THE VORTICITY DIRECTION DYADIC FOR THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

ALEJANDRO JOSÉ SOTO FRANCO

Abstract. We analyze the incompressible Navier–Stokes equations on $\mathbb{R}^3$ through the vorticity direction dyadic $\Xi = \xi \otimes \xi$ and an associated scale–invariant entropy that couples its nonlocal Calderón–Zygmund curvature with a Fisher-type term. Using a harmonic–analytic representation of the curvature operator, we establish a monotone expression for this entropy in Fourier space. For any parabolic blow-up sequence of a Leray–Hopf solution, the entropy passes to the ancient limit and the equality case in the monotonicity formula yields a linear frequency–space relation whose tempered solutions are Gaussian self-similar profiles. Such Gaussian dyadic profiles are incompatible with the divergence-free structure, the Biot–Savart decay, and the Leray–Hopf energy inequality, and hence any ancient blow-up limit must be trivial. This furnishes a geometric–harmonic spectral rigidity mechanism that precludes nontrivial blow-up limits for Leray–Hopf solutions on $\mathbb{R}^3$.

Contents

1. Introduction

Nearly two centuries after their formulation, the global regularity problem for the three-dimensional incompressible Navier–Stokes equations remains open [1–5]. Leray–Hopf weak solutions exist globally for all finite–energy initial data [2, 3], yet it is unknown whether such solutions can develop finite-time singularities. If blow-up were to occur, parabolic rescaling produces ancient limit profiles, but the classical formulation offers no mechanism forcing rigidity or triviality of these limits [6, 7]. A successful resolution must therefore identify a geometric quantity that: (i) survives weak and parabolic limits, (ii) admits a closed evolution compatible with harmonic analysis, and (iii) supports a monotonicity principle strong enough to enforce rigidity of all ancient blow-up limits.

A natural candidate in the vorticity formulation is the direction field $\xi = \omega/|\omega|$, which governs vortex stretching and whose spatial coherence has regularizing effects [8]. However, ξ is analytically unstable: it is undefined on the vacuum set $\{\omega = 0\}$, fails to be weakly compact, and does not behave well under parabolic rescaling. As a result, ξ cannot serve as a blow-up–stable geometric variable.

To overcome these obstructions, we introduce the *vorticity direction dyadic*

$$\Xi = \xi \otimes \xi \in \mathrm{Sym}_3^+,$$

a rank-one projector taking values in the Veronese surface $V_2(\mathbb{S}^2)$. Unlike ξ, the dyadic field admits weak-$*$ limits as an Sym_3^+-valued Radon measure and is preserved by the structure of the Navier–Stokes nonlinearity: Biot–Savart, the pressure projection, and the stretching term all act linearly or bilinearly on Ξ. This makes Ξ the minimal scale-invariant geometric lift of vorticity direction compatible with weak compactness.

We derive an intrinsic evolution equation for Ξ featuring a nonlocal Calderón–Zygmund curvature operator:

$$\mathcal{K}_{ij}[\Xi](x) = \mathrm{p.\,v.}\int_{\mathbb{R}^3} K_{iab}(x-y)\, K_{jcd}(x-y)\, \Xi_{ac}(y)\, \omega_b(y)\, \omega_d(y)\, \mathrm{d}y,$$

where K_{iab} is the kernel associated with the Biot–Savart gradient. This operator encodes the dyadic component of vortex stretching, is homogeneous of degree 2 under parabolic scaling, and is stable under the weak limits generated by Leray–Hopf solutions. In particular, the stretching and diffusion terms yield a closed evolution for Ξ modulo Calderón–Zygmund operators, a crucial feature for the harmonic–analytic approach developed here.

To detect concentration and extract rigidity we introduce a scale-invariant *dyadic entropy*

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \Big(\mathcal{K}_{ij}[\Xi]_\tau(x)\, \Xi_{\tau,ij}(x) + |\nabla \Xi_\tau(x)|^2\Big) |\omega_\tau(x)|\, G_\tau(x)\, \mathrm{d}x,$$

where $\tau = -t$ is backward time and G_τ is the backward heat kernel. The integrand couples a nonlocal curvature density with a Fisher-type Dirichlet term, weighted by the effective Gaussian density $\tau|\omega_\tau|\, G_\tau$. The choice of weights is dictated by the parabolic scaling symmetry: each term is dimensionless.

A key advantage of the dyadic formulation is its compatibility with Fourier analysis. The curvature operator admits the representation

$$\widehat{\mathcal{K}_{ij}[\Xi]}(\eta) = M_{ijacbd}(\eta)\, \widehat{\Xi_{ac}\, \omega_b \omega_d}(\eta), \qquad M(\eta) = m(\eta) \otimes m(\eta),$$

where $m(\eta)$ is the Calderón–Zygmund symbol for the Biot–Savart gradient. Weighted by $\widehat{G_\tau}(\eta) = e^{-\tau|\eta|^2}$, the entropy admits the frequency-space form

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \Big(M(\eta)\, \widehat{\Xi\omega\omega}(\eta,-\tau) : \overline{\widehat{\Xi}(\eta,-\tau)} + |\eta|^2\, |\widehat{\Xi}(\eta,-\tau)|^2\Big) e^{-\tau|\eta|^2}\, \mathrm{d}\eta.$$

Differentiating this expression and using the evolution law for Ξ produces a harmonic–analytic perfect-square identity:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}(\tau) = 2\tau^2 \int_{\mathbb{R}^3} \Big| i\eta\, \widehat{\Xi}(\eta,-\tau) + M(\eta)\widehat{\Xi\omega\omega}(\eta,-\tau) + \tfrac{1}{2\tau}\widehat{\Xi}(\eta,-\tau)\Big|^2 e^{-\tau|\eta|^2}\, \mathrm{d}\eta.$$

This is the dyadic Calderón–Zygmund analogue of Perelman's monotonicity formula for Ricci flow [9]: the entropy is nondecreasing and achieves equality only when the expression inside the square bracket vanishes identically.

The equality case yields a fully linear constraint in frequency space. Let $(\omega_\infty, \Xi_\infty)$ be an ancient limit obtained from parabolic blow-up around a hypothetical singularity [6, 7]. If $\mathcal{W}_\infty$ is constant on a time interval, then

$$i\eta\,\widehat{\Xi_\infty}(\eta,-\tau) + M(\eta)\,\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau) + \tfrac{1}{2\tau}\widehat{\Xi_\infty}(\eta,-\tau) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^3),$$

a first-order affine relation in η. Solutions of this equation compatible with the dyadic measure constraint are necessarily Gaussian:

$$|\omega_\infty(x,-\tau)| = C(\tau)e^{-|x|^2/4\tau}, \qquad \Xi_\infty(x,-\tau) = P,$$

where P is a rank-one projector. These dyadic Gaussian profiles violate incompressibility unless $C(\tau) \equiv 0$, forcing the ancient limit to be trivial.

Thus the dyadic entropy furnishes a *spectral rigidity principle*: any ancient solution saturating the entropy monotonicity must vanish identically. Since a genuine singularity would produce a nontrivial ancient blow-up limit, finite-time singularities cannot occur (Corollary 6.5).

2. Preliminaries

We work on $\mathbb{R}^3$ equipped with Lebesgue measure. Indices range from 1 to 3 and repeated indices are summed. All Fourier transforms are taken in the spatial variable unless stated otherwise. The viscosity is normalized to 1 throughout. We write $\mathcal{D}'(\Omega)$ for the space of distributions on an open set $\Omega \subset \mathbb{R}^3 \times \mathbb{R}$.

2.1. Vorticity, Leray–Hopf solutions, and Biot–Savart.

The vorticity is defined by

$$\omega_i = \varepsilon_{ijk}\,\partial_j u_k, \tag{2.1}$$

where ε_{ijk} is the Levi–Civita tensor. The incompressible Navier–Stokes equations are written in velocity form as

$$\partial_t u_i + u_j\partial_j u_i + \partial_i p = \Delta u_i, \quad \partial_i u_i = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)), \tag{2.2}$$

where the pressure is determined (up to a constant) by

$$-\Delta p = \partial_i\partial_j(u_i u_j) \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times (0,T)). \tag{2.3}$$

The initial datum of (2.2) is $u_0 \in L^2(\mathbb{R}^3)$, $\partial_i u_{0,i} = 0$ (divergence-free).

Definition 2.1 (Leray–Hopf solution)**.** A divergence-free field $u_i : \mathbb{R}^3 \times (0,T) \to \mathbb{R}$ is a *Leray–Hopf solution* if

$$u_i \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1),$$

the global energy inequality holds,

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s \;\le\; \|u_0\|_{L^2}^2, \tag{2.4}$$

u satisfies the Navier–Stokes equations (2.2) in the sense of distributions, and $u(t) \rightharpoonup u_0$ in $L^2(\mathbb{R}^3)$ as $t \downarrow 0$.

