CONFIGURATION-SPACE CURVATURE AND THE NAVIER–STOKES SINGULAR SET

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Abstract

At a singular point of a suitable weak solution of the three-dimensional Navier–Stokes equations, the partial-regularity theory of Caffarelli, Kohn and Nirenberg leaves two possibilities: either a configuration-space curvature concentrates, or the strain energy and the enstrophy come into balance. Working from the partial-regularity theory and the Liouville theory for divergence-free drifts, we control the first possibility coercively and constrain the second. The contribution is a geometric identity. The curvature of the L^2 metric on the group $\mathrm{SDiff}(\mathbb{T}^3)$ of volume-preserving diffeomorphisms is the pressure Hessian, the Gauss curvature of $\mathrm{SDiff}(\mathbb{T}^3)$ inside the flat group $\mathrm{Diff}(\mathbb{T}^3)$ whose second fundamental form is the pressure gradient; its L^2 size is the strain–enstrophy imbalance $\| \, |S|^2 - \frac{1}{2}|\omega|^2 \|$, so the configuration-space curvature is the pressure Hessian of the velocity-gradient dynamics. The imbalance carries a transport law in which the inertial production, the curvature work, and viscosity each redistribute it with zero net budget. Under a scale-critical L^3 bound, the local form of the Escauriaza–Seregin–Šverák condition, the balanced possibility forces a bounded ancient pressureless flow that a Liouville theorem for divergence-free drifts excludes, so every such singular point, the Type-I points among them, is curvature-concentrating. Beyond the critical class the imbalance is the structure of the nonlinearity the energy identity does not see, the structure an energy-preserving averaged equation discards, so the curvature is where regularity is to be decided.

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Contents

1 Introduction

1.1 The geometry of the configuration group

The configuration space of an ideal incompressible fluid on a closed Riemannian manifold M is the group $\mathrm{SDiff}(M)$ of volume-preserving diffeomorphisms, carrying the right-invariant L^2 metric. Arnold showed that the geodesics of this metric are the solutions of the Euler equations [Arn66], and Ebin and Marsden placed the resulting infinite-dimensional geometry on a rigorous analytic footing [EM70]. On the Lie algebra $\mathfrak{X}_\sigma$ of smooth divergence-free vector fields the Levi-Civita connection of the L^2 metric is the projection of the flat ambient derivative,

$$\nabla_u v = \mathbb{P}\big[(u \cdot \nabla)v\big], \tag{1}$$

with $\mathbb{P}$ the Leray projection onto divergence-free fields. Arnold computed the sectional curvatures of (1) and found them predominantly negative [Arn66; AK98]. Negative sectional curvature forces the exponential divergence of nearby geodesics through the Jacobi equation, and so measures the Lagrangian instability of the flow; the equivalence of this geometric instability with the Eulerian instability of the velocity field was established by Rouchon, Misiołek and Preston [Rou92; Mis93; Pre04]. The sign of these curvatures is carried by the pressure: for a pair of divergence-free fields the sectional value is, modulo terms that vanish on Fourier modes, the negative L^2 energy of an associated pressure gradient [AK98], so the incompressibility constraint is the source of the curvature.

1.2 The regularity problem

For the viscous problem on $\mathbb{T}^3$, Leray and Hopf constructed global weak solutions of finite energy [Ler34; Hop51], and whether every smooth datum gives a globally smooth solution is the subject of the Clay millennium problem [Fef06]. Smoothness follows from a range of scaling-critical conditions on the velocity, the Prodi–Serrin–Ladyzhenskaya class [Pro59; Ser62; Lad67], with the endpoint $L^\infty_t L^3_x$ secured by Escauriaza, Seregin and Šverák [ESŠ03b]; the vorticity counterpart for the inviscid problem is the criterion of Beale, Kato and Majda [BKM84]. Where global smoothness is open, the partial-regularity theory of Scheffer and of Caffarelli, Kohn and Nirenberg bounds the singular set, which has parabolic Hausdorff dimension at most one [Sch76; CKN82], with later proofs given by Lin and by Vasseur [Lin98; Vas07].

1.3 Vortex stretching and the pressure Hessian

The mechanism behind these results is vortex stretching, the growth of vorticity driven by the scalar $\alpha = \xi^{\top} S\,\xi$, with ξ the vorticity direction and S the strain. Constantin and Fefferman showed that a solution whose vorticity direction stays coherent on the set of high vorticity remains smooth [CF93], a geometric depletion of the nonlinearity developed further in the geometric statistics of turbulence and the constraints on singular Euler flows [Con94; Con96] and in the regularising effect of the vorticity direction for the viscous problem [BB02]. The strain that drives α evolves through the pressure Hessian $\nabla^2 p$, a nonlocal transform of the velocity gradient whose intractability is the central difficulty of the local models of the velocity-gradient tensor introduced by Vieillefosse and Cantwell, and of the analysis of vortex stretching by Ohkitani and Kishiba [Vie82; Can92; OK95]. The geometry of the configuration group and the analysis of vortex stretching have thus developed alongside one another, the first through the curvature of (1), the second through the pressure Hessian. The purpose of this paper is to show that these are one object.

1.4 Results

Throughout, u is smooth and divergence-free on the flat torus $\mathbb{T}^3$, with strain $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, vorticity $\omega = \operatorname{curl} u$, vorticity direction $\xi = \omega/|\omega|$ on $\{\omega \neq 0\}$, and pressure p determined by

$$-\Delta p = \partial_i u_j\,\partial_j u_i = |S|^2 - \tfrac{1}{2}|\omega|^2 =: q. \tag{2}$$

We write $\nabla^2 f$ for the Hessian of a function f, and $\mathcal{R}$ for the matrix Riesz transform with symbol $\eta_i\eta_j/|\eta|^2$, so that $\nabla^2 p = -\mathcal{R}q$.

The contribution is a reading of the Navier–Stokes singular set through the curvature of the configuration group. At a singular point of a suitable weak solution the partial-regularity theory leaves a dichotomy (Theorem 3): the curvature concentrates, or the flow comes into strain–enstrophy balance. Away from balance the curvature controls regularity coercively (Proposition 2), and in the scale-critical class the balanced alternative is excluded, so every scale-critical singular point, the Type-I points among them, is curvature-concentrating (Section 6). Beyond that class the balanced regime is supercritical, where the energy identity is blind and the curvature is the structure a regularity proof must read (Section 7.3). This reading rests on a geometric identity, that the curvature of $\mathrm{SDiff}(\mathbb{T}^3)$ is the pressure Hessian, which we establish first.

That the sectional curvature of $\mathrm{SDiff}(\mathbb{T}^3)$ is carried by the pressure is due to Arnold [Arn66; AK98], with the analytic foundation of Ebin and Marsden [EM70] and the curvature computations of Rouchon, Misiołek and Preston [Rou92; Mis93; Pre04]. The identity established here is at the level of the curvature operator: $R(u,\cdot)u$ is the Leray-projected pressure Hessian $-\mathbb{P}[(\nabla^2 p)\,\cdot]$, the same operator $\nabla^2 p = -\mathcal{R}q$ that drives the velocity-gradient dynamics of Vieillefosse, Cantwell, Ohkitani, Meneveau and Johnson [Vie82; Can92; OK95; Men11; Joh21]. This identifies the configuration-space curvature with the pressure Hessian of that dynamics, measures it by the strain–enstrophy imbalance, and turns it on the singular set through a transport law and a blow-up limit. The three theorems follow, one to a section, and the dichotomy is read through the dynamics. The geometry of Sections 2–4, the velocity-gradient dynamics it identifies, and the partial-regularity and divergence-free-drift Liouville theory that the singular-set reading draws on are established. The contribution is the operator identity that binds the geometry to the dynamics, and the reading of the singular set it gives.

Theorem 1 (Section 2) identifies the curvature with the pressure Hessian. The second fundamental form of $\mathrm{SDiff}(\mathbb{T}^3)$ in the flat group $\mathrm{Diff}(\mathbb{T}^3)$ is the pressure gradient, the flat ambient part of the curvature vanishes, and $R(u,v)u = -\mathbb{P}[(\nabla^2 p)\, v] - \mathbb{P}[(\nabla^2 q_v)\, u]$, with q_v the mixed pressure $\Delta q_v = \partial_i v_j\, \partial_j u_i$; the Gauss equation reproduces Arnold's sectional curvature formula. This is the Gauss equation for the constraint of incompressibility: the ambient group has no curvature, and the whole curvature of $\mathrm{SDiff}(\mathbb{T}^3)$ is carried by the second fundamental form, which is the pressure.

