x - \int_\Omega \nabla p \cdot \nabla \varphi \,\mathrm{d}x = \ell(\varphi) - a(p,\varphi) = 0$, so $\operatorname{div} v = 0$ in $\mathcal{D}'(\Omega)$. To verify that $v \in L^2_\sigma(\Omega)$ (the L^2-closure of smooth divergence-free fields, not merely the distributional kernel), one approximates v by mollification: $v_\varepsilon = \rho_\varepsilon * \tilde{v}$ satisfies $\operatorname{div} v_\varepsilon = \rho_\varepsilon * \operatorname{div} \tilde{v} = 0$ on any compact subset of Ω for ε sufficiently small, and $v_\varepsilon \to v$ in L^2. Multiplying by a smooth cutoff confirms $v \in L^2_\sigma(\Omega)$.

Uniqueness of the decomposition follows from the orthogonality already established: if $u = v_1 + \nabla p_1 = v_2 + \nabla p_2$ with $v_i \in L^2_\sigma(\Omega)$ and $\nabla p_i \in G(\Omega)$, then $v_1 - v_2 = \nabla(p_2 - p_1) \in L^2_\sigma(\Omega) \cap G(\Omega)$, and orthogonality forces $v_1 - v_2 = 0$. On simply connected domains (or $\mathbb{R}^n$), the de Rham theorem ensures that every curl-free L^2 vector field is a gradient, so $G(\Omega)$ coincides with the orthogonal complement $L^2_\sigma(\Omega)^\perp$, completing the proof. □

1.6.4 The Leray projector

Definition 1.22 (Leray projector[14])**.** The *Leray (or Helmholtz–Leray) projector* is the bounded linear operator

$$\mathbb{P} : L^2(\Omega;\mathbb{R}^n) \longrightarrow L^2_\sigma(\Omega)$$

defined by $\mathbb{P}u = v$, where $u = v + \nabla p$ is the Helmholtz decomposition of u. Equivalently, $\mathbb{P}$ is the orthogonal projection onto $L^2_\sigma(\Omega)$.

Proposition 1.23 (Properties of the Leray projector)**.** *The operator $\mathbb{P}$ satisfies:*

(i) Boundedness. $\|\mathbb{P}u\|_{L^2} \le \|u\|_{L^2}$ *for all* $u \in L^2(\Omega;\mathbb{R}^n)$.

(ii) Idempotency. $\mathbb{P}^2 = \mathbb{P}$.

(iii) Self-adjointness. $\langle \mathbb{P}u, v\rangle_{L^2} = \langle u, \mathbb{P}v\rangle_{L^2}$ *for all* $u, v \in L^2(\Omega;\mathbb{R}^n)$.

[14] Lean 4: `NavierStokes.lerayProjector`

(iv) Projection onto divergence-free fields. $\operatorname{div}(\mathbb{P}u) = 0$ *for all* u.

Proof. Properties (i)–(iii) are standard consequences of orthogonal projection in Hilbert space. Property (iv) holds by definition of $L^2_\sigma(\Omega)$. □

Remark 1.24 (Role in the Navier–Stokes equations). Applying $\mathbb{P}$ to the momentum equation $\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f$ eliminates the pressure gradient ∇p (which lies in $G(\Omega)$) and yields the projected equation

$$\partial_t u + \mathbb{P}\big[(u \cdot \nabla)u\big] - \nu\Delta u = \mathbb{P}f. \tag{1.11}$$

This is the form amenable to semigroup theory and the Galerkin method of Chapter 2.

Summary. The tools assembled in this chapter form the complete functional analytic toolkit for the weak theory of the Navier–Stokes equations. The Sobolev space $H^1_0(\Omega)$ provides the energy space for the velocity field; the subcritical embedding $H^1 \hookrightarrow L^6$ (for $n = 3$) controls the nonlinear term; Rellich–Kondrachov furnishes the compactness needed in Galerkin limits; the Poincaré inequality ensures coercivity; and the Helmholtz decomposition decouples velocity from pressure via the Leray projector $\mathbb{P}$. Chapter 2 combines these ingredients to construct Leray–Hopf weak solutions.

Chapter 2

Leray–Hopf Weak Solutions

In 1934, Jean Leray [5] proved that the three-dimensional incompressible Navier–Stokes equations admit global-in-time weak solutions for arbitrary L^2 initial data, and that these solutions satisfy a fundamental energy inequality. Two decades later, Eberhard Hopf [4] extended the construction to bounded domains. The solutions produced by their method, now called *Leray–Hopf weak solutions*, remain the only class of weak solutions known to exist globally in three dimensions. Whether every Leray–Hopf weak solution is in fact smooth is the central open question of the Clay Millennium Problem [3].

This chapter constructs Leray–Hopf weak solutions on $\mathbb{R}^3$ in full detail. We proceed from the strong formulation (Section 2.1) and its weak counterpart (Section 2.2) to the existence proof via the Galerkin method. Section 2.6 states the Clay Millennium Problem.

Throughout this chapter, the spatial dimension is $n = 3$. We use freely the functional analytic tools from Chapter 1: Sobolev spaces, embedding theorems, the Rellich–Kondrachov theorem, the Poincaré inequality, and the Helmholtz decomposition with the Leray projector.

2.1 The incompressible Navier–Stokes equations

2.1.1 Strong formulation

The motion of an incompressible viscous fluid with unit density occupying $\mathbb{R}^3$ is governed by the Navier–Stokes system: find a velocity field $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ and a pressure field $p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ satisfying

$$\partial_t u + (u \cdot \nabla)u \;=\; \nu\Delta u - \nabla p + f, \qquad x \in \mathbb{R}^3,\ t > 0, \tag{2.1}$$

$$\operatorname{div} u \;=\; 0, \qquad x \in \mathbb{R}^3,\ t > 0, \tag{2.2}$$

$$u(x, 0) \;=\; u_0(x), \qquad x \in \mathbb{R}^3, \tag{2.3}$$

together with the decay condition $u(x, t) \to 0$ as $|x| \to \infty$.

Remark 2.1 (Physical meaning of each term). In the momentum equation (2.1):

(i) $\partial_t u$ is the local (Eulerian) acceleration of the fluid;

(ii) $(u \cdot \nabla)u = \sum_{j=1}^{3} u_j \, \partial_j u$ is the *convective* (or *advective*) nonlinearity, encoding the transport of momentum by the flow itself;

(iii) $\nu \Delta u$ is the viscous diffusion term, with $\nu > 0$ the kinematic viscosity;

(iv) ∇p is the pressure gradient, which enforces the incompressibility constraint (2.2) (the pressure acts as a Lagrange multiplier);

(v) $f = f(x,t)$ is an external body force.

The constraint $\operatorname{div} u = 0$ expresses conservation of mass for a fluid of constant density: the velocity field is volume-preserving.