The velocity is recovered from vorticity through the Biot–Savart law,

$$u_i(x,t) = \text{p. v.}\int_{\mathbb{R}^3} B_{ij}(x-y)\,\omega_j(y,t)\,\mathrm{d}y, \tag{2.5}$$

where $B_{ij}(z)$ is homogeneous of degree -2, odd, and satisfies $z_i B_{ij}(z) = 0$. Differentiating (2.5) yields the Calderón–Zygmund representation

$$\partial_k u_i(x,t) = \text{p. v.} \int_{\mathbb{R}^3} K_{ijk}(x-y)\, \omega_j(y,t)\, \mathrm{d}y, \tag{2.6}$$

where K_{ijk} is homogeneous of degree -3, cancels on the sphere, and satisfies the Hörmander condition[1].

Definition 2.2 (Vorticity equation)**.** The vorticity evolution satisfies

$$\partial_t \omega_i + u_j \partial_j \omega_i = \omega_j \partial_j u_i + \Delta \omega_i \quad \text{in } \mathcal{D}'. \tag{2.7}$$

The pressure term disappears due to $\varepsilon_{ijk}\partial_j\partial_i p = 0$, reflecting that vorticity evolution is driven solely by *stretching* and *diffusion*. This decomposition will be relevant in Subsection 3.2.

Remark 2.3 (Formal identities)*.* One has formally (momentarily ignoring issues of differentiating distributional identities)

$$\partial_t \omega_i = \varepsilon_{ijk}\, \partial_j(\partial_t u_k), \qquad u_j \partial_j \omega_i = \varepsilon_{ijk}\, \partial_j(u_\ell \partial_\ell u_k),$$

and

$$\omega_j \partial_j u_i = \varepsilon_{jmn}\, (\partial_m u_n)(\partial_j u_i).$$

2.2. **Parabolic scaling and backward cylinders.**

Definition 2.4 (Parabolic scaling)**.** For $\lambda > 0$ define

$$u_i^{(\lambda)}(x,t) = \lambda\, u_i(\lambda x, \lambda^2 t), \qquad \omega_i^{(\lambda)}(x,t) = \lambda^2\, \omega_i(\lambda x, \lambda^2 t). \tag{2.8}$$

If (u,p) solves (2.2) on $(0,T)$, then $(u^{(\lambda)}, p^{(\lambda)})$ solves (2.2) on $(0, \lambda^{-2}T)$ with vorticity $\omega^{(\lambda)}$.

Definition 2.5 (Backward cylinders)**.**

$$Q_r(x_0,t_0) = \big\{(x,t) : |x - x_0| < r,\ \ t_0 - r^2 < t < t_0\big\}.$$

Note that $Q_r(x_0,t_0)$ rescales to $Q_1(0,0)$ under the parabolic map $(x,t) \mapsto \big((x - x_0)/r, (t-t_0)/r^2\big)$.

2.3. **The dyadic field and dyadic measure.**

Definition 2.6 (Dyadic field)**.** For a nonzero vector $w \in \mathbb{R}^3$, define the rank-one projector

$$\Phi_{ij}(w) = \frac{w_i}{|w|}\frac{w_j}{|w|}, \qquad \Phi_{ij}(0) = 0.$$

Given a vorticity field ω in the sense of (2.1), set

$$\Xi_{ij} = \Phi_{ij}(\omega). \tag{2.9}$$

Thus $\Xi(x) \in \mathrm{Sym}_3^+$ is a spatially varying projector identifying the vorticity direction[2]. We may define $\Phi(0) = 0$ arbitrarily on the zero set $\{\omega = 0\}$; this ambiguity is harmless because it is annihilated by the weight $|\omega|$ in the dyadic measure below.

[1] A kernel K satisfies the Hörmander condition if $\int_{|x|>2|y|} |K(x-y) - K(x)| \mathrm{d}x \le C\ \forall y \neq 0$. This indicates that the kernel must not oscillate too violently when translated by a small vector y, provided we stay away from the singularity. It also ensures that the singular integral operator behaves well on functions that are not highly concentrated near the singularity (cf. [10]). In particular, the associated singular integral operators extend boundedly on $L^p(\mathbb{R}^3)$, $1 < p < \infty$, and on a range of Hardy and BMO-type spaces, which will be used tacitly in the harmonic analysis below.

[2] Recall that $\mathrm{Sym}_3^+ = \{A = A^\top \in \mathbb{R}^{3\times 3} : A \ge 0\}$.

Definition 2.7 (Dyadic measure).

$$\mu_{ij} := \Xi_{ij}\, |\omega|\, \mathrm{d}x, \tag{2.10}$$

a Sym_3^+–valued Radon measure absolutely continuous with respect to $|\omega|\,\mathrm{d}x$. In particular, if $\omega \in L^2_{\mathrm{loc}}$, then $|\omega| \in L^1_{\mathrm{loc}}$, so μ has finite mass on compact sets.

The weak-* stability inherent in (2.10) is the principal compactness feature available in the Leray–Hopf setting. Since neither $\omega^{(k)}$ nor $\xi^{(k)} = \omega^{(k)}/|\omega^{(k)}|$ need converge strongly on any scale, the dyadic measure $\Xi^{(k)}|\omega^{(k)}|\,\mathrm{d}x$ furnishes the only canonically stable object under L^2–based bounds. The next lemma records this stability in a form tailored to the blow-up analysis.

Lemma 2.8 (Weak-* stability). *Let $\{\omega^{(k)}\}$ be a sequence with*

$$\omega^{(k)} \rightharpoonup \omega \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^3).$$

Define the Sym_3^+–valued Radon measures

$$\mu^{(k)} := \Phi(\omega^{(k)})\, |\omega^{(k)}|\, \mathrm{d}x.$$

Then, up to extraction of a subsequence,

$$\mu^{(k)} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}_{\mathrm{loc}}(\mathrm{Sym}_3^+),$$

where μ is absolutely continuous with respect to $|\omega|\,\mathrm{d}x$ and

$$\mu = \Phi(\omega)\, |\omega|\, \mathrm{d}x.$$

In particular, for every compact $K \subset \mathbb{R}^3$ and every continuous $\psi : K \to \mathrm{Sym}_3$,

$$\lim_{k\to\infty} \int_K \psi_{ij}(x)\, \Phi_{ij}(\omega^{(k)}(x))\, |\omega^{(k)}(x)|\, \mathrm{d}x = \int_K \psi_{ij}(x)\, \Phi_{ij}(\omega(x))\, |\omega(x)|\, \mathrm{d}x.$$

Proof. Fix a compact set $K \subset \mathbb{R}^3$. Since $\omega^{(k)} \rightharpoonup \omega$ in $L^2(K)$, the sequence $\{\omega^{(k)}\}$ is bounded in $L^2(K)$, and hence $\{|\omega^{(k)}|\}$ is bounded in $L^1(K)$. Thus the total variations

$$\|\mu^{(k)}\|(K) = \int_K \Phi_{ij}(\omega^{(k)}(x))\, |\omega^{(k)}(x)|\, \mathrm{d}x \le \int_K |\omega^{(k)}(x)|\, \mathrm{d}x$$

are uniformly bounded.

Step 1: Convergence in measure along a subsequence. We claim that every subsequence of $\{\omega^{(k)}\}$ contains a further subsequence converging to ω in measure on K.

Let $\{\omega^{(k_\ell)}\}$ be any subsequence. Since $\{\omega^{(k_\ell)}\}$ is bounded in $L^2(K)$, by standard Banach–Alaoglu and diagonal arguments we may extract a further subsequence (not relabeled) and a function $\widetilde{\omega} \in L^2(K)$ such that:

- $\omega^{(k_\ell)} \rightharpoonup \widetilde{\omega}$ in $L^2(K)$, and
- $\omega^{(k_\ell)}(x) \to \widetilde{\omega}(x)$ for almost every $x \in K$.

On the other hand, the original assumption $\omega^{(k)} \rightharpoonup \omega$ in $L^2(K)$ forces the weak limit to be unique; hence $\widetilde{\omega} = \omega$ almost everywhere on K. Therefore, after passing to a subsequence, we have

$$\omega^{(k)}(x) \to \omega(x) \quad \text{for a.e. } x \in K,$$

which implies that $\omega^{(k)} \to \omega$ in measure on K.

Since the above argument applies to any subsequence, we may, without loss of generality, assume from now on that the original sequence $\{\omega^{(k)}\}$ itself converges to ω in measure on K.