Theorem 2 (Section 4) measures it. The pointwise trace of the curvature is the strain–enstrophy imbalance, $\operatorname{tr} \nabla^2 p = \Delta p = -q$, and its L^2 size is that imbalance, $\|\nabla^2 p\|_{L^2(\mathbb{T}^3)} = \|q\|_{L^2(\mathbb{T}^3)}$. The curvature is large where strain and enstrophy are out of balance and vanishes on the pressureless flows $\{q \equiv 0\}$, the single-field form of the same-shell degeneracy that Arnold found for Fourier-mode pairs.

Theorem 3 (Section 5) reads this against regularity. Because $|\nabla u|^2 = q + |\omega|^2$ pointwise, at a singular point of a suitable weak solution either the curvature concentrates, $\limsup_{r\to 0} r^{-1} \int_{Q_r} |q| > 0$, or the scaled strain energy and half-enstrophy converge to a common value: the flow comes into strain–enstrophy balance, equivalently it becomes locally pressureless. Away from balance the converse holds: the curvature controls the local energy coercively (Proposition 2), so the two directions place the singular set where the curvature concentrates or the pressure depletes. The Constantin–Fefferman criterion controls the stretching scalar $\alpha = \xi^\top S\, \xi$ under a coherence hypothesis on ξ and reaches the regime of coherent vorticity; the curvature, through its imbalance, governs the complementary regime.

Two further results read the balanced alternative dynamically. Section 5 gives the imbalance a transport law, $(\partial_t + u \cdot \nabla) q = -6 \det(\nabla u) - 2\, S : \nabla^2 p + 2\nu \operatorname{tr}(\nabla u\, \Delta \nabla u)$, in which each term carries no net source, so balance demands their pointwise cancellation against a zero-sum budget. Section 6 makes this rigid: a balanced singular point would produce a bounded ancient solution with no pressure, $q \equiv 0$ and $\nabla p \equiv 0$. Under a Type-I bound the limit inherits the self-similar decay $|U| \le M|t|^{-1/2}$ and an elementary maximum principle forces it to vanish; under the larger scale-critical L^3 bound the divergence-free drift lies in the critical class and the Liouville theorem of Seregin, Silvestre, Šverák and Zlatoš [Ser+12] forces the same. Every scale-critical singular point is therefore curvature-concentrating. Beyond the critical class the balanced regime is supercritical, and the averaged-equation barrier of Tao [Tao16] shows it is closed to the energy identity and the scaling alone; the curvature is the structure of the Euler nonlinearity that resolves it, and the reduced pressureless Liouville is the gateway it leaves open.

Section 3 proves Theorem 1; the remaining sections are self-contained. The identities are confirmed by an independent symbolic and spectral computation, described in Appendix A.

2 The connection, the second fundamental form, and the Gauss equation

This section and the next supply the apparatus, the curvature of the configuration group and its size, which Sections 5 and 6 turn on the singular set.

Let $\mathfrak{X}_\sigma$ be the space of smooth divergence-free vector fields on $\mathbb{T}^3$ with the L^2 inner product $g(v, w) = \int_{\mathbb{T}^3} v \cdot w \, dx$. The Leray projection $\mathbb{P}$ is the orthogonal projection of L^2 onto $\mathfrak{X}_\sigma$, and $\mathbb{Q} = I - \mathbb{P} =$

$\nabla\Delta^{-1}\,\mathrm{div}$ is the complementary projection onto gradients. The group $\mathrm{Diff}(\mathbb{T}^3)$ of all diffeomorphisms carries a flat L^2 metric, whose geodesics are the pressureless straight-line motions; its volume-preserving subgroup $\mathrm{SDiff}(\mathbb{T}^3)$ is a submanifold whose induced Levi-Civita connection on $\mathfrak{X}_\sigma$ is the projection of the flat connection [EM70; Mis93; AK98],

$$\nabla_u v = \mathbb{P}\big[(u\cdot\nabla)v\big], \qquad u, v \in \mathfrak{X}_\sigma. \tag{3}$$

This is the metric connection in the convention of Misiołek and of Ebin and Marsden; the Gauss equation of Proposition 1 confirms that the sectional curvature it produces is Arnold's.

2.1 The second fundamental form

The normal part of the flat derivative is the second fundamental form of the constraint.

Lemma 1. *The second fundamental form of* $\mathrm{SDiff}(\mathbb{T}^3)$ *in* $\mathrm{Diff}(\mathbb{T}^3)$ *is*

$$\mathrm{I\!I}(a,b) = \mathbb{Q}\big[(a\cdot\nabla)b\big] = \nabla\phi_{ab}, \qquad \Delta\phi_{ab} = \partial_i a_j\,\partial_j b_i, \tag{4}$$

for $a, b \in \mathfrak{X}_\sigma$. *It is symmetric, and on the diagonal it is the pressure gradient,* $\mathrm{I\!I}(u,u) = -\nabla p$.

Proof. The flat derivative of b along a is $(a\cdot\nabla)b$; its normal (gradient) part is $\mathbb{Q}[(a\cdot\nabla)b] = \nabla\Delta^{-1}\,\mathrm{div}[(a\cdot\nabla)b]$, and $\mathrm{div}[(a\cdot\nabla)b] = \partial_i a_j\,\partial_j b_i$ since a is divergence-free. Symmetry follows because the Lie bracket $[a,b] = (a\cdot\nabla)b - (b\cdot\nabla)a$ is divergence-free, so $\mathbb{Q}[(a\cdot\nabla)b] = \mathbb{Q}[(b\cdot\nabla)a]$. For $a = b = u$ the source is $\partial_i u_j\,\partial_j u_i = q = -\Delta p$ by (2), hence $\phi_{uu} = -p$ and $\mathrm{I\!I}(u,u) = -\nabla p$. □

2.2 Curvature and the Gauss equation

The curvature of (3),

$$R(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w,$$

expands, for $u, v, w \in \mathfrak{X}_\sigma$, as

$$R(u,v)w = \mathbb{P}\big[(u\cdot\nabla)\mathbb{P}[(v\cdot\nabla)w]\big] - \mathbb{P}\big[(v\cdot\nabla)\mathbb{P}[(u\cdot\nabla)w]\big] - \mathbb{P}\big[([u,v]\cdot\nabla)w\big]. \tag{5}$$

Since the ambient group is flat, the curvature is governed entirely by the second fundamental form through the Gauss equation. This is the content of the main identity.

Theorem 1. *Let* u *be smooth and divergence-free on* $\mathbb{T}^3$. *The flat ambient part of the curvature vanishes identically,*

$$(u\cdot\nabla)(v\cdot\nabla)u - (v\cdot\nabla)(u\cdot\nabla)u - ([u,v]\cdot\nabla)u = 0, \tag{6}$$

and for every divergence-free v *the curvature operator is the Leray-projected pressure Hessian,*

$$R(u,v)u = -\mathbb{P}\big[(\nabla^2 p)\,v\big] - \mathbb{P}\big[(\nabla^2 q_v)\,u\big], \tag{7}$$

with q_v *the mixed pressure* $\Delta q_v = \partial_i v_j\,\partial_j u_i$.

Equation (7) names the curvature as the pressure Hessian $\nabla^2 p = -\mathcal{R}q$ acting on v, together with a mixed-pressure term quadratic in the velocity gradient. The sectional form of the same identity is Arnold's, recovered here from the second fundamental form.

Proposition 1. *For $u, v \in \mathfrak{X}_\sigma$ the sectional curvature satisfies the Gauss equation*

$$g\big(R(u,v)v,u\big) = \big\langle \mathrm{II}(u,u), \mathrm{II}(v,v)\big\rangle_{L^2} - \|\mathrm{II}(u,v)\|_{L^2}^2, \tag{8}$$

which is Arnold's sectional curvature formula with the pressure term $-\|\mathrm{II}(u,v)\|_{L^2}^2$ and remainder $B(u,v) = \langle \mathrm{II}(u,u), \mathrm{II}(v,v)\rangle_{L^2}$.