2.1.2 The convective term and its symmetry

The convective term $(u \cdot \nabla)u$ is the source of the mathematical difficulty. Written in component form, it reads

$$\big[(u \cdot \nabla)u\big]_i \;=\; \sum_{j=1}^{3} u_j \, \partial_j u_i, \qquad i = 1, 2, 3.$$

When u is divergence-free, this can be rewritten in conservative form:

$$(u \cdot \nabla)u \;=\; \operatorname{div}(u \otimes u), \tag{2.4}$$

since $\operatorname{div}(u \otimes u)_i = \sum_j \partial_j (u_j u_i) = \sum_j u_j \, \partial_j u_i + u_i \underbrace{\operatorname{div} u}_{=0}$. The conservative form (2.4) is better suited to the weak formulation because it requires only one derivative, distributed via integration by parts.

2.1.3 Dimensionless form and the Reynolds number

Introduce characteristic scales: a length L, a velocity U, and a time $T = L/U$. The dimensionless variables $x' = x/L$, $t' = t/T$, $u' = u/U$, $p' = p/(U^2)$, $f' = fL/U^2$ transform the Navier–Stokes equations into

$$\partial_{t'} u' + (u' \cdot \nabla')u' \;=\; \frac{1}{\mathrm{Re}} \, \Delta' u' - \nabla' p' + f', \tag{2.5}$$

where

$$\mathrm{Re} \;=\; \frac{UL}{\nu} \tag{2.6}$$

is the *Reynolds number*. The Reynolds number measures the ratio of inertial forces (the convective term) to viscous forces (the diffusion term). At low

Reynolds number ($\mathrm{Re} \ll 1$), viscosity dominates and the flow is laminar; at high Reynolds number ($\mathrm{Re} \gg 1$), the convective nonlinearity dominates and the flow may become turbulent.

Remark 2.2. The regularity problem is independent of the Reynolds number in the following sense: the Clay Millennium Problem asks whether smooth solutions remain smooth for *all* $\nu > 0$, not merely for small Reynolds numbers. The difficulty lies entirely in the three-dimensional geometry of the nonlinear term.

2.2 Weak formulation

The strong formulation (2.1)–(2.3) requires u to be twice differentiable in space and once in time. To construct solutions with less regularity, we pass to a weak formulation by testing against smooth, compactly supported, divergence-free vector fields.

2.2.1 Derivation of the weak form

Let $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0,T); \mathbb{R}^3)$ with $\operatorname{div} \varphi = 0$. Multiply the momentum equation (2.1) by φ, integrate over $\mathbb{R}^3 \times (0,T)$, and integrate by parts.

The time derivative term gives

$$\int_0^T \int_{\mathbb{R}^3} \partial_t u \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \;=\; -\int_0^T \int_{\mathbb{R}^3} u \cdot \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t \;-\; \int_{\mathbb{R}^3} u_0 \cdot \varphi(x,0) \, \mathrm{d}x,$$

using $\varphi(\cdot, T) = 0$ and $u(\cdot, 0) = u_0$.

The viscous term, after integration by parts, becomes

$$\int_0^T \int_{\mathbb{R}^3} \nu \Delta u \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \;=\; -\nu \int_0^T \int_{\mathbb{R}^3} \nabla u : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t,$$

where $\nabla u : \nabla \varphi = \sum_{i,j} \partial_j u_i \, \partial_j \varphi_i$.

The pressure term vanishes: $\int_{\mathbb{R}^3} \nabla p \cdot \varphi \, \mathrm{d}x = -\int_{\mathbb{R}^3} p \operatorname{div} \varphi \, \mathrm{d}x = 0$, since $\operatorname{div} \varphi = 0$. This is the key advantage of testing against divergence-free functions: the pressure is eliminated entirely.

The convective term, using the conservative form (2.4), integrates by parts to

$$\int_0^T \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \;=\; -\int_0^T \int_{\mathbb{R}^3} (u \otimes u) : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t,$$

where $(u \otimes u) : \nabla \varphi = \sum_{i,j} u_i u_j \, \partial_j \varphi_i$.

Collecting terms yields the weak formulation.

2.2.2 Definition of weak solution

Definition 2.3 (Weak solution of Navier–Stokes[1]). Let $u_0 \in L^2_\sigma(\mathbb{R}^3)$, $f \in L^2(0,T;H^{-1}(\mathbb{R}^3))$, and $0 < T \le \infty$. A vector field

$$u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1_0(\mathbb{R}^3))$$

with $\operatorname{div} u = 0$ in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ is a *weak solution* of the Navier–Stokes equations if for every $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0,T);\mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$,

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^3} u\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t \;-\; \int_{\mathbb{R}^3} u_0\cdot\varphi(x,0)\,\mathrm{d}x \\ &\quad+\int_0^T\int_{\mathbb{R}^3}(u\otimes u):\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \;+\;\nu\int_0^T\int_{\mathbb{R}^3}\nabla u:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad=\int_0^T \langle f,\varphi\rangle_{H^{-1}\times H^1_0}\,\mathrm{d}t. \end{aligned} \tag{2.7}$$

Remark 2.4 (Well-definedness of the nonlinear term). The nonlinear term $\int(u\otimes u):\nabla\varphi$ is well-defined when $u \in L^2(0,T;H^1_0(\mathbb{R}^3))$. Indeed, by the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ (Theorem 1.11 with $n=3$, $p=2$), $u(\cdot,t) \in L^6(\mathbb{R}^3)$ for a.e. t, so $u\otimes u \in L^3(\mathbb{R}^3)$ and $\nabla\varphi \in L^{3/2}(\mathbb{R}^3)$ (since φ is smooth and compactly supported), whence the integral is finite by Hölder's inequality. More precisely, $u\otimes u \in L^1(0,T;L^3(\mathbb{R}^3))$ and the pairing is controlled by the energy class norms.

Remark 2.5 (Pressure recovery). The pressure p does not appear in the weak formulation (2.7) because we test against divergence-free functions. Once a weak solution u is obtained, the pressure can be recovered (up to an additive function of time) by solving the Poisson equation $-\Delta p = \operatorname{div}\big[(u\cdot\nabla)u\big]$ in the sense of distributions. On $\mathbb{R}^3$, this is accomplished via the Calderón–Zygmund theory of singular integrals, yielding $\nabla p \in L^{3/2}(\mathbb{R}^3)$ when $u \in L^6(\mathbb{R}^3)$.

2.3 The Galerkin method

The Galerkin method reduces the Navier–Stokes initial-value problem to a sequence of finite-dimensional ordinary differential equations whose solutions approximate the desired weak solution. On a bounded domain $\Omega \subset \mathbb{R}^3$, one typically projects onto eigenfunctions of the Stokes operator, exploiting its compact resolvent and discrete spectrum. On $\mathbb{R}^3$ the Stokes operator has continuous spectrum, so that approach is unavailable. We instead construct an approximation basis directly from the separability of $L^2_\sigma(\mathbb{R}^3)$, then carry out the Galerkin procedure, derive uniform a priori estimates, and pass to the limit using a localisation argument that replaces the global compactness afforded by Rellich–Kondrachov on bounded domains.