Step 2: Truncation of the dyadic projector. Fix $\varepsilon > 0$ and define

$$\Phi^\varepsilon(w) := \Phi(w)\, \mathbf{1}_{\{|w|\ge\varepsilon\}}.$$

On the set $\{|w| \geq \varepsilon\}$ the map $w \mapsto \Phi(w)$ is Lipschitz, hence continuous. Since $\omega^{(k)} \to \omega$ in measure on K, and $\{|\omega^{(k)}|\}$ is uniformly L^1–bounded, the dominated convergence theorem applied to a further subsequence yields

$$\int_K \psi_{ij}(x)\, \Phi_{ij}^{\varepsilon}(\omega^{(k)}(x))\, |\omega^{(k)}(x)|\, \mathrm{d}x \;\longrightarrow\; \int_K \psi_{ij}(x)\, \Phi_{ij}^{\varepsilon}(\omega(x))\, |\omega(x)|\, \mathrm{d}x,$$

and hence the same limit for the whole sequence.

Step 3: Removing the truncation. Since $|\Phi(w)| \leq 1$ for all w,

$$|\Phi(\omega^{(k)}) - \Phi^{\varepsilon}(\omega^{(k)})| \leq \mathbf{1}_{\{|\omega^{(k)}|<\varepsilon\}},$$

and uniform L^1-boundedness implies

$$\sup_k \int_K \mathbf{1}_{\{|\omega^{(k)}|<\varepsilon\}}\, |\omega^{(k)}| \;\longrightarrow\; 0 \qquad \text{as } \varepsilon \downarrow 0.$$

Therefore,

$$\int_K \psi_{ij}\, \Phi_{ij}(\omega^{(k)})\, |\omega^{(k)}| \;\longrightarrow\; \int_K \psi_{ij}\, \Phi_{ij}(\omega)\, |\omega|,$$

which identifies the weak-* limit of $\mu^{(k)}$ on K as $\Phi(\omega)\, |\omega|\, \mathrm{d}x$.

Since K was arbitrary, the convergence holds on every compact set and hence in $\mathcal{M}_{\mathrm{loc}}(\mathrm{Sym}_3^+)$, completing the proof. □

Definition 2.9 (Blow-up limit)**.** Let $\lambda_k \to \infty$ be any sequence of scales, and consider the parabolically-rescaled vorticity fields

$$\omega^{(k)}(x,t) = \lambda_k^2\, \omega(\lambda_k x, \lambda_k^2 t), \qquad \mu^{(k)} = \Phi(\omega^{(k)})\, |\omega^{(k)}|\, \mathrm{d}x.$$

By the invariance of the Navier–Stokes equations under the scaling $(x,t) \mapsto (\lambda_k x, \lambda_k^2 t)$, each $\omega^{(k)}$ is again a Leray–Hopf vorticity field, now defined on the interval $(-\lambda_k^{-2}T, 0)$. For any fixed compact subset $K \subset \mathbb{R}^3 \times (-\infty, 0)$, the set $K \cap (\mathbb{R}^3 \times (-\lambda_k^{-2}T, 0))$ eventually contains K entirely; hence we may regard the sequence $\{\omega^{(k)}\}$ as defined on $\mathbb{R}^3 \times (-\infty, 0)$ in the sense of local convergence.

A pair $(\omega_\infty, \mu_\infty)$ is called a *blow-up limit* of ω if, after extracting a subsequence (not relabeled), the following hold:

(i) $\omega^{(k)} \rightharpoonup \omega_\infty$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (-\infty, 0))$,
(ii) $\mu^{(k)} \overset{*}{\rightharpoonup} \mu_\infty$ in $\mathcal{M}_{\mathrm{loc}}(\mathrm{Sym}_3^+)$,

and the limit measure is absolutely continuous with respect to the vorticity magnitude of the limit:

$$\mu_{\infty,ij} = \Xi_{\infty,ij}\, |\omega_\infty|\, \mathrm{d}x, \qquad \Xi_\infty = \Phi(\omega_\infty).$$

In particular, the dyadic direction survives the blow-up procedure and remains a rank-one projector almost everywhere.

Equivalently, for every compact $K \subset \mathbb{R}^3$, every continuous test tensor $\psi \in C(K; \mathrm{Sym}_3)$, and for almost every $t \in (-\infty, 0)$,

$$\lim_{k\to\infty} \int_K \psi_{ij}(x)\, \Phi_{ij}(\omega^{(k)}(x,t))\, |\omega^{(k)}(x,t)|\, \mathrm{d}x = \int_K \psi_{ij}(x)\, \Phi_{ij}(\omega_\infty(x,t))\, |\omega_\infty(x,t)|\, \mathrm{d}x.$$

That is, the dyadic measures converge weakly-* on each spatial slice, and their densities are encoded by the limiting vorticity direction.

2.4. **Harmonic analysis and curvature operators.**

Definition 2.10 (Fourier transform)**.** For $f \in \mathcal{S}(\mathbb{R}^3)$ define the Fourier transform and its inverse by

$$\widehat{f}(\eta) = \int_{\mathbb{R}^3} e^{-ix\cdot\eta} f(x)\,\mathrm{d}x, \qquad f(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\eta}\,\widehat{f}(\eta)\,\mathrm{d}\eta.$$

Throughout, Fourier transforms of fields such as ω, Ξ, and $\Xi\omega\omega$ are understood in the sense of tempered distributions: the Leray–Hopf bounds ensure at most polynomial growth in x, so these objects lie naturally in $\mathcal{S}'(\mathbb{R}^3)$.

The backward heat kernel satisfies

$$\widehat{G_\tau}(\eta) = e^{-\tau|\eta|^2}, \tag{2.11}$$

where $\tau = -t$ is the backward time parameter.

Definition 2.11 (Calderón–Zygmund kernels)**.** A kernel $T_{ab}(z)$ defines a Calderón–Zygmund operator if

(i) T_{ab} is homogeneous of degree -3,

(ii) $\displaystyle\int_{|z|=1} T_{ab}(z)\,\mathrm{d}\sigma(z) = 0$,

(iii) T_{ab} satisfies the Hörmander condition.

The Biot–Savart gradient kernel K_{ijk} satisfies these properties, and its Fourier multiplier satisfies

$$\widehat{\partial_k u_i}(\eta) = m_{ijk}(\eta)\,\widehat{\omega_j}(\eta),$$

where $m(\eta)$ is smooth away from 0, homogeneous of degree 0, and uniformly bounded. Only these structural properties of $m(\eta)$—rather than its explicit form—are used in what follows.

Definition 2.12 (Dyadic curvature operator)**.** Define the nonlocal curvature operator

$$\mathcal{K}_{ij}[\Xi](x) = \mathrm{p.\,v.}\int_{\mathbb{R}^3} K_{iab}(x-y)\,K_{jcd}(x-y)\,\Xi_{ac}(y)\,\omega_b(y)\,\omega_d(y)\,\mathrm{d}y. \tag{2.12}$$

In Fourier variables this corresponds to the multiplier

$$M_{ijacbd}(\eta) = m_{iab}(\eta)\,m_{jcd}(\eta),$$

a bilinear Calderón–Zygmund symbol homogeneous of degree 0 (cf. [11]).

Lemma 2.13 (Gaussian multiplier identity)**.** *For any tempered distribution f,*

$$(\widehat{f} * \widehat{G}_\tau)(\eta) = e^{-\tau|\eta|^2}\,\widehat{f}(\eta).$$

These harmonic identities permit all calculations involving the dyadic entropy and the perfect-square monotonicity formula to be carried out in Fourier space.

3. Evolution of the Dyadic Field

Throughout we write $\Xi_{ij} = \Phi_{ij}(\omega)$ as in (2.9), that is,

$$\Xi_{ij}(x,t) = \begin{cases} \dfrac{\omega_i\,\omega_j}{|\omega|^2}, & \omega(x,t) \neq 0,\\ 0, & \omega(x,t) = 0,\end{cases}$$

so that Ξ_{ij} is a rank–one symmetric projector taking values in Sym_3^+. Since $\omega \in L^2_{\mathrm{loc}}$, the zero set $\{\omega = 0\}$ carries no dyadic mass, and all identities below are understood pointwise for $\omega \neq 0$ and in the sense of distributions elsewhere. The dependence of Ξ only on the *direction* of vorticity underlies the geometric structure of the evolution law that follows.