Proof. The ambient group $\mathrm{Diff}(\mathbb{T}^3)$ is flat, so the Gauss equation for the submanifold $\mathrm{SDiff}(\mathbb{T}^3)$ reads $g(R(u,v)v,u) = \langle \mathrm{II}(u,u), \mathrm{II}(v,v)\rangle - \langle \mathrm{II}(u,v), \mathrm{II}(v,u)\rangle$, and II is symmetric by Lemma 1. With $\mathrm{II}(u,u) = -\nabla p$ this is Arnold's formula; the pressure energy $-\|\mathrm{II}(u,v)\|^2$ is its leading term and $B = \langle \mathrm{II}(u,u), \mathrm{II}(v,v)\rangle$ the remainder that vanishes on the Fourier-mode pairs of [AK98]. □

3 Proof of Theorem 1

Proof of Theorem 1. On scalar functions the flat directional derivatives satisfy

$$(u\cdot\nabla)(v\cdot\nabla) - (v\cdot\nabla)(u\cdot\nabla) = ([u,v]\cdot\nabla), \tag{9}$$

so applying (9) to each component of u gives the vanishing of the flat ambient part (6).

Expanding each inner Leray projection in (5) through $\mathbb{P} = I - \mathbb{Q}$, the three terms that carry no inner $\mathbb{Q}$ assemble into

$$\mathbb{P}\big[(u\cdot\nabla)(v\cdot\nabla)u - (v\cdot\nabla)(u\cdot\nabla)u - ([u,v]\cdot\nabla)u\big],$$

which vanishes by (6), and there remains

$$R(u,v)u = -\mathbb{P}\big[(u\cdot\nabla)\,\mathbb{Q}[(v\cdot\nabla)u]\big] + \mathbb{P}\big[(v\cdot\nabla)\,\mathbb{Q}[(u\cdot\nabla)u]\big]. \tag{10}$$

Incompressibility gives $\mathrm{div}[(u\cdot\nabla)u] = \partial_i u_j\,\partial_j u_i = q = -\Delta p$ by (2), so $\mathbb{Q}[(u\cdot\nabla)u] = \nabla\Delta^{-1}\,\mathrm{div}[(u\cdot\nabla)u] = -\nabla p$ and

$$(v\cdot\nabla)\,\mathbb{Q}[(u\cdot\nabla)u] = -(v\cdot\nabla)\nabla p = -(\nabla^2 p)\,v.$$

In the same way $\mathrm{div}[(v\cdot\nabla)u] = \partial_i v_j\,\partial_j u_i = \Delta q_v$, a divergence and so of zero mean, which defines q_v; hence $\mathbb{Q}[(v\cdot\nabla)u] = \nabla q_v$ and $(u\cdot\nabla)\,\mathbb{Q}[(v\cdot\nabla)u] = (\nabla^2 q_v)\,u$. Substituting into (10) gives (7). □

4 The curvature is the strain–enstrophy imbalance

Theorem 1 expresses the curvature through the pressure Hessian $\nabla^2 p = -\mathcal{R}q$. We now measure it. The relevant scalar is the strain–enstrophy imbalance $q = |S|^2 - \frac{1}{2}|\omega|^2$ of (2).

Theorem 2. *Let u be smooth and divergence-free on $\mathbb{T}^3$. The pressure Hessian and the imbalance q are related by*

$$\mathrm{tr}\,\nabla^2 p = \Delta p = -q, \qquad \|\nabla^2 p\|_{L^2(\mathbb{T}^3)} = \|q\|_{L^2(\mathbb{T}^3)}, \qquad |q| \le \sqrt{3}\,|\nabla^2 p| \;\; \textit{pointwise}, \tag{11}$$

and $c_s\|q\|_{L^s} \le \|\nabla^2 p\|_{L^s} \le C_s\|q\|_{L^s}$ for $1 < s < \infty$. Consequently the curvature vanishes precisely on the pressureless fields $\{q \equiv 0\}$, equivalently $|S|^2 = \frac{1}{2}|\omega|^2$ pointwise.

Proof. The trace of the pressure Hessian is $\operatorname{tr}\nabla^2 p = \Delta p = -q$ by (2); this is a pointwise identity. Writing $\nabla^2 p = -\mathcal{R}q$ with $\mathcal{R}$ the matrix Riesz transform of symbol $\eta_i\eta_j/|\eta|^2$, Plancherel gives

$$\|\nabla^2 p\|_{L^2}^2 = \int \frac{\sum_{i,j}\eta_i^2\eta_j^2}{|\eta|^4}\,|\hat{q}(\eta)|^2\,\mathrm{d}\eta = \int |\hat{q}(\eta)|^2\,\mathrm{d}\eta = \|q\|_{L^2}^2,$$

since $\sum_{i,j}\eta_i^2\eta_j^2 = |\eta|^4$. The pointwise bound $|q| = |\operatorname{tr}\nabla^2 p| \le \sqrt{3}\,|\nabla^2 p|$ is Cauchy–Schwarz. The matrix Riesz transform is bounded on L^s for $1 < s < \infty$, which gives $\|\nabla^2 p\|_{L^s} \le C_s\|q\|_{L^s}$; the pointwise trace bound gives $\|q\|_{L^s} \le \sqrt{3}\,\|\nabla^2 p\|_{L^s}$. Finally $\nabla^2 p \equiv 0$ forces $\mathcal{R}q \equiv 0$, hence $\hat{q} \equiv 0$ on the nonzero frequencies and $q \equiv 0$ since q has zero mean; conversely $q \equiv 0$ gives $\nabla^2 p \equiv 0$. On the torus $q \equiv 0$ is $\Delta p \equiv 0$, that is p constant. □

The curvature is large where strain and enstrophy are out of balance, and zero where the field is pressureless. The latter is the single-field counterpart of Arnold's same-shell degeneracy [AK98]: the curvature of a two-plane of Fourier modes vanishes when the modes have equal frequency, because the pressure they generate vanishes, and here the curvature of a single field vanishes when its own pressure vanishes. The identification $\|\nabla^2 p\|_{L^2} = \|q\|_{L^2}$ is the rigorous content of the statement that the curvature density is the strain–enstrophy imbalance; the pointwise comparison fails because $\nabla^2 p$ is nonlocal, but its trace, the local part of the curvature, is $-q$.

5 Curvature concentration and the singular set

We read Theorem 2 against the partial-regularity theory. Write $Q_r = Q_r(x_0, t_0) = B_r(x_0)\times(t_0 - r^2, t_0)$ for the parabolic cylinder. The starting point is the pointwise identity and bound

$$|\nabla u|^2 = q + |\omega|^2, \qquad |q| \le |\nabla u|^2, \qquad q = |S|^2 - \tfrac{1}{2}|\omega|^2, \tag{12}$$

the identity since $|\nabla u|^2 = |S|^2 + \frac{1}{2}|\omega|^2$, and the bound since $|q| \le |S|^2 + \frac{1}{2}|\omega|^2$. By Theorem 2 the imbalance q is the local part of the curvature, $q = -\operatorname{tr}\nabla^2 p$. The bound in (12) is the coercivity that lets the curvature control the local energy away from the balanced set $\{q = 0\}$.

We use the ε-regularity criterion of Caffarelli, Kohn and Nirenberg in the following form [CKN82]: there is an absolute constant $\varepsilon_0 > 0$ such that if

$$\limsup_{r\to 0}\ \frac{1}{r}\int_{Q_r} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t < \varepsilon_0, \tag{13}$$

then (x_0, t_0) is a regular point. Equivalently, a singular point has $\limsup_{r\to 0} r^{-1}\int_{Q_r}|\nabla u|^2 \ge \varepsilon_0$. A suitable weak solution is smooth on the complement of its singular set, which carries zero parabolic one-dimensional measure, so the pointwise identities of Sections 2–4 hold on the regular set where the scaled quantities below are evaluated.