[1] Lean 4: `NavierStokes.WeakNSSolution`

2.3.1 Approximation basis on $\mathbb{R}^3$

The space $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ is dense in $L^2_\sigma(\mathbb{R}^3)$ (a consequence of the Helmholtz decomposition, Theorem 1.21). Since $L^2_\sigma(\mathbb{R}^3)$ is a separable Hilbert space, this dense subspace contains a countable dense subset $\{\varphi_k\}_{k=1}^\infty$. Applying the Gram–Schmidt procedure to $\{\varphi_k\}_{k=1}^\infty$ in the $L^2(\mathbb{R}^3)$ inner product produces an orthonormal sequence $\{w_k\}_{k=1}^\infty$ in $L^2_\sigma(\mathbb{R}^3)$ satisfying:

(i) each w_k belongs to $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and is divergence-free;

(ii) $\langle w_j, w_k\rangle_{L^2} = \delta_{jk}$ for all $j, k \in \mathbb{N}$;

(iii) $\overline{\operatorname{span}\{w_k : k \in \mathbb{N}\}} = L^2_\sigma(\mathbb{R}^3)$.

Property (i) holds because each w_k is a finite linear combination of the φ_j, all of which are smooth, compactly supported, and divergence-free; the linear combination inherits each property. Property (iii) follows because $\operatorname{span}\{w_k\}$ contains $\operatorname{span}\{\varphi_k\}$ (each φ_k can be recovered from $w_1, \ldots, w_k$ by inverting the Gram–Schmidt triangular system), and the latter is dense in $L^2_\sigma(\mathbb{R}^3)$ by construction.

Remark 2.6 (Bounded-domain alternative). On a bounded domain Ω with smooth boundary, the Stokes operator $A = -\mathbb{P}\Delta$ has compact resolvent[2] and therefore possesses a complete orthonormal system of eigenfunctions in $L^2_\sigma(\Omega)$ with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$. This eigenbasis is the standard choice for Galerkin approximation in the bounded setting; see Temam [7], Chapter III. On $\mathbb{R}^3$ the Stokes operator is still well-defined as a self-adjoint operator on $L^2_\sigma(\mathbb{R}^3)$, but its spectrum is purely continuous, so eigenfunctions in the classical sense do not exist. The Gram–Schmidt construction above circumvents this difficulty entirely.

2.3.2 Finite-dimensional approximation

For each $m \in \mathbb{N}$, define the Galerkin subspace

$$V_m \;=\; \operatorname{span}\{w_1, \ldots, w_m\} \;\subset\; L^2_\sigma(\mathbb{R}^3), \tag{2.8}$$

and let $P_m \colon L^2_\sigma(\mathbb{R}^3) \to V_m$ denote the orthogonal projection. We seek an approximate solution of the form

$$u_m(t) \;=\; \sum_{k=1}^{m} c_k^{(m)}(t)\, w_k, \tag{2.9}$$

where the coefficient functions $c_k^{(m)} \colon [0, T_m) \to \mathbb{R}$ are to be determined. The trilinear form

$$b(u, v, w) \;=\; \int_{\mathbb{R}^3} (u \cdot \nabla) v \cdot w \,\mathrm{d}x \tag{2.10}$$

[2] Lean 4: `NavierStokes.StokesOperator`.

is well-defined whenever the integrand is summable. We require u_m to satisfy the projected equations: for each $k = 1, \dots, m$,

$$\begin{aligned} \langle \partial_t u_m, w_k \rangle_{L^2} \;+\; \nu\, \langle \nabla u_m, \nabla w_k \rangle_{L^2} \;+\; b(u_m, u_m, w_k) \;&=\; \langle f, w_k \rangle_{L^2}\,, \\ u_m(0) \;&=\; P_m u_0. \end{aligned} \tag{2.11}$$

The antisymmetry of the trilinear form is the key structural property underpinning all energy estimates.

Lemma 2.7 (Antisymmetry of the trilinear form)**.** *Let $u \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\operatorname{div} u = 0$, and let $v \in H^1(\mathbb{R}^3;\mathbb{R}^3)$. Then*

$$b(u, v, v) \;=\; 0. \tag{2.12}$$

More generally, $b(u, v, w) = -b(u, w, v)$ for all $v, w \in H^1(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. Write out the trilinear form componentwise:

$$b(u, v, v) \;=\; \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} u_j\, \partial_j v_i\, v_i \,\mathrm{d}x \;=\; \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_j\, \partial_j\, |v|^2 \,\mathrm{d}x.$$

Since u is smooth and compactly supported, integration by parts yields

$$\frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_j\, \partial_j\, |v|^2 \,\mathrm{d}x \;=\; -\frac{1}{2} \int_{\mathbb{R}^3} (\operatorname{div} u)\, |v|^2 \,\mathrm{d}x \;=\; 0,$$

where the boundary terms vanish because u is compactly supported and $v \in H^1(\mathbb{R}^3;\mathbb{R}^3)$, and the final equality uses $\operatorname{div} u = 0$. The general antisymmetry identity $b(u, v, w) = -b(u, w, v)$ follows by polarisation: apply the vanishing result to $v + w$ and $v - w$, then subtract. □

Substituting the ansatz (2.9) into (2.11) and using the orthonormality $\langle w_j, w_k \rangle_{L^2} = \delta_{jk}$ yields an ODE system for the coefficient vector

$$c^{(m)}(t) \;=\; \big(c_1^{(m)}(t), \dots, c_m^{(m)}(t)\big)^\top \in \mathbb{R}^m :$$

$$\dot{c}_k^{(m)} \;=\; -\nu \sum_{j=1}^{m} A_{kj}\, c_j^{(m)} \;-\; \sum_{i,j=1}^{m} B_{kij}\, c_i^{(m)}\, c_j^{(m)} \;+\; f_k(t), \quad k = 1, \dots, m,$$

where $A_{kj} = \langle \nabla w_j, \nabla w_k \rangle_{L^2}$, $B_{kij} = b(w_i, w_j, w_k)$, and $f_k(t) = \langle f(t), w_k \rangle_{L^2}$. The right-hand side is locally Lipschitz in $c^{(m)}$ (it is quadratic) and measurable in t, so the Cauchy–Lipschitz theorem applies.

Proposition 2.8 (Local existence of Galerkin approximations)**.** *For each $m \in \mathbb{N}$, there exists $T_m > 0$ and a unique absolutely continuous function $c^{(m)} \colon [0, T_m) \to \mathbb{R}^m$ satisfying the ODE system above with initial data $c_k^{(m)}(0) = \langle u_0, w_k \rangle_{L^2}$. The corresponding $u_m \in C([0, T_m); V_m)$ satisfies* (2.11) *for a.e. $t \in (0, T_m)$.*

Proof. The coefficients A_{kj} are constants (each w_k is smooth and compactly supported, so $\langle \nabla w_j, \nabla w_k \rangle_{L^2} < \infty$). Likewise, the B_{kij} are finite constants because $w_i, w_j, w_k \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$. The forcing terms $f_k(t) = \langle f(t), w_k \rangle_{L^2}$ are in $L^2(0,T)$ whenever $f \in L^2(0,T; H^{-1}(\mathbb{R}^3))$, and in particular are locally integrable. The right-hand side of the ODE is therefore a locally Lipschitz function of $c^{(m)}$ plus a locally integrable function of t, so the Cauchy–Lipschitz (Picard–Lindelöf) theorem gives a unique maximal solution on some interval $[0, T_m)$. □

2.3.3 A priori estimates

The following energy estimate is the heart of the Galerkin method. It provides bounds that are uniform in m and sufficient to extract a convergent subsequence. For simplicity of exposition, we set $f = 0$; the case of nonzero forcing requires only minor modifications (absorbing f via the Cauchy–Schwarz and Young inequalities).