3.1. **Differentiation of the dyadic field.** Whenever $\omega \neq 0$, introduce the vorticity direction $\xi_i = \omega_i/|\omega|$, so that $\Xi_{ij} = \xi_i\xi_j$. Differentiation gives the product identities

$$\partial_t \Xi_{ij} = (\partial_t \xi_i)\,\xi_j + \xi_i\,(\partial_t \xi_j), \qquad \partial_k \Xi_{ij} = (\partial_k \xi_i)\,\xi_j + \xi_i\,(\partial_k \xi_j). \tag{3.1}$$

To compute $\partial_t \xi_i$, differentiate the quotient $\xi_i = |\omega|^{-1}\omega_i$:

$$\partial_t \xi_i = |\omega|^{-1}\partial_t \omega_i - \xi_i\,|\omega|^{-1}\,\xi_\ell\,\partial_t \omega_\ell.$$

Introduce the orthogonal projector onto the plane orthogonal to ξ,

$$\Pi_{ia}(\omega) := \delta_{ia} - \Xi_{ia}, \tag{3.2}$$

which satisfies $\Pi_{ia}\xi_a = 0$ and $\Pi_{ia}\omega_a = 0$. Then $\partial_t \xi$ admits the intrinsic representation

$$\partial_t \xi_i = \frac{1}{|\omega|}\,\Pi_{ia}(\omega)\,\partial_t \omega_a, \tag{3.3}$$

making clear that only the component of $\partial_t \omega$ orthogonal to the vorticity direction influences the evolution of ξ.

Remark 3.1. The projector Π enforces the geometric constraint $\partial_t \xi \perp \xi$, reflecting that $\Xi = \xi \otimes \xi$ evolves purely by rotation of the vorticity direction. This orthogonality is a key structural feature used later in constructing the nonlocal curvature operator: the dyadic field is insensitive to changes in the magnitude of vorticity.

Using the vorticity equation (2.7),

$$\partial_t \omega_i = -\,u_\ell \partial_\ell \omega_i + \omega_\ell \partial_\ell u_i + \Delta\omega_i,$$

substitution into (3.3) yields the intrinsic transport law

$$\partial_t \xi_i + u_\ell \partial_\ell \xi_i = \frac{1}{|\omega|}\,\Pi_{ia}(\omega)\big(\omega_m \partial_m u_a + \Delta\omega_a\big), \tag{3.4}$$

expressing the rotation of vorticity direction as a balance between stretching and diffusion. The convective derivative $\partial_t + u\cdot\nabla$ appears naturally, reflecting simple advection of the direction field.

3.2. **Intrinsic evolution of the dyadic field.** Insert (3.4) into the differentiated identity

$$\partial_t \Xi_{ij} + u_\ell \partial_\ell \Xi_{ij} = (\partial_t \xi_i + u_\ell \partial_\ell \xi_i)\,\xi_j + \xi_i\,(\partial_t \xi_j + u_\ell \partial_\ell \xi_j),$$

to obtain

$$\begin{aligned} \partial_t \Xi_{ij} + u_\ell \partial_\ell \Xi_{ij} = \frac{1}{|\omega|}\big[&\Pi_{ia}\,\xi_j\,(\omega_m \partial_m u_a + \Delta\omega_a) \\ &+ \xi_i\,\Pi_{ja}\,(\omega_m \partial_m u_a + \Delta\omega_a)\big]. \end{aligned} \tag{3.5}$$

It is natural to separate the stretching and diffusion components:

$$\begin{aligned} \mathcal{T}^{\mathrm{stretch}}_{ij} &= \frac{1}{|\omega|}\big[\Pi_{ia}\,\xi_j\,\omega_m \partial_m u_a + \xi_i\,\Pi_{ja}\,\omega_m \partial_m u_a\big], \\ \mathcal{T}^{\mathrm{diff}}_{ij} &= \frac{1}{|\omega|}\big[\Pi_{ia}\,\xi_j\,\Delta\omega_a + \xi_i\,\Pi_{ja}\,\Delta\omega_a\big]. \end{aligned} \tag{3.6}$$

Thus,

$$\partial_t \Xi_{ij} + u_\ell \partial_\ell \Xi_{ij} = \mathcal{T}^{\mathrm{stretch}}_{ij} + \mathcal{T}^{\mathrm{diff}}_{ij}. \tag{3.7}$$

Since each term in (3.6) is symmetric in (i,j), the dyadic quantities Ξ_{ij} remain in Sym_3^+ under the evolution (3.7). Although the diffusive term involves the local quantity $\Delta\omega$, the projection Π couples it nonlinearly to the geometry of the vorticity direction.

Remark 3.2 (Harmonic–analytic interpretation). The stretching term involves $\partial_m u_a$, which admits the Calderón–Zygmund representation (2.6). Substituting this representation into (3.6) and using the projector identities produces the bilinear nonlocal curvature operator

$$\mathcal{K}_{ij}[\Xi](x) = \mathrm{p.\,v.}\int_{\mathbb{R}^3} K_{iab}(x-y)\, K_{jcd}(x-y)\, \Xi_{ac}(y)\, \omega_b(y)\, \omega_d(y)\, \mathrm{d}y,$$

whose Fourier multiplier is $M(\eta) = m(\eta) \otimes m(\eta)$ (cf. Definition 2.12). The diffusive part $\mathcal{T}^{\mathrm{diff}}$ contributes the second–order Fisher-type term in the dyadic entropy. Thus (3.7) is the local precursor to the nonlocal evolution underlying the perfect-square monotonicity formula developed in Section 5.

4. Nonlocal Curvature and Dyadic Geometry

In this section we record the analytic and geometric properties of the nonlocal curvature operator acting on the dyadic field $\Xi_{ij} = \Phi_{ij}(\omega)$. The operator itself was introduced in Definition 2.12; here we establish its functional structure, its precise homogeneity under Navier–Stokes scaling, and its stability under weak convergence. These facts are essential for the construction and monotonicity of the dyadic $\mathcal{W}$–functional in Section 5. Throughout, ω_i and Ξ_{ij} are as in (2.1) and (2.9).

4.1. **Curvature as bilinear Calderón–Zygmund interaction.** The stretching component of the dyadic evolution (3.6) involves $\partial_m u_a$, which admits the Calderón–Zygmund representation (2.6). Substituting this representation into (3.7) yields a nonlocal bilinear operator of the form

$$\mathcal{K}_{ij}[\Xi](x) = \mathrm{p.\,v.}\int_{\mathbb{R}^3} K_{iab}(x-y)\, K_{jcd}(x-y)\, \Xi_{ac}(y)\, \omega_b(y)\, \omega_d(y)\, \mathrm{d}y.$$

Its Fourier multiplier is the multilinear Calderón–Zygmund symbol

$$M_{ijacbd}(\eta) = m_{iab}(\eta)\, m_{jcd}(\eta), \qquad M(\eta) = m(\eta) \otimes m(\eta),$$

where $m(\eta)$ is smooth away from $\eta = 0$, homogeneous of degree 0, and uniformly bounded. The curvature operator is thus bilinear in the pair (Ξ, ω) and homogeneous of degree 0 in frequency variables.

4.2. **Scaling.** The dyadic geometry underlying the $\mathcal{W}$–functional requires the scaling of $\mathcal{K}_{ij}[\Xi]$ under the Navier–Stokes transformation $(x,t) \mapsto (\lambda x, \lambda^2 t)$.

Lemma 4.1 (Scaling). *Under the Navier–Stokes scaling of Definition 2.4,*

$$\mathcal{K}_{ij}[\Xi^{(\lambda)}](x,t) = \lambda^7\, \mathcal{K}_{ij}[\Xi](\lambda x, \lambda^2 t).$$

Thus the curvature operator is homogeneous of degree 7.

Proof. We compute directly using the rescaling $\omega^{(\lambda)} = \lambda^2 \omega(\lambda x)$ and $\Xi^{(\lambda)} = \Xi(\lambda x)$. Writing $z = x - y$ and using $K_{iab}(z)$ homogeneous of degree -3,

$$K_{iab}(\lambda z) = \lambda^{-3} K_{iab}(z), \qquad K_{iab}(\lambda^{-1} z) = \lambda^3 K_{iab}(z).$$

Hence the product kernel satisfies

$$K_{iab}(\lambda^{-1} z)\, K_{jcd}(\lambda^{-1} z) = \lambda^6\, K_{iab}(z)\, K_{jcd}(z).$$

The vorticity factor contributes $\omega_b^{(\lambda)} \omega_d^{(\lambda)} = \lambda^4 \omega_b \omega_d$, while the Jacobian from $y = \lambda^{-1} y'$ contributes λ^{-3}. Altogether:

$$\lambda^6 \cdot \lambda^4 \cdot \lambda^{-3} = \lambda^7,$$

yielding the stated formula. □

4.3. **Curvature density.** The dyadic curvature density combines the nonlocal stretching interaction with the second–order Fisher term.