Proposition 2 (coercive criterion)**.** *Let u be a suitable weak solution of the Navier–Stokes equations on $\mathbb{T}^3$, let (x_0, t_0) be a point, and let $\delta \in (0, 1]$. Suppose the flow is δ-imbalanced near (x_0, t_0),*

$$\int_{Q_r} |q| \ \ge\ \delta \int_{Q_r} |\nabla u|^2 \qquad \textit{for all small } r, \tag{14}$$

and the curvature does not concentrate, $\limsup_{r\to 0} r^{-1}\int_{Q_r}|q| < \delta\,\varepsilon_0$, with ε_0 the Caffarelli–Kohn–Nirenberg constant. Then (x_0, t_0) is a regular point.

Proof. By (14), $\int_{Q_r}|\nabla u|^2 \le \delta^{-1}\int_{Q_r}|q|$, so

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}|\nabla u|^2 \;\le\; \delta^{-1}\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}|q| \;<\; \varepsilon_0,$$

and the ε-regularity criterion of Caffarelli, Kohn and Nirenberg [CKN82] gives regularity. □

The criterion is the coercive half of the picture: through its local trace q, the curvature controls the full gradient energy with constant δ^{-1}, so long as the flow stays δ-imbalanced. The hypothesis (14) is sharp, since its failure is the approach to balance. The other half is what a singularity forces.

Theorem 3 (balance at a singular point)**.** *Let u be a suitable weak solution of the Navier–Stokes equations on $\mathbb{T}^3$ and let (x_0, t_0) be a singular point. Then either the curvature concentrates,*

$$\limsup_{r\to 0}\;\frac{1}{r}\int_{Q_r}|q|\,\mathrm{d}x\,\mathrm{d}t > 0, \tag{15}$$

or the scaled strain energy and half-enstrophy share a common positive limit,

$$\limsup_{r\to 0}\;\frac{1}{r}\int_{Q_r}|S|^2 = \limsup_{r\to 0}\;\frac{1}{r}\int_{Q_r}\tfrac{1}{2}|\omega|^2 \;\ge\; \tfrac{1}{2}\varepsilon_0. \tag{16}$$

Proof. By the ε-regularity criterion [CKN82] a singular point has $\limsup_{r\to 0} r^{-1}\int_{Q_r}|\nabla u|^2 \ge \varepsilon_0$. Suppose (15) fails, so $r^{-1}\int_{Q_r}|q| \to 0$. By (12),

$$\frac{1}{r}\int_{Q_r}|\omega|^2 = \frac{1}{r}\int_{Q_r}|\nabla u|^2 - \frac{1}{r}\int_{Q_r} q,$$

and $r^{-1}\left|\int_{Q_r} q\right| \le r^{-1}\int_{Q_r}|q| \to 0$, so $\limsup_{r\to 0} r^{-1}\int_{Q_r}|\omega|^2 \ge \varepsilon_0$. Since $|S|^2 = q + \frac{1}{2}|\omega|^2$, the scaled strain energy differs from the scaled half-enstrophy by $r^{-1}\int_{Q_r} q \to 0$, which gives (16). □

Together, Proposition 2 and Theorem 3 characterise the role of the curvature at the singular set: away from balance the curvature controls regularity, and a singular point with bounded curvature is approached through strain–enstrophy balance. By Theorem 2 the balanced set is precisely $\{q \equiv 0\} = \{\Delta p \equiv 0\}$, so the curvature concentrates at the singularity unless the flow becomes locally pressureless, $\Delta p \to 0$ in scaled average.

The balanced regime is not free: the imbalance carries a transport law whose inertial and curvature terms must cancel locally to sustain it.

Proposition 3 (imbalance production)**.** *For a smooth solution of the Navier–Stokes equations on $\mathbb{T}^3$, the imbalance $q = |S|^2 - \frac{1}{2}|\omega|^2$ is transported by*

$$(\partial_t + u\cdot\nabla)\,q = -6\det(\nabla u) \;-\; 2\,S\!:\!\nabla^2 p \;+\; 2\nu\,\mathrm{tr}\big(\nabla u\,\Delta\nabla u\big), \tag{17}$$

writing $\mathrm{tr}(\nabla u\,\Delta\nabla u) = \partial_i u_j\,\Delta\partial_j u_i$ for the matrix trace, and each of the three terms carries zero net imbalance over the torus,

$$\int_{\mathbb{T}^3}\det(\nabla u)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} S\!:\!\nabla^2 p\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3}\mathrm{tr}\big(\nabla u\,\Delta\nabla u\big)\,\mathrm{d}x = 0. \tag{18}$$

Proof. The velocity gradient $A = \nabla u$ evolves by $(\partial_t + u \cdot \nabla)A = -A^2 - \nabla^2 p + \nu \Delta A$, obtained by differentiating the momentum equation. Since $q = \mathrm{tr}(A^2)$,

$$(\partial_t + u \cdot \nabla)\, q = 2\,\mathrm{tr}\big(A\,(\partial_t + u \cdot \nabla)A\big) = -2\,\mathrm{tr}(A^3) - 2\,\mathrm{tr}(A\,\nabla^2 p) + 2\nu\,\mathrm{tr}(A\,\Delta A).$$

For the trace-free gradient $\mathrm{tr}(A^3) = 3\det A$, and only the symmetric part of A pairs with the Hessian, $\mathrm{tr}(A\,\nabla^2 p) = S : \nabla^2 p$; this gives (17). The determinant of a gradient is a null Lagrangian, so its integral over the torus vanishes; $\int_{\mathbb{T}^3} S : \nabla^2 p = \int_{\mathbb{T}^3} p\,\partial_i \partial_j S_{ij} = \frac{1}{2}\int_{\mathbb{T}^3} p\,\Delta(\mathrm{div}\,u) = 0$; and, in Fourier variables, $\int_{\mathbb{T}^3} \mathrm{tr}(\nabla u\,\Delta \nabla u) = -\sum_k |k|^2\,|k \cdot \hat{u}(k)|^2 = 0$ since $k \cdot \hat{u}(k) = 0$ for the divergence-free field. These are the three identities of (18). □

The production law is the dynamic form of the dichotomy. The total imbalance is conserved: the strain and half-enstrophy carry equal mean, $\int_{\mathbb{T}^3} |S|^2 = \frac{1}{2}\int_{\mathbb{T}^3} |\omega|^2$, so $\int_{\mathbb{T}^3} q = 0$ at every time, and by (18) each of the inertial term $-6\det(\nabla u)$, the curvature work $-2\,S : \nabla^2 p$, and the viscous term redistributes the imbalance with zero net budget. The imbalance is produced locally and cancelled locally. A singular point approached through balance, where q vanishes in scaled average, therefore requires the inertial production and the curvature work to cancel pointwise up to the singular time against this zero-sum budget. The next section shows that this balance is rigid: under a Type-I bound it forces a pressureless blow-up limit, and the production law then governs that limit.

6 The balanced blow-up limit

The dichotomy of Section 5 leaves one regime the curvature does not detect, the balanced regime (16) where $r^{-1}\int_{Q_r} |q| \to 0$. We show this regime is rigid in the scale-critical class: a balanced singular point produces a bounded ancient solution of the Navier–Stokes equations with no pressure, the production law (17) governs that limit, and a Liouville theorem excludes it. The exclusion holds for every Type-I point by an elementary maximum principle, and for the strictly larger scale-critical L^3 class, the local form of the Escauriaza–Seregin–Šverák condition, through the Liouville theorem for divergence-free drifts.

6.1 Exclusion in the scale-critical class

Recall the parabolic rescaling at $z_0 = (x_0, t_0)$,

$$u^\lambda(x,t) = \lambda\, u(x_0 + \lambda x,\ t_0 + \lambda^2 t), \qquad p^\lambda(x,t) = \lambda^2\, p(x_0 + \lambda x,\ t_0 + \lambda^2 t), \tag{19}$$

which preserves the equations and the scaled dissipation, $\int_{Q_1} |\nabla u^\lambda|^2 = \lambda^{-1}\int_{Q_\lambda} |\nabla u|^2$, and likewise $\int_{Q_1} |q^\lambda| = \lambda^{-1}\int_{Q_\lambda} |q|$. The solution obeys a Type-I bound at z_0 if $|u(x,t)| \le M\,(t_0 - t)^{-1/2}$ on a backward parabolic neighbourhood, so that $|u^\lambda| \le M|t|^{-1/2}$ uniformly in λ.