Theorem 2.9 (Energy estimate for Galerkin approximations)**.** *For $f = 0$, the Galerkin approximation u_m satisfies*

$$\frac{1}{2} \|u_m(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u_m(s)\|_{L^2}^2 \, \mathrm{d}s = \frac{1}{2} \|P_m u_0\|_{L^2}^2 \le \frac{1}{2} \|u_0\|_{L^2}^2 \tag{2.13}$$

for all $t \in [0, T_m)$.

Proof. Multiply the Galerkin equation (2.11) by $c_k^{(m)}(t)$ and sum over $k = 1, \dots, m$. Since $u_m = \sum_k c_k^{(m)} w_k$, the left-hand side becomes $\langle \partial_t u_m, u_m \rangle_{L^2} + \nu \|\nabla u_m\|_{L^2}^2 + b(u_m, u_m, u_m)$. By Lemma 2.7, $b(u_m, u_m, u_m) = 0$. The first term equals $\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|_{L^2}^2$. With $f = 0$, the right-hand side vanishes, giving

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_m(t)\|_{L^2}^2 + \nu \|\nabla u_m(t)\|_{L^2}^2 = 0.$$

Integrating from 0 to t and using $u_m(0) = P_m u_0$ yields (2.13). The inequality $\|P_m u_0\|_{L^2} \le \|u_0\|_{L^2}$ follows from the fact that P_m is an orthogonal projection. □

Corollary 2.10 (Global existence and uniform bounds)**.** *The Galerkin approximation u_m extends to all of $[0, T]$ for every $T > 0$, and the following uniform bounds hold (independent of m):*

$$\sup_{t \in [0,T]} \|u_m(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2, \qquad \int_0^T \|\nabla u_m(t)\|_{L^2}^2 \, \mathrm{d}t \le \frac{1}{2\nu} \|u_0\|_{L^2}^2. \tag{2.14}$$

Proof. The energy estimate (2.13) shows that $\|u_m(t)\|_{L^2} \le \|u_0\|_{L^2}$ for all $t \in [0, T_m)$. In terms of the coefficient vector $c^{(m)}(t)$, this reads $\left|c^{(m)}(t)\right| = \|u_m(t)\|_{L^2} \le \|u_0\|_{L^2}$, so $c^{(m)}(t)$ remains bounded. By the continuation criterion for ODE solutions (a solution that remains bounded extends beyond its maximal interval), $T_m = +\infty$. The bound on $\int \|\nabla u_m\|^2$ follows directly from (2.13) by dropping the non-negative first term. □

We also need a bound on the time derivatives to apply compactness arguments.

Lemma 2.11 (Time derivative bound)**.** *The sequence* $\{\partial_t u_m\}_{m=1}^{\infty}$ *satisfies*

$$\sup_{m\in\mathbb{N}} \|\partial_t u_m\|_{L^{4/3}(0,T;\,H^{-1}(\mathbb{R}^3))} \;\le\; C\big(\nu, \|u_0\|_{L^2}\big). \tag{2.15}$$

Proof. From (2.11), for any $\phi \in H^1_0(\mathbb{R}^3;\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$,

$$\langle \partial_t u_m, \phi\rangle_{H^{-1},H^1_0} \;=\; -\nu\,\langle \nabla u_m, \nabla\phi\rangle_{L^2} \;-\; b(u_m, u_m, \phi).$$

The viscous term satisfies $|\nu\,\langle \nabla u_m, \nabla\phi\rangle_{L^2}| \le \nu\,\|\nabla u_m\|_{L^2}\,\|\nabla\phi\|_{L^2}$. For the nonlinear term, Hölder's inequality with the Sobolev embedding $H^1_0(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ (Theorem 1.11) and interpolation give

$$|b(u_m, u_m, \phi)| \;\le\; C\,\|u_m\|_{L^2}^{1/2}\,\|\nabla u_m\|_{L^2}^{3/2}\,\|\nabla\phi\|_{L^2}\,.$$

Dividing by $\|\nabla\phi\|_{L^2}$ and taking the supremum over ϕ yields

$$\begin{aligned}\|\partial_t u_m(t)\|_{H^{-1}} \;\le\; &\nu\,\|\nabla u_m(t)\|_{L^2}\\ &+\; C\,\|u_0\|_{L^2}^{1/2}\,\|\nabla u_m(t)\|_{L^2}^{3/2},\end{aligned}$$

using $\|u_m\|_{L^2} \le \|u_0\|_{L^2}$ from Corollary 2.10. Raising to the power 4/3 and integrating in time, the dominant term $\|\nabla u_m\|^{3/2}$ raised to the power 4/3 gives $\|\nabla u_m\|^2$, so

$$\int_0^T \|\partial_t u_m(t)\|_{H^{-1}}^{4/3}\,\mathrm{d}t \;\le\; C\Big(\nu, \|u_0\|_{L^2}\,, \int_0^T \|\nabla u_m\|_{L^2}^2\,\mathrm{d}t\Big),$$

and the right-hand side is bounded uniformly in m by (2.14). □

2.3.4 Passage to the limit

We now extract a subsequence of Galerkin approximations that converges to a weak solution of the Navier–Stokes equations on $\mathbb{R}^3 \times (0, T)$. The principal difficulty, compared with the bounded-domain case, is that the embedding $H^1_0(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is continuous but not compact. We overcome this by a localisation and diagonal argument.

Theorem 2.12 (Aubin–Lions, local version)**.** *Let $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ for $R > 0$. Suppose $\{v_m\}_{m=1}^\infty$ is uniformly bounded in $L^2(0,T;H_0^1(\mathbb{R}^3))$ with $\{\partial_t v_m\}$ uniformly bounded in $L^{4/3}(0,T;H^{-1}(\mathbb{R}^3))$. Then for each fixed $R > 0$, the restrictions $\{v_m|_{B_R}\}$ are relatively compact in $L^2(0,T;L^2(B_R))$.*

Proof. Fix $R > 0$. The restriction map $H_0^1(\mathbb{R}^3) \to H^1(B_R)$ is continuous and the embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ is compact by the Rellich–Kondrachov theorem (Theorem 1.14), since B_R is a bounded Lipschitz domain. Likewise, $L^2(B_R) \hookrightarrow H^{-1}(B_R)$ continuously. The restrictions $v_m|_{B_R}$ are therefore uniformly bounded in $L^2(0,T;H^1(B_R))$, and $\partial_t(v_m|_{B_R})$ are uniformly bounded in $L^{4/3}(0,T;H^{-1}(B_R))$ (since restriction commutes with time differentiation in the distributional sense, and the restriction of an $H^{-1}(\mathbb{R}^3)$ distribution to B_R has no larger $H^{-1}(B_R)$ norm). The classical Aubin–Lions lemma (with the triple $H^1(B_R) \Subset L^2(B_R) \hookrightarrow H^{-1}(B_R)$ and exponents $p = 2$, $q = 4/3$) yields relative compactness of $\{v_m|_{B_R}\}$ in $L^2(0,T;L^2(B_R))$. $\square$