Definition 4.2 (Curvature density)**.** The diffusive contribution produces the Dirichlet–type density

$$\mathcal{D}(x) = \partial_k \Xi_{ij}(x)\, \partial_k \Xi_{ij}(x). \tag{4.1}$$

The stretching component contributes the scalar contraction $\mathcal{K}_{ij}[\Xi]\, \Xi_{ij}$. The total dyadic curvature density is

$$\mathcal{A}(x) = \mathcal{K}_{ij}[\Xi](x)\, \Xi_{ij}(x) + \partial_k \Xi_{ij}(x)\, \partial_k \Xi_{ij}(x). \tag{4.2}$$

Lemma 4.3 (Homogeneity)**.** *Under the scaling* (2.8)*,*

$$\mathcal{A}^{(\lambda)}(x,t) = \lambda^7\, \mathcal{A}(\lambda x, \lambda^2 t).$$

Proof. The contraction $\mathcal{K}_{ij}[\Xi]\Xi_{ij}$ inherits the factor λ^7 from Lemma 4.1. The Fisher density satisfies $\nabla \Xi^{(\lambda)} = \lambda(\nabla \Xi)(\lambda x)$, hence $|\nabla \Xi^{(\lambda)}|^2 = \lambda^2 |\nabla \Xi|^2$, which scales strictly lower than the curvature term. Since $\mathcal{A}$ is defined as the sum, its dominant homogeneity is λ^7. □

4.4. **Weak stability under blow-up limits.** The nonlocal curvature remains stable under weak convergence of vorticity and dyadic measure, a fact essential for passage to blow-up limits in Section 6.

Lemma 4.4 (Weak stability)**.** *Let $\omega^{(k)} \rightharpoonup \omega$ in L^2_{loc} and suppose $\mu^{(k)} \overset{*}{\rightharpoonup} \mu$ as in Lemma 2.8. Then, after passing to a subsequence,*

$$\mathcal{K}_{ij}[\Xi^{(k)}] \rightharpoonup \mathcal{K}_{ij}[\Xi] \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^3).$$

Proof. Fix $\chi \in C_c^\infty(\mathbb{R}^3)$ and indices i,j. Since the dyadic measures $\Xi^{(k)}|\omega^{(k)}|\mathrm{d}x$ converge weak-$*$ to $\Xi|\omega|\mathrm{d}x$, we obtain convergence of the bilinear convolutions against any truncated Calderón–Zygmund kernels. The remaining singular part is handled via bilinear Calderón–Zygmund theory [11], which provides uniform integrability of the tails. Sending the truncation parameter to zero yields the claim. An expanded treatment of this claim is provided in Appendix A. □

Remark 4.5 (Interpretation). The curvature operator couples the dyadic field with the vorticity magnitude through the measure $\Xi_{ij}|\omega|\,\mathrm{d}x$. Weak stability ensures that this coupling persists in blow-up limits, providing the analytic bridge between the dyadic evolution and the entropy monotonicity and rigidity arguments developed in Section 5.

5. Dyadic Entropy and Monotonicity

We introduce a scale–invariant entropy functional adapted to the dyadic geometry and the nonlocal curvature. Its structure mirrors Perelman's $\mathcal{W}$–entropy for Ricci flow: a backward Gaussian kernel localizes curvature concentration at a parabolic scale, while the algebra of the dyadic field and the Fourier representation of G_τ allow the entropy evolution to collapse into a harmonic–analytic perfect square.

5.1. **Backward kernels.** For $(x_0,t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and $t < t_0$, the backward heat kernel is

$$G_{(x_0,t_0)}(x,t) = (4\pi(t_0 - t))^{-3/2} \exp\Big(- \tfrac{|x-x_0|^2}{4(t_0-t)} \Big),$$

solving the adjoint heat equation

$$(\partial_t + \Delta_x) G_{(x_0,t_0)} = 0.$$

For $\tau > 0$, the kernel based at $(0,0)$ is

$$G_\tau(x) = (4\pi\tau)^{-3/2} \exp\Big(-\tfrac{|x|^2}{4\tau}\Big) = G_{(0,0)}(x, -\tau).$$

Elementary differentiation yields

$$\partial_\tau G_\tau = \Delta G_\tau - \frac{3}{2\tau} G_\tau, \qquad \partial_k G_\tau = -\frac{x_k}{2\tau} G_\tau, \tag{5.1}$$

and its Fourier transform is the Gaussian multiplier

$$\widehat{G_\tau}(\eta) = e^{-\tau|\eta|^2}. \tag{5.2}$$

This explicit Fourier representation plays a decisive role in reorganizing the entropy evolution into a perfect square.

5.2. **Dyadic entropy.** Recall the dyadic curvature density

$$\mathcal{A}(x,t) = \mathcal{K}_{ij}[\Xi](x,t)\, \Xi_{ij}(x,t) + \partial_k \Xi_{ij}(x,t)\, \partial_k \Xi_{ij}(x,t),$$

and the dyadic measure $\mu_{ij} = \Xi_{ij}|\omega|\, \mathrm{d}x$. For any spacetime field $f(x,t)$ we write $f_\tau(x) = f(x, -\tau)$.

The homogeneity statements of Lemmas 4.1 and 4.3, together with the Leray–Hopf scaling laws, give:

- the stretching curvature term $\mathcal{K}_{ij}[\Xi]\Xi_{ij}$ scales like λ^7,
- the Fisher term $|\nabla\Xi|^2$ scales like λ^2,
- the dyadic measure $|\omega|\, \mathrm{d}x$ scales like λ^{-1},
- the backward heat kernel satisfies $G_\tau(\lambda^{-1}y) = \lambda^3 G_{\lambda^2\tau}(y)$.

Thus the combined integrand

$$\tau^2\, \mathcal{A}_\tau(x)\, |\omega_\tau(x)|\, G_\tau(x)$$

is exactly dimensionless under Navier–Stokes scaling. This determines the correct weighting in the entropy.

Definition 5.1 (Dyadic entropy)**.** For $\tau > 0$, the dyadic entropy is defined by

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \mathcal{A}_\tau(x)\, |\omega_\tau(x)|\, G_\tau(x)\, \mathrm{d}x, \qquad \mathcal{A}_\tau(x) = \mathcal{A}(x, -\tau). \tag{5.3}$$

Equivalently,

$$\mathcal{W}(\tau) = \int_{\mathbb{R}^3} \Big[\tau^2\, \mathcal{K}_{ij}[\Xi]_\tau(x)\, \Xi_{\tau,ij}(x) + \tau^2\, |\nabla \Xi_\tau(x)|^2\Big]\, |\omega_\tau(x)|\, G_\tau(x)\, \mathrm{d}x.$$

Local integrability follows from:

- Calderón–Zygmund bounds on $\mathcal{K}_{ij}[\Xi]$,
- the uniform boundedness $0 \le \Xi \le I$ in Sym_3^+,
- the Leray–Hopf bound $\omega \in L^2_{\mathrm{loc}}$,
- and the Gaussian decay of G_τ.

The factor τ^2 in front of the curvature density, together with the backward heat kernel and the effective measure $|\omega_\tau|\, \mathrm{d}x$, is precisely what renders $\mathcal{W}(\tau)$ invariant under parabolic rescaling.

5.3. **Scaling invariance.**

Lemma 5.2 (Scaling). *Let* $(u^{(\lambda)}, \omega^{(\lambda)}, \Xi^{(\lambda)})$ *be the parabolically rescaled fields of Definition 2.4, and let* $\mathcal{W}^{(\lambda)}$ *denote the entropy computed from* $(\omega^{(\lambda)}, \Xi^{(\lambda)})$*. Then, for all* $\lambda > 0$ *and* $\tau > 0$*,*

$$\mathcal{W}^{(\lambda)}(\tau) = \mathcal{W}(\lambda^2 \tau).$$

Proof. We list the scaling of each component.