Theorem 4 (balanced blow-up limit)**.** *Let u be a suitable weak solution of the Navier–Stokes equations on $\mathbb{T}^3$ that obeys a Type-I bound at a singular point z_0 and is approached through balance, $\lim_{r\to 0} r^{-1}\int_{Q_r} |q| = 0$. Then a subsequence of the rescalings (19) converges in C^2_{loc} to a bounded ancient solution U of the Navier–Stokes equations on $\mathbb{R}^3 \times (-\infty, 0)$ with*

$$q_U = |S_U|^2 - \tfrac{1}{2}|\omega_U|^2 \equiv 0, \qquad \nabla p_U \equiv 0, \tag{20}$$

so that U solves $\partial_t U + (U \cdot \nabla)U = \nu \Delta U$ with $\mathrm{div}\,U = 0$. The limit inherits the Type-I decay $|U(x,t)| \le M|t|^{-1/2}$ on $\mathbb{R}^3 \times (-\infty, 0)$, and $\int_{Q_1} |\nabla U|^2 \ge \varepsilon_0$.

Proof. The Type-I bound gives $|u^\lambda| \le M|t|^{-1/2}$ uniformly in λ, so $\{u^\lambda\}$ is bounded on every compact subset of $\mathbb{R}^3 \times (-\infty, 0)$. For a suitable weak solution this local boundedness propagates to uniform local bounds on the derivatives through the ε-regularity theory and the Type-I blow-up procedure of Seregin and Šverák [SŠ09; NRŠ96; CKN82], so a subsequence u^{λ_k} converges in C^2_{loc} to a bounded ancient solution U of the Navier–Stokes equations on $\mathbb{R}^3 \times (-\infty, 0)$ in the sense of [Koc+09]. Choose the subsequence to realise the singular lower bound $\limsup_{r\to 0} r^{-1} \int_{Q_r} |\nabla u|^2 \ge \varepsilon_0$ of (13); the scaled dissipation is invariant and lower semicontinuous under the convergence, so $\int_{Q_1} |\nabla U|^2 \ge \varepsilon_0$ and U is not constant. For the imbalance, the scaling $\int_{Q_R} |q^\lambda| = R\,\Theta_q(R\lambda)$ with $\Theta_q(\rho) = \rho^{-1} \int_{Q_\rho} |q|$, together with the balance hypothesis $\Theta_q(\rho) \to 0$, gives $\int_{Q_R} |q_U| = \lim_k \int_{Q_R} |q^{\lambda_k}| = 0$ for every R, so $q_U \equiv 0$ on $\mathbb{R}^3 \times (-\infty, 0)$. The bound $|u^\lambda| = \lambda|u| \le \lambda M(\lambda^2|t|)^{-1/2} = M|t|^{-1/2}$ passes to the limit, giving the Type-I decay $|U(x,t)| \le M|t|^{-1/2}$. Finally $q_U \equiv 0$ gives $\nabla^2 p_U = -\mathcal{R} q_U \equiv 0$ by (11); the limit is a mild ancient solution in the sense of [Koc+09], whose pressure is recovered from U with no spatially harmonic part, so $\nabla p_U \equiv 0$ and the momentum equation reduces to the stated advection–diffusion system. □

On the limit the production law sharpens to a pointwise constraint. Since $q_U \equiv 0$ gives $(\partial_t + U\cdot\nabla) q_U = 0$, and the curvature term in (17) carries the factor $\nabla^2 p_U \equiv 0$, the identity becomes

$$3 \det(\nabla U) = \nu \operatorname{tr}\big(\nabla U\, \Delta \nabla U\big) \qquad \text{on } \mathbb{R}^3 \times (-\infty, 0). \tag{21}$$

The pressureless limit thus carries two rigidities at once: the algebraic balance $|S_U|^2 = \frac{1}{2}|\omega_U|^2$ and the differential constraint (21). Since $q_U \equiv 0$ also gives $\Delta q_U = 0$, the product-rule identity $\Delta q = 2\operatorname{tr}(\nabla u\, \Delta \nabla u) + 2\,(\partial_k \partial_i u_j)(\partial_k \partial_j u_i)$ rewrites the constraint as

$$3 \det(\nabla U) = -\nu\, (\partial_k \partial_i U_j)(\partial_k \partial_j U_i) = -\nu \sum_k \operatorname{tr}\big((\partial_k \nabla U)^2\big). \tag{22}$$

Each $\partial_k \nabla U$ is trace-free, so the right side has no definite sign and the determinant is sign-indefinite: the differential constraint alone leaves the limit unforced. Closure comes from the boundedness of the ancient flow together with a decay or critical bound on the drift, which we now establish.

Lemma 2 (Liouville for the pressureless Type-I limit)**.** *A smooth divergence-free solution U of the pressureless system*

$$\partial_t U_i + U \cdot \nabla U_i = \nu \Delta U_i, \qquad i = 1, 2, 3, \tag{23}$$

on $\mathbb{R}^3 \times (-\infty, 0)$ that obeys the Type-I decay $|U(x,t)| \le M|t|^{-1/2}$ is identically zero.

Proof. Each component U_i solves the scalar equation (23), a uniformly parabolic equation whose drift U is divergence-free. On every strip $\mathbb{R}^3 \times [t_1, t_2]$ with $t_2 < 0$ the drift is bounded by $M|t_2|^{-1/2}$ and U_i is bounded, so the maximum principle for bounded solutions applies: $t \mapsto \sup_{\mathbb{R}^3} U_i(\cdot, t)$ is non-increasing and $t \mapsto \inf_{\mathbb{R}^3} U_i(\cdot, t)$ is non-decreasing. The Type-I decay gives $|U_i(\cdot, t)| \le M|t|^{-1/2} \to 0$ as $t \to -\infty$, so both the supremum and the infimum tend to zero, and by monotonicity

$$\sup_{\mathbb{R}^3} U_i(\cdot, t) \le 0 \le \inf_{\mathbb{R}^3} U_i(\cdot, t) \qquad \text{for every } t < 0,$$

whence $U_i \equiv 0$. Thus $U \equiv 0$. □

The maximum principle uses the Type-I decay only through its vanishing as $t \to -\infty$. The same conclusion holds without pointwise decay whenever the limit lies in the scale-critical class $L^\infty_t L^3_x$, through the Liouville theorem for divergence-free drifts of Seregin, Silvestre, Šverák and Zlatoš [Ser+12]. This is the local form of the critical condition of Escauriaza, Seregin and Šverák [ESŠ03b], and it contains the Type-I points while reaching strictly beyond them.

Lemma 3 (Liouville for the pressureless critical limit)**.** *A bounded smooth divergence-free solution U of the pressureless system* (23) *on $\mathbb{R}^3 \times (-\infty, 0)$ that lies in the scale-critical class*

$$\sup_{t<0} \|U(\cdot,t)\|_{L^3(\mathbb{R}^3)} < \infty \tag{24}$$

is constant.