Theorem 2.13 (Convergence of Galerkin approximations)**.** *There exist a subsequence $\{u_{m_j}\}_{j=1}^\infty$ and a function*

$$u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H_0^1(\mathbb{R}^3;\mathbb{R}^3))$$

such that:

(i) $u_{m_j} \rightharpoonup u$ *weakly in* $L^2(0,T;H_0^1(\mathbb{R}^3))$;

(ii) $u_{m_j} \overset{*}{\rightharpoonup} u$ *weak-$*$ in* $L^\infty(0,T;L^2(\mathbb{R}^3))$;

(iii) $u_{m_j} \to u$ *strongly in* $L^2(0,T;L^2(K))$ *for every compact set* $K \subset \mathbb{R}^3$.

Proof. The uniform bounds (2.14) show that $\{u_m\} \subset L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H_0^1(\mathbb{R}^3))$, both bounded. By the Banach–Alaoglu theorem, there exist a subsequence (still denoted $\{u_m\}$) and u in the indicated spaces satisfying (i) and (ii).

For (iii), we use a diagonal argument. By Theorem 2.12 applied with $R = 1$, there is a subsequence $\{u_m^{(1)}\}$ such that $u_m^{(1)}|_{B_1} \to u|_{B_1}$ strongly in $L^2(0,T;L^2(B_1))$. Applying Theorem 2.12 again with $R = 2$ to the subsequence $\{u_m^{(1)}\}$, we extract a further subsequence $\{u_m^{(2)}\}$ converging strongly on B_2. Continuing inductively, for each $n \in \mathbb{N}$ we obtain a subsequence $\{u_m^{(n)}\}$ of $\{u_m^{(n-1)}\}$ converging strongly in $L^2(0,T;L^2(B_n))$. The diagonal subsequence $u_{m_j} = u_j^{(j)}$ then converges strongly in $L^2(0,T;L^2(B_n))$ for every $n \in \mathbb{N}$. Since every compact $K \subset \mathbb{R}^3$ is contained in some B_n, conclusion (iii) follows. $\square$

It remains to verify that the limit u satisfies the weak formulation of the Navier–Stokes equations.

Theorem 2.14 (Passage to the limit)**.** *The function u obtained in Theorem 2.13 is a weak solution of the Navier–Stokes equations on $\mathbb{R}^3 \times (0,T)$ in the sense of* (2.7) *(with $f = 0$).*

Proof. Fix $\phi \in C_c^\infty(\mathbb{R}^3 \times [0,T); \mathbb{R}^3)$ with $\operatorname{div} \phi = 0$. Choose $R > 0$ large enough that $\operatorname{supp} \phi \subset B_R \times [0,T)$. For each m, let $\phi_m(t) = \sum_{k=1}^m \langle \phi(t), w_k \rangle\, w_k$ be the projection of ϕ onto V_m. Since $\{w_k\}$ is complete in $L^2_\sigma(\mathbb{R}^3)$, $\phi_m \to \phi$ in every relevant norm.

The Galerkin equation (2.11) with test function ϕ_m, after integration by parts in time, gives

$$
\begin{aligned}
&-\int_0^T \langle u_m, \partial_t \phi_m \rangle_{L^2}\, \mathrm{d}t \;-\; \langle P_m u_0, \phi_m(0) \rangle_{L^2} \\
&\qquad + \int_0^T b(u_m, u_m, \phi_m)\, \mathrm{d}t \;+\; \nu \int_0^T \langle \nabla u_m, \nabla \phi_m \rangle_{L^2}\, \mathrm{d}t \;=\; 0.
\end{aligned}
$$

We pass each term to the limit along the subsequence $\{u_{m_j}\}$.

The linear terms converge by weak and weak-$*$ convergence. For the time-derivative term, $u_m \rightharpoonup u$ in $L^2(0,T;L^2)$ and $\partial_t \phi_m \to \partial_t \phi$ strongly, so

$$
\int_0^T \langle u_m, \partial_t \phi_m \rangle\, \mathrm{d}t \;\to\; \int_0^T \langle u, \partial_t \phi \rangle\, \mathrm{d}t.
$$

Similarly, weak convergence of ∇u_m in $L^2(0,T;L^2)$ gives

$$
\int_0^T \langle \nabla u_m, \nabla \phi_m \rangle\, \mathrm{d}t \;\to\; \int_0^T \langle \nabla u, \nabla \phi \rangle\, \mathrm{d}t.
$$

The initial data term converges because $P_m u_0 \to u_0$ in L^2 and $\phi_m(0) \to \phi(0)$.

The nonlinear term is the critical passage. Since $\operatorname{supp} \phi \subset B_R \times [0,T)$,

$$
\int_0^T b(u_m, u_m, \phi_m)\, \mathrm{d}t \;=\; -\int_0^T \int_{B_R} (u_m \otimes u_m) : \nabla \phi_m \, \mathrm{d}x\, \mathrm{d}t.
$$

We decompose $u_m \otimes u_m - u \otimes u = (u_m - u) \otimes u_m + u \otimes (u_m - u)$. The strong convergence $u_{m_j} \to u$ in $L^2(0,T;L^2(B_R))$ from Theorem 2.13(iii), combined with the uniform bound $\|u_m\|_{L^\infty(0,T;L^2)} \le \|u_0\|_{L^2}$, gives $(u_{m_j} - u) \otimes u_{m_j} \to 0$ in $L^1(0,T;L^1(B_R))$. For the second factor, $u \otimes (u_{m_j} - u) \to 0$ by the weak convergence $u_{m_j} \rightharpoonup u$ in $L^2(0,T;H^1_0)$, since $(u \cdot \nabla)\phi$ is a fixed bounded function supported in $B_R \times [0,T)$. Since $\nabla \phi_m \to \nabla \phi$ in $L^\infty(0,T;L^\infty(\mathbb{R}^3))$, we conclude

$$
\int_0^T b(u_{m_j}, u_{m_j}, \phi_{m_j})\, \mathrm{d}t \;\to\; \int_0^T b(u, u, \phi)\, \mathrm{d}t.
$$

Combining all terms, u satisfies (2.7). □

2.4 Energy inequality

The Galerkin approximations satisfy an energy *equality* (Theorem 2.9), but after passage to the limit, only an *inequality* survives.

2.4.1 Statement and derivation

Theorem 2.15 (Energy inequality[3])**.** *Let u be the weak solution obtained in Theorem 2.14. Then for almost every $t \in (0, T)$,*

$$\frac{1}{2}\,\|u(t)\|_{L^2}^2 \;+\; \nu \int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s \;\leq\; \frac{1}{2}\,\|u_0\|_{L^2}^2\,. \tag{2.16}$$

Proof. The Galerkin approximations satisfy the energy equality

$$\frac{1}{2}\,\|u_m(t)\|_{L^2}^2 \;+\; \nu \int_0^t \|\nabla u_m(s)\|_{L^2}^2 \,\mathrm{d}s \;=\; \frac{1}{2}\,\|P_m u_0\|_{L^2}^2$$

for all $t \in [0, T]$, by Theorem 2.9. We pass each side to the limit separately.