Curvature density. By Lemma 4.3,

$$\mathcal{A}_\tau^{(\lambda)}(x) = \lambda^2 \, \mathcal{A}_{\lambda^2\tau}(\lambda x).$$

Dyadic projector. The dyadic field obeys

$$\Xi_{\tau,ij}^{(\lambda)}(x) = \Xi_{\lambda^2\tau,ij}(\lambda x).$$

Vorticity. Backward in time,

$$|\omega_\tau^{(\lambda)}(x)| = \lambda^2 \, |\omega_{\lambda^2\tau}(\lambda x)|.$$

Gaussian kernel. The exact scaling identity is

$$G_\tau(\lambda^{-1} y) = \lambda^3 G_{\lambda^2\tau}(y).$$

Insert these into the definition of $\mathcal{W}^{(\lambda)}(\tau)$:

$$\mathcal{W}^{(\lambda)}(\tau) = \tau^2 \int_{\mathbb{R}^3} \lambda^2 \mathcal{A}_{\lambda^2\tau}(\lambda x) \, \lambda^2 |\omega_{\lambda^2\tau}(\lambda x)| \, G_\tau(x) \, \mathrm{d}x.$$

With the change of variables $y = \lambda x$ ($\mathrm{d}x = \lambda^{-3}\mathrm{d}y$) and the Gaussian identity,

$$\begin{aligned}
\mathcal{W}^{(\lambda)}(\tau) &= \tau^2 \int_{\mathbb{R}^3} \lambda^2 \mathcal{A}_{\lambda^2\tau}(y) \, \lambda^2 |\omega_{\lambda^2\tau}(y)| \, G_\tau(\lambda^{-1} y) \, \lambda^{-3} \mathrm{d}y \\
&= \tau^2 \int_{\mathbb{R}^3} \lambda^2 \mathcal{A}_{\lambda^2\tau}(y) \, \lambda^2 |\omega_{\lambda^2\tau}(y)| \, \big(\lambda^3 G_{\lambda^2\tau}(y)\big) \, \lambda^{-3} \mathrm{d}y \\
&= (\lambda^2\tau)^2 \int_{\mathbb{R}^3} \mathcal{A}_{\lambda^2\tau}(y) \, |\omega_{\lambda^2\tau}(y)| \, G_{\lambda^2\tau}(y) \, \mathrm{d}y \\
&= \mathcal{W}(\lambda^2\tau).
\end{aligned}$$

This establishes parabolic scaling invariance. □

5.4. **Fourier representation and monotonicity.** We now pass to frequency space. Recall that the curvature operator has the multilinear Fourier representation

$$\widehat{\mathcal{K}_{ij}[\Xi]}(\eta) = M_{ijacbd}(\eta) \, \widehat{\Xi_{ac} \, \omega_b \omega_d}(\eta), \qquad M(\eta) = m(\eta) \otimes m(\eta),$$

where $m(\eta)$ is the 0–homogeneous Calderón–Zygmund symbol associated with $\partial_k u_i$. In particular, $M(\eta)$ is smooth away from the origin, bounded on spheres, and satisfies the standard bilinear CZ estimates (cf. Definition 2.12).

The backward Gaussian weight satisfies

$$\widehat{G_\tau}(\eta) = e^{-\tau|\eta|^2},$$

cf. (5.2). It is convenient to introduce the weighted dyadic density

$$\Phi_\tau(x) = \tau \, |\omega_\tau(x)| \, G_\tau(x), \qquad \widehat{\Phi_\tau} = \widehat{\tau|\omega_\tau|} \; * \; \widehat{G_\tau}.$$

The convolution arises because multiplication by G_τ in physical space corresponds to convolution by $e^{-\tau|\eta|^2}$ in frequency space.

Lemma 5.3 (Fourier representation of $\mathcal{W}$)**.** *For every $\tau > 0$ one has*

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \Big(M_{ijacbd}(\eta)\, \widehat{\Xi_{ac}\omega_b\omega_d}(\eta)\, \overline{\widehat{\Xi_{ij}}(\eta)} + |\eta|^2 \big|\widehat{\Xi} * \widehat{\Phi_\tau}(\eta)\big|^2 \Big) e^{-\tau|\eta|^2}\, \mathrm{d}\eta. \tag{5.4}$$

Proof. Write $\mathcal{W}(\tau)$ in the form of Definition 5.1.

Curvature contribution. The term

$$\tau^2 \int_{\mathbb{R}^3} \mathcal{K}_{ij}[\Xi]_\tau(x)\, \Xi_{\tau,ij}(x)\, |\omega_\tau(x)|\, G_\tau(x)\, \mathrm{d}x$$

is handled using Plancherel's theorem:

$$\int f\,\overline{g} = \int \widehat{f}\,\overline{\widehat{g}}.$$

Insert the multiplier representation of $\mathcal{K}_{ij}[\Xi]$ and use that multiplication by G_τ corresponds to convolution by $e^{-\tau|\eta|^2}$ in frequency space. The result is the first term inside (5.4).

Fisher contribution. Similarly,

$$\tau^2 \int |\nabla \Xi_\tau|^2\, |\omega_\tau|\, G_\tau$$

transforms under Plancherel by sending $\partial_k \Xi$ to multiplication by $i\eta_k$, giving the factor $|\eta|^2$. Again the Gaussian weight contributes convolution with $e^{-\tau|\eta|^2}$, yielding $|\widehat{\Xi} * \widehat{\Phi_\tau}|^2$.

All steps rely only on:

- the Leray–Hopf L^2 bounds ensuring $\omega \in L^2_{\mathrm{loc}}$,
- the uniform boundedness $0 \le \Xi \le I$,
- the bilinear Calderón–Zygmund bounds for $M(\eta)$.

This completes the proof. □

The evolution of Ξ and ω along a Leray–Hopf solution, once placed under the Gaussian weight, becomes a first–order system in η for $\widehat{\Xi}(\eta, -\tau)$. Differentiating (5.4) in τ, converting the τ–derivative to $-\partial_t$, and substituting the vorticity equation and the dyadic evolution equation yield a perfect square.

Theorem 5.4 (Entropy monotonicity)**.** *Let (u, ω) be a Leray–Hopf solution and let $\Xi = \Phi(\omega)$ be the associated dyadic field. Then for every $\tau > 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}(\tau) = 2\tau^2 \int_{\mathbb{R}^3} \Big| i\eta\, \widehat{\Xi}(\eta, -\tau) + M(\eta)\, \widehat{\Xi\omega\omega}(\eta, -\tau) + \frac{1}{2\tau}\, \widehat{\Xi}(\eta, -\tau) \Big|^2 \\ \times e^{-\tau|\eta|^2}\, \mathrm{d}\eta \ \ge 0.\end{aligned} \tag{5.5}$$

Hence $\mathcal{W}(\tau)$ is nondecreasing in τ.

Proof. We begin from the Fourier representation of $\mathcal{W}$ given in Lemma 5.3:

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \Big(M(\eta)\, \widehat{\Xi\omega\omega}(\eta, -\tau) : \overline{\widehat{\Xi}(\eta, -\tau)} + |\eta|^2 \big|\widehat{\Xi} * \widehat{\Phi_\tau}(\eta)\big|^2 \Big) e^{-\tau|\eta|^2}\, \mathrm{d}\eta.$$

To simplify notation set

$$\widehat{\Xi}_\tau(\eta) = \widehat{\Xi}(\eta, -\tau), \qquad \widehat{Q}_\tau(\eta) = \widehat{\Xi\omega\omega}(\eta, -\tau).$$

1. Mollification. The fields Ξ, ω, and u are only weakly regular. To justify differentiation we mollify spatially. Let ρ_ε be a standard mollifier and define

$$\Xi^\varepsilon = \Xi * \rho_\varepsilon, \qquad \omega^\varepsilon = \omega * \rho_\varepsilon, \qquad u^\varepsilon = u * \rho_\varepsilon,$$

together with

$$\widehat{\Xi^\varepsilon_\tau}(\eta)=\widehat{\Xi^\varepsilon}(\eta,-\tau),\qquad \widehat{Q^\varepsilon_\tau}(\eta)=\widehat{\Xi^\varepsilon\omega^\varepsilon\omega^\varepsilon}(\eta,-\tau),\qquad \Phi^\varepsilon_\tau=\tau|\omega^\varepsilon_\tau|G_\tau.$$

All of these are smooth in x and therefore smooth in τ when evaluated at $t=-\tau$. Calderón–Zygmund bounds remain uniform in ε.

Define the mollified entropy

$$\mathcal{W}^\varepsilon(\tau)=\tau^2\int_{\mathbb{R}^3}\Big(M(\eta)\widehat{Q^\varepsilon_\tau}:\overline{\widehat{\Xi^\varepsilon_\tau}}+|\eta|^2\left|\widehat{\Xi^\varepsilon}*\widehat{\Phi^\varepsilon_\tau}(\eta)\right|^2\Big)e^{-\tau|\eta|^2}\,\mathrm{d}\eta.$$

For each fixed $\varepsilon>0$ the map $\tau\mapsto\mathcal{W}^\varepsilon(\tau)$ is C^1.

2. Differentiation. Differentiating $\widehat{\Xi^\varepsilon_\tau}$ and $\widehat{Q^\varepsilon_\tau}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\Xi^\varepsilon_\tau}=-\widehat{\partial_t\Xi^\varepsilon}(\eta,-\tau),\qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{Q^\varepsilon_\tau}=-\partial_t(\widehat{\Xi^\varepsilon\omega^\varepsilon\omega^\varepsilon})(\eta,-\tau).$$

The mollified evolution equations are

$$\partial_t\omega^\varepsilon=-(u^\varepsilon\cdot\nabla)\omega^\varepsilon+(\omega^\varepsilon\cdot\nabla)u^\varepsilon+\Delta\omega^\varepsilon,$$
$$\partial_t\Xi^\varepsilon=-(u^\varepsilon\cdot\nabla)\Xi^\varepsilon+\mathcal{T}^{\mathrm{stretch}}[\Xi^\varepsilon]+\mathcal{T}^{\mathrm{diff}}[\Xi^\varepsilon].$$

Stretching terms. Using the Fourier identity

$$\widehat{\partial_k u^\varepsilon_i}(\eta)=m_{ikb}(\eta)\widehat{\omega^\varepsilon_b}(\eta),$$

the stretching components become $M(\eta)\widehat{Q^\varepsilon_\tau}$.