Proof. Each component U_i solves $\partial_t U_i + U \cdot \nabla U_i = \nu \Delta U_i$, a uniformly parabolic equation with the divergence-free drift $b = U$. Write $b = \operatorname{div} d$ for the skew-symmetric tensor potential d of U. Then

$$\operatorname{div}(d\,\nabla U_i) = (\partial_j d_{jk})\,\partial_k U_i + d_{jk}\,\partial_j \partial_k U_i = U_k\,\partial_k U_i = U \cdot \nabla U_i,$$

since $\partial_j d_{jk} = b_k = U_k$ and the skew tensor annihilates the symmetric second derivative. Hence $U \cdot \nabla U_i = \operatorname{div}(d\,\nabla U_i)$ and U_i solves the divergence-form equation $\partial_t U_i = \operatorname{div}\big((\nu\mathbb{I} - d)\,\nabla U_i\big)$ with leading coefficient $\nu\mathbb{I} - d$ of uniformly elliptic symmetric part and skew part $-d$. The potential d is a matrix Riesz potential of U, and the endpoint estimate $I_1\colon L^3(\mathbb{R}^3) \to \mathrm{BMO}(\mathbb{R}^3)$ gives, from (24),

$$d \in L^\infty\big((-\infty,0); \mathrm{BMO}(\mathbb{R}^3; \mathbb{R}^{3\times 3})\big).$$

The Liouville theorem of Seregin, Silvestre, Šverák and Zlatoš [Ser+12] states that, for such a leading coefficient, the only bounded ancient suitable weak solutions are the constants, so each U_i is constant and U is constant. □

Theorem 5 (no balanced Type-I singularity)**.** *A suitable weak solution of the Navier–Stokes equations on $\mathbb{T}^3$ that obeys a Type-I bound has no balanced singular point. Every Type-I singular point is curvature-concentrating,*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r} |q| \,\mathrm{d}x\,\mathrm{d}t > 0. \tag{25}$$

Proof. Suppose a Type-I singular point were approached through balance. By Theorem 4 the rescalings converge to a pressureless limit U that obeys the Type-I decay $|U(x,t)| \le M|t|^{-1/2}$ and satisfies $\int_{Q_1} |\nabla U|^2 \ge \varepsilon_0$. Lemma 2 forces $U \equiv 0$, so $\int_{Q_1} |\nabla U|^2 = 0$, a contradiction. The balanced alternative of Theorem 3 therefore cannot occur at a Type-I singular point, which leaves (25). □

The critical Liouville lemma extends the exclusion from the pointwise Type-I points to the scale-critical class, the local form of the condition of Escauriaza, Seregin and Šverák.

Theorem 6 (no balanced scale-critical singularity)**.** *Let u be a suitable weak solution of the Navier–Stokes equations on $\mathbb{T}^3$ with a singular point z_0 at which the scale-critical bound*

$$\limsup_{r\to 0} \sup_{t_0 - r^2 < t < t_0} \|u(\cdot,t)\|_{L^3(B_r(x_0))} < \infty \tag{26}$$

holds. Then z_0 is curvature-concentrating, $\limsup_{r\to 0} r^{-1} \int_{Q_r} |q| > 0$. The class (26) *contains the Type-I points and is the local form of the Escauriaza–Seregin–Šverák critical condition.*

Proof. Suppose z_0 were approached through balance. Under (26) the rescalings (19) satisfy $\|u^\lambda(\cdot,t)\|_{L^3(B_R)} = \|u(\cdot,t_0+\lambda^2 t)\|_{L^3(B_{\lambda R}(x_0))}$, bounded uniformly in λ for every fixed R as $\lambda\to 0$, so $\{u^\lambda\}$ is bounded in $L^\infty_t L^3_x$ on every cylinder. The ε-regularity theory in the critical class [ESŠ03b; SŠ09] upgrades this to uniform local bounds on u^λ and its derivatives, so a subsequence converges in C^2_{loc} to a bounded smooth ancient solution U on $\mathbb{R}^3\times(-\infty,0)$ with $\sup_{t<0}\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}<\infty$ and $\int_{Q_1}|\nabla U|^2 \ge \varepsilon_0$. The balance hypothesis gives $q_U\equiv 0$ as in Theorem 4, so $\nabla^2 p_U = -\mathcal{R}q_U \equiv 0$; the limit is a mild ancient solution in the sense of [Koc+09], whose pressure is recovered from U with no spatially harmonic part, so $\nabla p_U\equiv 0$ and the momentum equation reduces to the pressureless system (23). By Lemma 3 the limit is constant, contradicting $\int_{Q_1}|\nabla U|^2\ge\varepsilon_0$. □

6.2 Beyond the critical class

Lemmas 2 and 3 close the balanced alternative for the scale-critical class, where the drift U lies in $L^\infty_t L^3_x$ and so in $L^\infty_t \mathrm{BMO}^{-1}_x$. At a singular point outside that class the rescaling still produces a bounded ancient pressureless flow U with $q_U\equiv 0$, but the drift is now only bounded, with no spatial decay, and the divergence-free Liouville theorem no longer applies. We record the structure that bears on this supercritical Liouville problem: it carries the balance $q_U\equiv 0$, the constraint (21), and an energy supersolution.

Lemma 4 (the gradient energy is the enstrophy)**.** *On the limit U the balance $q_U\equiv 0$ gives*

$$|\nabla U|^2 = |\omega_U|^2, \qquad |S_U|^2 = \tfrac{1}{2}|\omega_U|^2. \tag{27}$$

Proof. By the gradient split (12), $|\nabla U|^2 = q_U + |\omega_U|^2 = |\omega_U|^2$, and $q_U = |S_U|^2 - \frac{1}{2}|\omega_U|^2 = 0$ is the second equality. □

Lemma 5 (Cayley–Hamilton rigidity)**.** *On the balanced limit the velocity gradient $A=\nabla U$ satisfies the pointwise identity*

$$(\nabla U)^3 = \det(\nabla U)\,\mathbb{I}. \tag{28}$$

Its eigenvalues are $\{c,\, c\zeta,\, c\zeta^2\}$ with $\zeta = e^{2\pi i/3}$ and $c^3 = \det(\nabla U)$, so the gradient is nilpotent where the pressureless flow has no inertial production and is conjugate to a fixed cube-root pattern elsewhere.

Proof. The characteristic polynomial of the 3×3 matrix A is $\lambda^3 - (\operatorname{tr} A)\lambda^2 + \frac{1}{2}\big((\operatorname{tr} A)^2 - \operatorname{tr} A^2\big)\lambda - \det A$. Incompressibility gives $\operatorname{tr} A = 0$ and the balance $q_U = \operatorname{tr} A^2 = 0$, so the polynomial reduces to $\lambda^3 - \det A$ and its roots are the three cube roots of $\det A$. The Cayley–Hamilton theorem gives (28). □

The drift equation carries one more piece of structure, unconditional and not visible to the energy identity. It localises the whole Liouville question onto a single quantity, the oscillation of the limit in the infinite past.

Lemma 6 (componentwise monotonicity)**.** *For a bounded ancient pressureless limit U, each component has monotone spatial extremes: $t\mapsto \sup_{\mathbb{R}^3} U_i(\cdot,t)$ is non-increasing and $t\mapsto \inf_{\mathbb{R}^3} U_i(\cdot,t)$ is non-decreasing on $(-\infty,0)$, so the oscillation*

$$\operatorname{osc}_i(t) = \sup_{\mathbb{R}^3} U_i(\cdot,t) - \inf_{\mathbb{R}^3} U_i(\cdot,t) \tag{29}$$

is non-increasing in time. Hence U is constant if and only if the past oscillation vanishes, $\lim_{t\to-\infty}\operatorname{osc}_i(t) = 0$ for each i.

Proof. Each component solves $\partial_t U_i + U \cdot \nabla U_i = \nu \Delta U_i$, a uniformly parabolic equation with the divergence-free drift U and no zeroth-order term. For a bounded solution on $\mathbb{R}^3 \times [t_1, t_2]$ the weak maximum principle gives $\sup_{\mathbb{R}^3} U_i(\cdot, t) \le \sup_{\mathbb{R}^3} U_i(\cdot, t_1)$ and $\inf_{\mathbb{R}^3} U_i(\cdot, t) \ge \inf_{\mathbb{R}^3} U_i(\cdot, t_1)$ for $t_1 \le t \le t_2$, which is the monotonicity, and osc_i is non-increasing as the difference of a non-increasing and a non-decreasing function. If $\mathrm{osc}_i(t) \to 0$ as $t \to -\infty$ then, being non-increasing and non-negative, osc_i vanishes identically, so each $U_i(\cdot, t)$ is constant in space; the momentum equation gives $\partial_t U_i = 0$, and U is constant. The converse is immediate. □

The Type-I decay $|U(\cdot, t)| \le M|t|^{-1/2}$ sends both extremes to zero as $t \to -\infty$, so it forces $\mathrm{osc}_i(-\infty) = 0$ and recovers Lemma 2. In the scale-critical class the divergence-free drift Liouville theorem closes the same gap. The supercritical obstruction is the past oscillation $\mathrm{osc}_i(-\infty)$, a single number per component that the energy identity leaves undetermined.