The right-hand side converges: $\|P_m u_0\|_{L^2} \to \|u_0\|_{L^2}$ as $m \to \infty$, since $P_m u_0 \to u_0$ in $L^2(\Omega)$ (completeness of the eigenfunctions $\{w_k\}$).

For the left-hand side, the strong convergence $u_m \to u$ in $L^2(0, T; L^2)$ from Theorem 2.13(iii) gives, after passing to a subsequence, $u_m(t) \to u(t)$ in L^2 for a.e. t. For such t,

$$\|u(t)\|_{L^2} \;=\; \lim_{m\to\infty} \|u_m(t)\|_{L^2}\,.$$

For the dissipation term, weak convergence $\nabla u_m \rightharpoonup \nabla u$ in $L^2(0, T; L^2)$ and weak lower semicontinuity of the norm yield

$$\int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s \;\leq\; \liminf_{m\to\infty} \int_0^t \|\nabla u_m(s)\|_{L^2}^2 \,\mathrm{d}s.$$

Combining, for a.e. t,

$$\begin{aligned}
&\frac{1}{2}\,\|u(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s \\
&\qquad\leq\; \liminf_{m\to\infty} \Big[\frac{1}{2}\,\|u_m(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u_m\|_{L^2}^2 \,\mathrm{d}s\Big] \\
&\qquad=\; \frac{1}{2}\,\|u_0\|_{L^2}^2\,.
\end{aligned}$$

This establishes (2.16). □

[3] Lean 4: `NavierStokes.energyInequality`

2.4.2 Why equality may fail

Remark 2.16 (Energy equality versus inequality). The passage from equality (for Galerkin approximations) to inequality (for the limit) arises because weak and weak-$*$ limits do not preserve norms in general: one has only the lower semicontinuity $\|u\| \le \liminf \|u_m\|$, not equality. Equality would hold if the convergence were *strong* in $L^\infty(0,T;L^2)$, but the Aubin–Lions lemma provides strong convergence only in $L^2(0,T;L^2)$ (i.e., integrated in time, not pointwise in time).

Physically, the energy inequality states that kinetic energy can only decrease (or be dissipated by viscosity), never spontaneously increase. The strict inequality would correspond to *anomalous dissipation*, a phenomenon related to turbulence and the Onsager conjecture. Whether every Leray–Hopf weak solution satisfies energy *equality* remains open in three dimensions. In two dimensions, energy equality is known to hold because the solutions are unique and regular.

2.4.3 The energy inequality with general initial times

Remark 2.17 (Strong energy inequality). The *strong energy inequality* asserts that for almost every $s \ge 0$ and all $t \ge s$,

$$\frac{1}{2}\,\|u(t)\|_{L^2}^2 \;+\; \nu \int_s^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;\le\; \frac{1}{2}\,\|u(s)\|_{L^2}^2\,. \tag{2.17}$$

This is a stronger statement than (2.16) because it holds with arbitrary initial time s (not just $s = 0$). The Leray–Hopf construction in fact produces solutions satisfying (2.17); the proof uses the same weak lower semicontinuity argument applied to the Galerkin energy equality restarted at time s.

2.5 Leray–Hopf weak solutions

We now collect the properties established in the preceding sections into a single definition.

2.5.1 The definition

Definition 2.18 (Leray–Hopf weak solution[4]). Let $u_0 \in L^2_\sigma(\mathbb{R}^3)$ and $0 < T \le \infty$. A vector field $u : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ is a *Leray–Hopf weak solution* of the Navier–Stokes equations with initial data u_0 if:

(i) *Regularity class.* $u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1_0(\mathbb{R}^3;\mathbb{R}^3))$.

(ii) *Weak formulation.* For every $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0,T);\mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$, the identity (2.7) holds (with $f = 0$).

[4] Lean 4: `NavierStokes.LerayHopfSolution`

(iii) *Energy inequality.* For almost every $t \in (0, T)$,

$$\frac{1}{2}\,\|u(t)\|_{L^2}^2 \;+\; \nu \int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s \;\leq\; \frac{1}{2}\,\|u_0\|_{L^2}^2\,.$$

(iv) *Weak continuity at $t = 0$.* $u(t) \rightharpoonup u_0$ in $L^2(\mathbb{R}^3)$ as $t \to 0^+$.

Remark 2.19 (Weak versus strong continuity in time). Condition (iv) requires only that $u(t)$ converges *weakly* to u_0 as $t \to 0^+$. In fact, the energy inequality (iii) upgrades this to strong continuity: the weak convergence $u(t) \rightharpoonup u_0$ implies $\|u_0\|_{L^2} \leq \liminf_{t\to 0^+} \|u(t)\|_{L^2}$, while the energy inequality gives $\limsup_{t\to 0^+} \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. Together, $\|u(t)\|_{L^2} \to \|u_0\|_{L^2}$, and in a Hilbert space, weak convergence together with convergence of norms implies strong convergence. Thus $u(t) \to u_0$ strongly in $L^2(\mathbb{R}^3)$ as $t \to 0^+$.

Remark 2.20 (Time regularity). A Leray–Hopf weak solution u belongs to $C_w([0, T); L^2(\mathbb{R}^3))$, the space of weakly continuous functions from $[0, T)$ to $L^2(\mathbb{R}^3)$. This means that for every $\psi \in L^2(\mathbb{R}^3)$, the map $t \mapsto \langle u(t), \psi\rangle_{L^2}$ is continuous on $[0, T)$. The weak continuity follows from the weak formulation (2.7) by choosing test functions of the form $\varphi(x, t) = \eta(t)\,\psi(x)$ with $\eta \in C_c^\infty([0, T))$.

2.5.2 The existence theorem

Theorem 2.21 (Existence of Leray–Hopf weak solutions[5])**.** *For every $u_0 \in L^2_\sigma(\mathbb{R}^3)$ and every $T > 0$ (including $T = \infty$), there exists a Leray–Hopf weak solution u of the Navier–Stokes equations on $\mathbb{R}^3 \times (0, T)$ with initial data u_0, in the sense of Definition 2.18.*

Proof. The proof assembles the results of the preceding sections. We give the construction on a bounded domain Ω, then extend to $\mathbb{R}^3$.

The Galerkin approximations u_m are defined by (2.9) and (2.11), using the orthonormal basis of $L^2_\sigma(\mathbb{R}^3)$ constructed via Gram–Schmidt in Section 2.3. By Proposition 2.8, each u_m exists locally, and by Corollary 2.10, each u_m extends to $[0, T]$ with the uniform bounds (2.14). By Theorem 2.13, a subsequence converges to a limit u in the senses (i)–(iii) of that theorem. By Theorem 2.14, u satisfies the weak formulation. By Theorem 2.15, u satisfies the energy inequality. This verifies conditions (i)–(iii) of Definition 2.18.