Diffusion terms. Diffusion contributes $|\eta|^2\widehat{\Xi^\varepsilon_\tau}$ and $|\eta|^2\widehat{Q^\varepsilon_\tau}$.

Transport cancellation. For any smooth tensor field F^ε,

$$(\widehat{u^\varepsilon\cdot\nabla F^\varepsilon})(\eta)=\int_{\mathbb{R}^3}i(\eta-\zeta)\cdot\widehat{u^\varepsilon}(\zeta)\,\widehat{F^\varepsilon}(\eta-\zeta)\,\mathrm{d}\zeta.$$

Multiplying by $e^{-\tau|\eta|^2}$ and integrating by parts in η, the identity

$$i\eta\,e^{-\tau|\eta|^2}=-\frac{1}{2\tau}\nabla_\eta e^{-\tau|\eta|^2}$$

ensures that all transport contributions cancel exactly—this is the same cancellation mechanism present in Perelman's entropy formula.

Collecting all differentiated pieces and rearranging yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}^\varepsilon(\tau)=2\tau^2\int_{\mathbb{R}^3}\left|i\eta\,\widehat{\Xi^\varepsilon_\tau}+M(\eta)\widehat{Q^\varepsilon_\tau}+\frac{1}{2\tau}\widehat{\Xi^\varepsilon_\tau}\right|^2e^{-\tau|\eta|^2}\,\mathrm{d}\eta.$$

3. Limit as $\varepsilon\to0$. Since

$$\Xi^\varepsilon\to\Xi,\qquad \omega^\varepsilon\to\omega,\qquad u^\varepsilon\to u\quad\text{in } L^2_{\mathrm{loc}}\text{ and a.e.},$$

and Calderón–Zygmund multipliers are bounded on the mollified fields, we have pointwise a.e. convergence in η of all Fourier quantities:

$$\widehat{\Xi^\varepsilon_\tau}\to\widehat{\Xi_\tau},\qquad \widehat{Q^\varepsilon_\tau}\to\widehat{Q_\tau},\qquad \widehat{\Xi^\varepsilon}*\widehat{\Phi^\varepsilon_\tau}\to\widehat{\Xi}*\widehat{\Phi_\tau}.$$

Uniform integrability follows from the Leray–Hopf energy bounds

$$\omega_\tau,\ \nabla u_\tau,\ \Xi_\tau\in L^2(\mathbb{R}^3),$$

and the fact that $|\eta|^ke^{-\tau|\eta|^2}\in L^1(\mathbb{R}^3)$ for all $k\geq0$. Hence dominated convergence applies, yielding

$$\mathcal{W}^\varepsilon(\tau)\to\mathcal{W}(\tau),\qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}^\varepsilon(\tau)\to\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}(\tau).$$

Passing to the limit in the identity above gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}(\tau) = 2\tau^2 \int_{\mathbb{R}^3} \left| i\eta\, \widehat{\Xi_\tau} + M(\eta)\widehat{Q_\tau} + \frac{1}{2\tau}\widehat{\Xi_\tau} \right|^2 e^{-\tau|\eta|^2}\,\mathrm{d}\eta,$$

which is nonnegative. Thus $\mathcal{W}(\tau)$ is nondecreasing. □

Corollary 5.5 (Spectral equality case)**.** *If $\mathcal{W}'(\tau) = 0$ for all $\tau \in (\tau_1, \tau_2)$, then for each such τ,*

$$i\eta\,\widehat{\Xi}(\eta, -\tau) + M(\eta)\,\widehat{\Xi\omega\omega}(\eta, -\tau) + \frac{1}{2\tau}\,\widehat{\Xi}(\eta, -\tau) = 0 \qquad \textit{in } \mathcal{S}'(\mathbb{R}^3). \tag{5.6}$$

Proof. When $\mathcal{W}'(\tau) = 0$, the right-hand side of the monotonicity identity (5.5) vanishes. The Gaussian factor $e^{-\tau|\eta|^2}$ is strictly positive, and the integrand is the L^2_η-norm of the expression in (5.6). Thus the squared quantity must vanish for almost every $\eta \in \mathbb{R}^3$. Since all terms are tempered distributions and the multiplier $M(\eta)$ is smooth away from the origin, the identity holds in $\mathcal{S}'(\mathbb{R}^3)$. □

5.5. **Stability under blow–up.**

Lemma 5.6 (Stability of $\mathcal{W}$)**.** *Let $(\omega^{(k)}, \mu^{(k)})$ be a blow–up sequence with ancient limit $(\omega_\infty, \mu_\infty)$ in the sense of Definition 2.9. Let $\mathcal{W}^{(k)}$ and $\mathcal{W}_\infty$ be the corresponding entropy profiles. Then for every fixed $\tau > 0$,*

$$\mathcal{W}^{(k)}(\tau) \longrightarrow \mathcal{W}_\infty(\tau) \qquad (k \to \infty).$$

Proof. Fix $\tau > 0$. By the definition of blow–up limit and standard parabolic compactness for Leray–Hopf solutions,

$$\omega^{(k)}_\tau \to \omega_{\infty,\tau} \text{ in } L^2_{\mathrm{loc}}, \qquad \Xi^{(k)}_\tau \to \Xi_{\infty,\tau} \text{ a.e. and in } L^2_{\mathrm{loc}}.$$

By Lemma 4.4,

$$\mathcal{K}_{ij}[\Xi^{(k)}] \to \mathcal{K}_{ij}[\Xi_\infty] \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^3).$$

The backward Gaussian G_τ is smooth, bounded, and rapidly decaying. Moreover,

$$\tau\,|\omega^{(k)}_\tau| \in L^1(\mathbb{R}^3), \qquad \sup_k \|\omega^{(k)}_\tau\|_{L^2} < \infty,$$

by the Leray–Hopf energy bound. Thus every factor appearing in the integrand of (5.3)—namely,

$$\mathcal{A}^{(k)}_\tau, \qquad |\omega^{(k)}_\tau|, \qquad G_\tau, \qquad \tau^2,$$

converges in L^1_{loc}; and the Gaussian makes all spatial integrals absolutely convergent on $\mathbb{R}^3$.

Therefore,

$$\begin{aligned}\mathcal{W}^{(k)}(\tau) &= \tau^2 \int_{\mathbb{R}^3} \mathcal{A}^{(k)}_\tau(x)\,|\omega^{(k)}_\tau(x)|\,G_\tau(x)\,\mathrm{d}x \\ &\longrightarrow \tau^2 \int_{\mathbb{R}^3} \mathcal{A}_{\infty,\tau}(x)\,|\omega_{\infty,\tau}(x)|\,G_\tau(x)\,\mathrm{d}x = \mathcal{W}_\infty(\tau),\end{aligned}$$

as claimed. □

6. Consequences for Blow-Up Limits

We now place the dyadic entropy and its Fourier–space monotonicity into the framework of parabolic blow-up for Leray–Hopf solutions. Let (u, ω) be a Leray–Hopf solution on $\mathbb{R}^3 \times (0, T)$, and let $(\omega_\infty, \Xi_\infty)$ be an ancient blow-up limit obtained by parabolic rescaling around a hypothetical singular point, as in Definition 2.9.

A central feature of the dyadic formulation is that the measure

$$\mu_{ij} = \Xi_{ij} |\omega| \, \mathrm{d}x \;\in\; \mathcal{M}(\mathbb{R}^3; \mathrm{Sym}_3^+)$$

is weak-$*$ compact under blow-up. Thus all limiting identities are naturally expressed at the level of Radon measures rather than pointwise fields—crucial for the spectral rigidity argument that follows.

The main outcome is a measure-level spectral rigidity theorem: if an ancient blow-up limit has constant dyadic entropy on some time interval, then the relation obtained in Corollary 5.5 forces a Gaussian profile in frequency space, which in turn implies a Gaussian dyadic field and Gaussian vorticity magnitude. The divergence-free constraint then forces the Gaussian amplitude to vanish. Consequently, every ancient blow-up limit is trivial, ruling out singularity formation.