Lemma 7 (energy supersolution)**.** *The kinetic energy $\Phi = \frac{1}{2}|U|^2$ of the limit is a bounded ancient supersolution of the drift–heat operator with bounded divergence-free drift U,*

$$\big(\partial_t + U \cdot \nabla - \nu\Delta\big)\Phi = -\nu|\nabla U|^2 = -\nu|\omega_U|^2 \le 0. \tag{30}$$

Proof. With $\nabla p_U \equiv 0$ the momentum equation gives $(\partial_t + U \cdot \nabla)\frac{1}{2}|U|^2 = U \cdot \nu\Delta U$, and $\nu\Delta\frac{1}{2}|U|^2 = \nu\, U \cdot \Delta U + \nu|\nabla U|^2$, whose difference is (30); the last equality is Lemma 4. □

The operator $\partial_t + U \cdot \nabla - \nu\Delta$ is uniformly parabolic in divergence form, since $U \cdot \nabla\Phi = \mathrm{div}(U\Phi)$ for the divergence-free drift, and its coefficients are bounded since U is bounded. The supersolution already constrains the limit unconditionally.

Lemma 8 (vanishing mean dissipation)**.** *Write $Q_R = B_R \times (-R^2, 0)$ for the parabolic cylinder of radius R in the ancient domain. The bounded limit U satisfies*

$$\int_{Q_R} |\nabla U|^2 \le C\, R^4, \qquad C = C\big(\nu, \|U\|_{L^\infty}\big), \tag{31}$$

so the mean dissipation $|Q_R|^{-1} \int_{Q_R} |\nabla U|^2 = O(R^{-1})$ tends to zero as $R \to \infty$.

Proof. Let η be a cutoff with $\eta \equiv 1$ on Q_R, $\mathrm{supp}\,\eta \subset Q_{2R}$, $0 \le \eta \le 1$, $|\nabla\eta| \le C/R$ and $|\partial_t\eta| \le C/R^2$. Multiplying the energy identity (30) by η^2, integrating over $\mathbb{R}^3 \times (-\infty, 0)$ and moving derivatives onto η^2 by parts, with $\Phi = \frac{1}{2}|U|^2$ and $\mathrm{div}\, U = 0$,

$$\nu \int |\nabla U|^2 \eta^2 = \int \Phi\, \partial_t(\eta^2) + \int \Phi\, U \cdot \nabla(\eta^2) - \nu \int \nabla\Phi \cdot \nabla(\eta^2).$$

Using $\Phi \le \frac{1}{2}\|U\|_\infty^2$ and $|\nabla\Phi| \le \|U\|_\infty |\nabla U|$, the first two integrals are bounded on Q_{2R} by $C\|U\|_\infty^2 R^3$ and $C\|U\|_\infty^3 R^4$, and the third by $\frac{\nu}{2}\int |\nabla U|^2\eta^2 + C\nu\|U\|_\infty^2 R^3$ through the Cauchy–Schwarz and Young inequalities. Absorbing the gradient term on the left gives (31). □

The vorticity of the limit carries the structure on which the critical theory turns, and it does so unconditionally.

Lemma 9 (vorticity differential inequality)**.** *The bounded ancient limit is smooth with bounded U and ∇U, and its vorticity $\omega_U = \operatorname{curl} U$ obeys the viscous vorticity equation $\partial_t\omega_U + U\cdot\nabla\omega_U = \omega_U\cdot\nabla U + \nu\Delta\omega_U$, hence the differential inequality*

$$|\partial_t\omega_U - \nu\Delta\omega_U| \le C\big(|\omega_U| + |\nabla\omega_U|\big), \qquad C = \max\big(\|U\|_{L^\infty}, \|\nabla U\|_{L^\infty}\big), \tag{32}$$

on $\mathbb{R}^3 \times (-\infty, 0)$.

Proof. Each component solves $\partial_t U_i + U\cdot\nabla U_i = \nu\Delta U_i$ with bounded drift U, so the De Giorgi–Nash–Moser theory gives $U \in C^\alpha$ and the Schauder estimates bootstrap to bounded U and ∇U. Taking the curl of the momentum equation removes any pressure gradient and gives the stated vorticity equation, using $\operatorname{curl}(U\cdot\nabla U) = U\cdot\nabla\omega_U - \omega_U\cdot\nabla U$ for the divergence-free field. The bounds on U and ∇U give (32). □

Inequality (32) is the hypothesis of the backward-uniqueness theory of Escauriaza, Seregin and Šverák [ESŠ03a; ESŠ03b]. Suppose the vorticity decays at spatial infinity at some time, $\omega_U(x,t) \to 0$ as $|x| \to \infty$ uniformly for $t \le t_0$. The Carleman backward-uniqueness estimate then forces $\omega_U \equiv 0$ on an exterior $\{|x| > R\} \times (-\infty, t_0)$, spatial unique continuation propagates the zero, and $\omega_U \equiv 0$; a bounded divergence-free curl-free field is constant, so U is constant. Every ingredient of this argument holds unconditionally for the bounded ancient limit: the differential inequality, the boundedness, and the parabolic regularity. The one input it requires is the decay of ω_U at spatial infinity, and the pressureless structure does not supply it. A shear $U = (f(y,t), 0, 0)$ with $\partial_t f = \nu\,\partial_y^2 f$ is pressureless and divergence-free with vorticity $\omega_U = (0, 0, -\partial_y f)$ independent of x and z, carrying no decay, while the maximum principle of Lemma 6 still makes it constant through the caloric Liouville property. The past oscillation $\operatorname{osc}_i(-\infty)$ and the spatial decay of the vorticity are the two faces of the supercritical wall.

A non-constant such limit concentrates its gradient: by Lemma 8 the dissipation vanishes in mean over large cylinders. The obstruction to a Liouville theorem is one of scale. The scale-critical class $L_t^\infty L_x^3$ is the setting in which the divergence-free drift lies in $L_t^\infty \mathrm{BMO}_x^{-1}$ and the parabolic Harnack inequality of Seregin, Silvestre, Šverák and Zlatoš holds, which is what Lemma 3 uses. This threshold is sharp: a divergence-free drift marginally beyond BMO^{-1} already admits a non-constant bounded ancient solution [Ser+12]. The remaining case is the supercritical one, a limit that is bounded but carries no spatial decay, so its drift sits in L_x^∞ alone, where the scale-invariant size $R\|U\|_\infty$ grows with R and the Harnack constant degrades. The Liouville property for bounded ancient pressureless flows with drift in L_x^∞ is the open three-dimensional case, settled in the two-dimensional and axisymmetric classes where the drift is effectively lower-dimensional [Koc+09]. The constraint (21), the balance (27), and the Cayley–Hamilton rigidity (28) are the extra structure the general bounded drift does not carry, and they are the data of the equation that an energy-preserving averaging discards (Section 7.3).

7 Discussion

7.1 Comparison with the coherence criterion

The Constantin–Fefferman criterion [CF93] controls the stretching scalar $\alpha = \xi^\top S\,\xi$ under a coherence hypothesis on the vorticity direction ξ, and so secures regularity throughout the regime of coherent vorticity. The curvature governs the complementary regime, and it does so through data independent of the coherence of ξ. To see the independence, take a columnar field $u = (\,-\partial_y\psi,\ \partial_x\psi,\ 0\,)$ with stream function $\psi = \psi(x, y)$ on $\mathbb{T}^3$, independent of z. Its vorticity is $\omega = (0, 0, \Delta\psi)$, so the vorticity direction $\xi = \pm e_3$ is constant and the coherence hypothesis holds with $\nabla\xi \equiv 0$. The imbalance $q = |S|^2 - \frac{1}{2}|\omega|^2$ of such a field is generically nonzero, so its curvature does not vanish. Such a two-dimensional field is globally regular, so the example separates the two data and does not exhibit a singular regime. Coherence of the vorticity direction and the strain–enstrophy imbalance are independent data, and the curvature reads the second where the criterion of Constantin and Fefferman reads the first.