For condition (iv), weak continuity at $t = 0$: the energy inequality gives $\limsup_{t\to 0^+} \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$, so $(u(t))$ is bounded in L^2 as $t \to 0^+$. By the Banach–Alaoglu theorem, every sequence $t_k \to 0^+$ has a subsequence along which $u(t_k)$ converges weakly in L^2. To identify the weak limit as u_0, choose a test function $\varphi(x, t) = \eta(t)\,\psi(x)$ in (2.7) with $\psi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$

[5] Lean 4: `NavierStokes.lerayHopf_existence`

divergence-free and $\eta \in C_c^\infty([0,\varepsilon))$ with $\eta(0) = 1$. Letting $\varepsilon \to 0$, the weak formulation reduces to $\langle u(0^+), \psi\rangle_{L^2} = \langle u_0, \psi\rangle_{L^2}$, confirming that the weak limit is u_0. Since the limit is the same along every subsequence, the full net converges: $u(t) \rightharpoonup u_0$ as $t \to 0^+$.

For the extension to $\mathbb{R}^3$, let $\Omega_k = B_k(0)$ for $k \in \mathbb{N}$. For each k, the construction above yields a Leray–Hopf weak solution $u^{(k)}$ on $\Omega_k \times (0,T)$ (extending u_0 by zero outside Ω_k if $\operatorname{supp} u_0$ is compact, or projecting onto $L^2_\sigma(\Omega_k)$ in general). The energy bounds are uniform in k: $\|u^{(k)}\|_{L^\infty(0,T;L^2)} + \|u^{(k)}\|_{L^2(0,T;H_0^1)} \leq C(\|u_0\|_{L^2}, \nu)$. A diagonal argument (extracting successive subsequences on $\Omega_1, \Omega_2, \ldots$ and using the Cantor diagonal construction) produces a limit u defined on $\mathbb{R}^3 \times (0,T)$ that inherits all four properties of Definition 2.18. The divergence-free condition passes to the limit because $\operatorname{div} u^{(k)} = 0$ in $\mathcal{D}'(\Omega_k)$ for each k, and any test function $\varphi \in C_c^\infty(\mathbb{R}^3)$ is supported in Ω_k for k sufficiently large. □

2.5.3 Non-uniqueness and the role of the energy inequality

Remark 2.22 (Uniqueness remains open). The Leray–Hopf existence theorem does not assert uniqueness. Whether Leray–Hopf weak solutions are unique for given initial data $u_0 \in L^2_\sigma(\mathbb{R}^3)$ is one of the major open problems in mathematical fluid dynamics. In two dimensions, uniqueness holds (the solution is in fact smooth for all time); in three dimensions, uniqueness is unknown.

The energy inequality plays a selection role: it excludes "parasitic" weak solutions that spontaneously gain energy. Without the energy inequality, non-uniqueness of weak solutions is known to occur. The convex integration techniques of De Lellis and Székelyhidi produce L^∞-weak solutions of the Euler equations ($\nu = 0$) with prescribed energy profiles, including solutions that violate the energy inequality. For $\nu > 0$, Buckmaster and Vicol (2019) constructed non-unique weak solutions in a regularity class strictly below the Leray–Hopf class. The energy inequality is therefore not merely a technical artefact but a physically meaningful selection criterion.

Remark 2.23 (Serrin-type regularity criteria). Although uniqueness in the full Leray–Hopf class is open, additional integrability conditions on u do imply both regularity and uniqueness. The classical *Serrin condition* states that if a Leray–Hopf weak solution satisfies $u \in L^q(0,T;L^r(\mathbb{R}^3))$ with $\frac{2}{q}+\frac{3}{r} = 1$ and $r > 3$, then u is smooth on $(0,T]$ and is the unique Leray–Hopf solution with that initial data. The Serrin condition is a *scaling-critical* condition: the Navier–Stokes equations are invariant under the rescaling $u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t)$, and the Serrin norm $\|u\|_{L^q_t L^r_x}$ is invariant under this rescaling when $\frac{2}{q} + \frac{3}{r} = 1$.

2.6 The Clay Millennium Problem

The existence of Leray–Hopf weak solutions guarantees that the Navier–Stokes equations have *some* kind of solution for all time. The Clay Millennium Problem asks whether these solutions (or, more precisely, strong solutions with smooth initial data) remain smooth.

2.6.1 The official problem statement

The Clay Mathematics Institute formulated the Navier–Stokes existence and smoothness problem in 2000 as one of the seven Millennium Prize Problems, with a prize of one million dollars. The precise mathematical statement, due to Fefferman [3], poses four related problems. We first set up the common hypotheses and then state each problem.

Fefferman considers the Navier–Stokes equations on $\mathbb{R}^3$ with viscosity $\nu > 0$, external force f, and initial velocity u_0. A solution (p, u) is called *physically reasonable* if it is smooth and has bounded energy:

$$p, u \in C^\infty(\mathbb{R}^3 \times [0, \infty)), \tag{2.18}$$

$$\int_{\mathbb{R}^3} |u(x,t)|^2 \, \mathrm{d}x < C \qquad \text{for all } t \geq 0 \quad \text{and some constant } C > 0. \tag{2.19}$$

For the whole-space problems, Fefferman requires the initial data and force to be smooth with rapid decay: for every multi-index α, every $m \geq 0$, and every $K > 0$,

$$|\partial_x^\alpha u_0(x)| \;\leq\; C_{\alpha K} \, (1 + |x|)^{-K} \qquad \text{on } \mathbb{R}^3, \tag{2.20}$$

$$|\partial_x^\alpha \partial_t^m f(x,t)| \;\leq\; C_{\alpha m K} \, (1 + |x| + t)^{-K} \qquad \text{on } \mathbb{R}^3 \times [0, \infty). \tag{2.21}$$

For the periodic problems on the torus $\mathbb{R}^3/\mathbb{Z}^3$, these decay conditions are replaced by periodicity:

$$u_0(x + e_j) = u_0(x), \quad f(x + e_j, t) = f(x,t), \quad p(x + e_j, t) = p(x,t), \tag{2.22}$$

for $1 \leq j \leq 3$, where e_j are the standard unit vectors, and the solution $u(x,t) = u(x + e_j, t)$ is required to be periodic as well.

With these conventions, Fefferman poses the following four problems. A positive answer to the Millennium Problem consists of proving either (A) or (B); a negative answer consists of proving either (C) or (D).

Conjecture 2.24 (Existence and smoothness on $\mathbb{R}^3$ (Problem A))**.** *Take $\nu > 0$. Let u_0 be any smooth, divergence-free vector field satisfying* (2.20)*, and take $f \equiv 0$. Then there exist smooth functions $p(x,t)$ and $u(x,t)$ on $\mathbb{R}^3 \times [0, \infty)$ satisfying the Navier–Stokes equations* (2.1)*–*(2.3) *together with* (2.18) *and* (2.19)*.*

Conjecture 2.25 (Existence and smoothness on $\mathbb{R}^3/\mathbb{Z}^3$ (Problem B))**.** *Take $\nu > 0$. Let u_0 be any smooth, divergence-free vector field satisfying* (2.22)*, and take $f \equiv 0$. Then there exist smooth functions $p(x,t)$ and $u(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying the Navier–Stokes equations* (2.1)*–*(2.3) *together with* (2.22) *(for the solution),* (2.18)*, and* (2.19)*.*

Fefferman also poses the complementary breakdown problems.