7.2 The balanced regime

The regime of (16) is the one in which the strain energy and the enstrophy concentrate together, $|S|^2$ and $\frac{1}{2}|\omega|^2$ sharing a common scaled limit, equivalently the imbalance q and the pressure both depleting in scaled average. The velocity-gradient dynamics has long associated the approach to intense events with a depletion of the nonlinearity and a balancing of strain against rotation: the local models of the velocity-gradient tensor of Vieillefosse and Cantwell, and the analysis of vortex stretching of Ohkitani and Kishiba, all turn on the nonlocal pressure Hessian that Theorem 1 identifies with the curvature [Vie82; Can92; OK95; Con94]. Theorem 3 places that picture inside the configuration-space geometry: by Theorem 2 the curvature is the imbalance, so the balanced regime is the regime in which the curvature of $\mathrm{SDiff}(\mathbb{T}^3)$ stays finite at the singular point.

The result is structural. The coercive criterion of Proposition 2 secures regularity while the flow stays away from balance, and the production law (17) shows the balanced regime is sustained only by a cancellation of the inertial source against the curvature work. Theorems 5 and 6 turn that cancellation into rigidity: in the scale-critical class the balanced regime is a bounded ancient pressureless limit whose divergence-free drift is critical, excluded by a Liouville theorem, by an elementary maximum principle at a Type-I point and by the divergence-free drift theorem in the larger class. The geometry thus locates the difficulty precisely. In the scale-critical class the only undetected behaviour, strain–enstrophy balance, is ruled out, so every scale-critical singular point is curvature-concentrating. Beyond that class the balanced regime is the supercritical Liouville problem for bounded ancient pressureless flows, open in three dimensions and settled in the symmetry classes where the swirl-free flow is already regular.

7.3 Structure beyond the energy identity

The exclusion of the balanced regime in the scale-critical class (Theorem 6) rests on the Liouville theorem for divergence-free drifts, which belongs to the critical regularity theory reachable from the energy identity and the scaling. The balanced regime in full generality is supercritical, and there a structural input is required. The averaged equation of Tao makes this precise: there is a bilinear operator $\tilde{B}$ that obeys the energy cancellation $\langle \tilde{B}(u, u), u\rangle = 0$, shares the scaling and the upper-bound estimates of the Euler operator $B(u, u) = \mathbb{P}[(u \cdot \nabla)u]$, and yet admits a smooth solution that blows up in finite time [Tao16]. Any argument resting on the energy identity, the scaling, and upper bounds for the nonlinearity applies equally to $\tilde{B}$, so it cannot reach global regularity for B. A proof must read structure of B that the averaging removes.

The curvature supplies such structure. The velocity gradient $A = \nabla u$ evolves by $(\partial_t + u \cdot \nabla)A = -A^2 - \nabla^2 p + \nu\Delta A$, in which the self-stretching $-A^2$ and the pressure Hessian $\nabla^2 p = -\mathcal{R}q$ are the bilinear and nonlocal data of the Euler operator. The production law (17) is their trace through the imbalance, and the imbalance $q = |S|^2 - \frac{1}{2}|\omega|^2$ is the size of the curvature (Theorem 2). The averaging that produces $\tilde{B}$ replaces the pressure projection by a multiplier of order zero and rotates the pointwise product structure, so it carries neither the Riesz identity $\nabla^2 p = -\mathcal{R}q$ nor the self-stretching determinant $\det(\nabla u)$. The curvature, and the balanced regime it governs, are data of the unaveraged equation lying outside the reach of the energy identity. The scale-critical exclusion occupies the side of the dichotomy the critical theory already reaches; the curvature locates the remaining, supercritical side and reads it through the pressure.

That supercritical side reduces to a single question (Section 6.2): whether a bounded ancient pressureless flow, $\partial_t U + U \cdot \nabla U = \nu \Delta U$ with $q_U \equiv 0$ and $\nabla p_U \equiv 0$, is constant. The reduction $q_U \equiv 0 \Leftrightarrow \nabla p_U \equiv 0$ is the Riesz identity again, so the reduced problem inherits the pressure structure that the averaging does not preserve. The barrier leaves this gateway open to a structural argument, and the constraint (21), the balance (27), the Cayley–Hamilton rigidity (28), and the energy supersolution (30) are the structural data it carries. The maximum principle localises the question further: by Lemma 6 the limit is constant precisely when its oscillation in the infinite past vanishes, so the supercritical obstruction is one number per component, beyond the reach of the energy identity.

8 Conclusion

The curvature of the volume-preserving diffeomorphism group is the pressure Hessian (Theorem 1), and the Gauss equation places its origin in the second fundamental form of incompressibility. Its size is the strain–enstrophy imbalance (Theorem 2), so it vanishes on the pressureless flows and reproduces Arnold's same-shell degeneracy. Read against the partial-regularity theory, the curvature controls regularity away from balance (Proposition 2), and a singular point with bounded curvature is approached through balance, equivalently through the depletion of the pressure (Theorem 3).

The geometry thus identifies the balanced, pressureless regime as the route to singularity it does not see, and the production law (17) shows that route is sustained only by a fine cancellation. In the scale-critical class Theorem 6 renders the balanced regime as a bounded ancient pressureless flow whose divergence-free drift is critical, which the Liouville theorem for such drifts excludes, so every scale-critical singular point, the Type-I points among them, is curvature-concentrating. Beyond the critical class the balanced regime is supercritical. There the energy identity is blind, and the averaged equation of Tao [Tao16] shows that regularity is closed to any argument that the energy identity and the scaling can supply. The curvature is the structure of the nonlinearity that lies past this barrier: the production law, the constraint (21), the Cayley–Hamilton rigidity (28), and the energy supersolution (30) are data an energy-preserving averaging discards. They reduce the supercritical balanced regime to one question, whether a bounded ancient pressureless flow is constant, and they are the structural input with which to answer it.

A Symbolic and spectral verification

The identities of Sections 2–5 are reproduced by an independent computation, available in the project repository recorded in the data and code availability statement. The check runs on divergence-free test fields from distinct families, an Arnold–Beltrami flow and a perturbation of it, a pair of anti-parallel vortex tubes, and a helical blob, so that the agreement is not an artefact of a single symmetry. It establishes the following.

1. The advective commutator (9), and with it the vanishing of the flat ambient part (6), holds symbolically for arbitrary smooth fields.

2. The reduced identity (7) reproduces the curvature (5) to relative error 6×10^{-14} on the band-limited fields, the residual tracking the spectral truncation floor.

3. The trace identity $\operatorname{tr} \nabla^2 p = -q$ and the gradient split (12) hold pointwise to order 10^{-15}.

4. The Gauss equation (8) holds to order 10^{-16}.

5. The identity $\|\nabla^2 p\|_{L^2} = \|q\|_{L^2}$ holds to ten significant figures, and the pointwise bound $|q| \le \sqrt{3}\,|\nabla^2 p|$ is confirmed.

6. The coercivity bound $|q| \le |\nabla u|^2$ of (12) holds pointwise, and the no-net-work identity $\int_{\mathbb{T}^3} S : \nabla^2 p = 0$ of (18) holds to order 10^{-9}.

7. The production law (17) is reproduced by two independent computations of the material derivative, by the definition through the Navier–Stokes right-hand side and by the right side of (17), agreeing to relative error 10^{-13} on the band-limited fields; the three zero-mean identities (18), for the determinant, the curvature work, and $\operatorname{tr}(\nabla u\, \Delta \nabla u)$, hold to machine precision.

8. The energy supersolution identity (30), $(\partial_t + U \cdot \nabla - \nu\Delta)\frac{1}{2}|U|^2 = -\nu|\nabla U|^2$, holds to order 10^{-6} on the sharp-profile fields and to machine precision on the band-limited ones. The product-rule identity $\Delta q = 2\operatorname{tr}(\nabla u\, \Delta \nabla u) + 2(\partial_k \partial_i u_j)(\partial_k \partial_j u_i)$ behind the reformulation (22) holds to machine precision, and the contraction $(\partial_k \partial_i u_j)(\partial_k \partial_j u_i)$ takes both signs, so the constraint carries no one-sided bound on the determinant.

The spectral checks use a Fourier pseudo-spectral discretisation on a 48^3 grid; the symbolic check in (i) is exact.

Declarations

Competing interests. The author declares that he has no competing interests.

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Data and code availability. This work has no associated datasets. The symbolic and spectral verification described in Appendix A is reproduced by an openly available script, `sympy/cf2_curvature_pressure_channel_verify.py`, in the project repository at `github.com/alejandro-soto-franco/navier-stokes`.

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