Conjecture 2.26 (Breakdown on $\mathbb{R}^3$ (Problem C))**.** *Take $\nu > 0$. Then there exist a smooth, divergence-free vector field u_0 on $\mathbb{R}^3$ and a smooth force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying* (2.20) *and* (2.21)*, for which there exist no physically reasonable solutions* (2.18)*–*(2.19) *of the Navier–Stokes equations on $\mathbb{R}^3 \times [0,\infty)$.*

Conjecture 2.27 (Breakdown on $\mathbb{R}^3/\mathbb{Z}^3$ (Problem D))**.** *Take $\nu > 0$. Then there exist a smooth, divergence-free vector field u_0 on $\mathbb{R}^3$ and a smooth force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying* (2.22)*, for which there exist no physically reasonable solutions of the Navier–Stokes equations on $\mathbb{R}^3 \times [0,\infty)$.*

Remark 2.28 (Interpretation of the problem). Problems (A) and (B) ask two things simultaneously: *existence* (there is a smooth solution for all time) and *smoothness* (the solution does not develop singularities). The decay condition (2.20) ensures that u_0 lies in every Sobolev space $H^k(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$ for all p, so the initial data is as regular as one could wish. The bounded energy condition (2.19) (note: bounded by a *constant*, not merely finite at each time) is a natural physical requirement and connects the problem to the Leray–Hopf framework.

Problems (A) and (B) set $f = 0$ to give solvers "reasonable leeway" (Fefferman's words). The breakdown problems (C) and (D) allow a nonzero force, since exhibiting blowup with no external forcing would be strictly harder. A negative answer to the Millennium Problem via (C) or (D) would demonstrate initial data and forces for which smooth, bounded-energy solutions simply do not exist globally.

2.6.2 Relation to the Leray–Hopf framework

Remark 2.29 (From Leray–Hopf to the Clay Problem). The Leray–Hopf theory provides a global weak solution u for any $u_0 \in L^2_\sigma(\mathbb{R}^3)$, but this solution may not be smooth. Problem (A) asks whether, for smooth and rapidly decaying u_0, the weak solution is in fact a classical (smooth) solution. Since smooth solutions are unique in the Leray–Hopf class (by the weak-strong uniqueness principle), a positive answer would imply that the Leray–Hopf weak solution *is* the unique smooth solution.

The connection is made precise by the following chain of implications. If u_0 is smooth and rapidly decaying, classical local existence theory (Fujita–Kato, 1964) produces a unique smooth solution u on some time interval

$[0, T^*)$. This smooth local solution is also a Leray–Hopf weak solution (it satisfies all four conditions of Definition 2.18). Problem (A) reduces to showing that $T^* = \infty$, i.e., the smooth local solution never breaks down.

2.6.3 What is known

Remark 2.30 (Partial results). The following results are known, falling short of resolving Problem (A):

(i) *Local existence.* For $u_0 \in H^1(\mathbb{R}^3)$ (or more generally in various critical spaces), there exists $T^* > 0$ and a unique smooth solution on $[0, T^*)$. If $\|u_0\|_{H^1}$ is sufficiently small (relative to ν), then $T^* = \infty$.

(ii) *Leray–Hopf global existence.* For $u_0 \in L^2_\sigma(\mathbb{R}^3)$, there exists a global weak solution satisfying the energy inequality (Theorem 2.21).

(iii) *Partial regularity.* The Caffarelli–Kohn–Nirenberg theorem (1982) states that for any suitable Leray–Hopf weak solution, the set of singular points (space-time points where u is not locally bounded) has one-dimensional parabolic Hausdorff measure zero. In particular, the singular set cannot contain a space-time curve.

(iv) *Serrin-type criteria.* If a Leray–Hopf weak solution satisfies the additional integrability condition $u \in L^q(0, T; L^r(\mathbb{R}^3))$ with $2/q + 3/r \leq 1$ and $r \geq 3$, then u is smooth on $(0, T]$ (see Remark 2.23).

(v) *Two dimensions.* In two spatial dimensions, the Leray–Hopf weak solution is unique, smooth for all time, and satisfies the energy equality. The three-dimensional case remains open.

2.6.4 What remains open

Remark 2.31 (The gap). The fundamental difficulty is the *supercritical* nature of the three-dimensional Navier–Stokes equations. The natural energy space $L^\infty(0, T; L^2) \cap L^2(0, T; H^1_0)$ lies below the scaling-critical threshold: the Navier–Stokes scaling $u_\lambda(x, t) = \lambda\, u(\lambda x, \lambda^2 t)$ preserves the $L^3(\mathbb{R}^3)$ norm, but the energy norm $\|u\|_{L^2}$ is not invariant (it scales as $\lambda^{-1/2}$). The energy bounds are therefore too weak to control the nonlinearity and prevent concentration of energy at small scales.

All known proofs of regularity (in two dimensions, or in three dimensions under additional hypotheses) exploit either the conservation of enstrophy (two dimensions) or integrability at or above the critical scaling (Serrin-type conditions). In three dimensions without additional assumptions, neither mechanism is available.

Resolving the Clay Problem requires proving one of Problems (A)–(D) above. A positive answer (Problem A or B) would show that smooth data always produce globally smooth solutions. A negative answer (Problem C or D) would exhibit specific data for which no physically reasonable solution exists globally. As of the time of writing, neither direction has produced a definitive result, and the problem remains one of the deepest open questions in mathematical analysis.

Summary. This chapter has established the existence of Leray–Hopf weak solutions for the three-dimensional incompressible Navier–Stokes equations with arbitrary L^2 initial data (Theorem 2.21). The construction proceeds via the Galerkin method: finite-dimensional approximations using eigenfunctions of the Stokes operator satisfy a priori energy estimates (Theorem 2.9), compactness arguments (Aubin–Lions, Rellich–Kondrachov) extract convergent subsequences (Theorem 2.13), and the strong L^2 convergence suffices to pass the nonlinear term to the limit (Theorem 2.14). The resulting weak solution satisfies the energy inequality (Theorem 2.15) but not necessarily energy equality. The Leray–Hopf framework connects directly to the Clay Millennium Problem (Conjecture 2.24): a positive resolution would show that, for smooth initial data, the Leray–Hopf weak solution is in fact the unique smooth solution for all time.

Chapter 3

The Biot–Savart Kernel and Leray Projection

Chapter 4

The Biot–Savart Connection

Chapter 5

Curvature and Classification

Chapter 6

Topological Invariants

Chapter 7

The Incompressibility Obstruction

Chapter 8

The Singularity Result

Appendix A

Lean Formalization Reference

Appendix B

Symbolic Verification

Appendix C

Numerical Experiments and Statistical Analysis

Appendix D

Rust Solver Documentation

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