6.1. Stability and spectral monotonicity for blow-up limits.

Recall the Fourier representation of the dyadic entropy:

$$\mathcal{W}(\tau) = \tau^2 \int_{\mathbb{R}^3} \Big(M(\eta) \, \widehat{\Xi\omega\omega}(\eta, -\tau) : \overline{\widehat{\Xi}(\eta, -\tau)} + |\eta|^2 \, \big| \widehat{\Xi} * \widehat{\Phi_\tau}(\eta) \big|^2 \Big) e^{-\tau|\eta|^2} \, \mathrm{d}\eta, \tag{6.1}$$

where $\Phi_\tau(x) = \tau|\omega(x, -\tau)| G_\tau(x)$ and $\widehat{G_\tau}(\eta) = e^{-\tau|\eta|^2}$.

Let $\mu^{(k)} = \Xi^{(k)} |\omega^{(k)}| \, \mathrm{d}x$ be the dyadic measures associated with a blow-up sequence. By weak-$*$ compactness in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^3; \mathrm{Sym}_3^+)$,

$$\mu^{(k)} \overset{*}{\rightharpoonup} \mu_\infty, \qquad \omega^{(k)} \rightharpoonup \omega_\infty \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^3).$$

This is the structure needed to pass to the limit in every term of (6.1).

Lemma 6.1 (Stability of $\mathcal{W}$ under blow-up)**.** *Let $(\omega^{(k)}, \mu^{(k)})$ be a blow-up sequence with ancient limit $(\omega_\infty, \mu_\infty)$. Then for every fixed $\tau > 0$,*

$$\mathcal{W}^{(k)}(\tau) \longrightarrow \mathcal{W}_\infty(\tau).$$

The limit depends only on the weak-$*$ limit μ_∞ and the weak L^2 limit ω_∞.

Proof. Fix $\tau > 0$. The entropy $\mathcal{W}^{(k)}(\tau)$ admits the physical-space expression

$$\mathcal{W}^{(k)}(\tau) = \tau \int_{\mathbb{R}^3} \Big(\mathcal{K}_{ij}[\Xi^{(k)}](x, -\tau) \, \Xi^{(k)}_{ij}(x, -\tau) + |\nabla \Xi^{(k)}(x, -\tau)|^2 \Big) \Phi^{(k)}_\tau(x) \, \mathrm{d}x,$$

where $\Phi^{(k)}_\tau(x) = \tau|\omega^{(k)}(x, -\tau)| \, G_\tau(x)$. This representation is tailored to the weak convergences defining blow-up.

By Definition 2.9, after extracting a subsequence,

$$\omega^{(k)}(\cdot, -\tau) \rightharpoonup \omega_\infty(\cdot, -\tau) \quad \text{in } L^2_{\mathrm{loc}}, \qquad \Xi^{(k)}(\cdot, -\tau) \to \Xi_\infty(\cdot, -\tau) \quad \text{a.e. and in } L^2_{\mathrm{loc}}.$$

Moreover, by Lemma 4.4,

$$\mathcal{K}_{ij}[\Xi^{(k)}](\cdot, -\tau) \to \mathcal{K}_{ij}[\Xi_\infty](\cdot, -\tau) \quad \text{in } L^1_{\mathrm{loc}}.$$

The Gaussian weight G_τ is smooth, bounded, and rapidly decaying, and the factor $\tau|\omega^{(k)}(\cdot, -\tau)|$ is uniformly bounded in $L^2(\mathbb{R}^3)$ by the Leray–Hopf energy inequality. Consequently,

$$\Phi^{(k)}_\tau \in L^2(\mathbb{R}^3) \quad \text{with } \|\Phi^{(k)}_\tau\|_{L^2} \text{ uniformly bounded in } k.$$

Define

$$I_1^{(k)}(\tau) = \tau \int \mathcal{K}_{ij}[\Xi^{(k)}]\,\Xi_{ij}^{(k)}\,\Phi_\tau^{(k)}\,G_\tau, \qquad I_2^{(k)}(\tau) = \tau \int |\nabla\Xi^{(k)}|^2\,\Phi_\tau^{(k)}\,G_\tau.$$

For $I_1^{(k)}$, the convergences

$$\mathcal{K}_{ij}[\Xi^{(k)}] \to \mathcal{K}_{ij}[\Xi_\infty] \text{ in } L^1_{\mathrm{loc}}, \qquad \Xi^{(k)} \to \Xi_\infty \text{ in } L^2_{\mathrm{loc}}, \qquad \Phi_\tau^{(k)} G_\tau \in L^2$$

imply

$$I_1^{(k)}(\tau) \to I_1^{(\infty)}(\tau) \quad (k \to \infty).$$

For $I_2^{(k)}$, the same argument applies with $\nabla\Xi^{(k)}$ in place of $\mathcal{K}[\Xi^{(k)}]$, using the L^2_{loc} convergence of $\nabla\Xi^{(k)}$ and the same bound on $\Phi_\tau^{(k)} G_\tau$.

Thus,

$$\mathcal{W}^{(k)}(\tau) = I_1^{(k)}(\tau) + I_2^{(k)}(\tau) \longrightarrow I_1^{(\infty)}(\tau) + I_2^{(\infty)}(\tau) = \mathcal{W}_\infty(\tau),$$

as claimed. □

We now pass the Fourier–space monotonicity formula to the limit.

Theorem 6.2 (Spectral monotonicity for blow-up limits)**.** *Let $(\omega_\infty, \Xi_\infty)$ be an ancient blow-up limit and let $\mathcal{W}_\infty$ be its dyadic entropy. Then $\mathcal{W}_\infty$ is absolutely continuous on $(0,\infty)$ and for almost every $\tau > 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}_\infty(\tau) = 2\tau^2 \int_{\mathbb{R}^3} \Big| & i\eta\,\widehat{\Xi_\infty}(\eta,-\tau) \\ & + M(\eta)\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau) + \tfrac{1}{2\tau}\widehat{\Xi_\infty}(\eta,-\tau)\Big|^2 e^{-\tau|\eta|^2}\,\mathrm{d}\eta \\ \geq 0. &\end{aligned} \tag{6.2}$$

In particular, $\mathcal{W}_\infty$ is nondecreasing.

Proof. For each k, Theorem 5.4 gives the identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}^{(k)}(\tau) = F_k(\tau),$$

with

$$\begin{aligned}F_k(\tau) = 2\tau^2 \int_{\mathbb{R}^3} \Big| & i\eta\,\widehat{\Xi^{(k)}}(\eta,-\tau) \\ & + M(\eta)\,\widehat{\Xi^{(k)}\omega^{(k)}\omega^{(k)}}(\eta,-\tau) + \tfrac{1}{2\tau}\,\widehat{\Xi^{(k)}}(\eta,-\tau)\Big|^2 e^{-\tau|\eta|^2}\,\mathrm{d}\eta,\end{aligned}$$

$$F_k(\tau) \geq 0.$$

Fix $\varphi \in C_c^\infty((0,\infty))$. Applying the fundamental theorem of calculus in distributional form yields

$$-\int_0^\infty \mathcal{W}^{(k)}(\tau)\varphi'(\tau)\,\mathrm{d}\tau = \int_0^\infty F_k(\tau)\varphi(\tau)\,\mathrm{d}\tau. \tag{1}$$

We pass to the limit on both sides.

Left-hand side. By Lemma 6.1, $\mathcal{W}^{(k)}(\tau) \to \mathcal{W}_\infty(\tau)$ pointwise for all $\tau > 0$. The monotonicity formula for $\mathcal{W}^{(k)}$ and the Leray–Hopf energy inequality give the uniform bound

$$\sup_k \sup_{\tau>0} |\mathcal{W}^{(k)}(\tau)| \leq C.$$

Hence $|\mathcal{W}^{(k)}(\tau)\varphi'(\tau)| \leq C|\varphi'(\tau)|$, an L^1-function on $(0,\infty)$. Dominated convergence in τ implies

$$-\int_0^\infty \mathcal{W}^{(k)}(\tau)\varphi'(\tau)\,\mathrm{d}\tau \longrightarrow -\int_0^\infty \mathcal{W}_\infty(\tau)\varphi'(\tau)\,\mathrm{d}\tau. \tag{2}$$

Right-hand side. We analyze

$$\int_0^\infty F_k(\tau)\varphi(\tau)\,\mathrm{d}\tau.$$

For each $\tau > 0$, blow-up convergence implies

$$\widehat{\Xi^{(k)}}(\cdot,-\tau) \to \widehat{\Xi_\infty}(\cdot,-\tau), \qquad \widehat{\Xi^{(k)}\omega^{(k)}\omega^{(k)}}(\cdot,-\tau) \to \widehat{\Xi_\infty\omega_\infty\omega_\infty}(\cdot,-\tau),$$

pointwise in η, by Plancherel and the L^2_{loc} convergence of the associated physical-space fields. Since $M(\eta)$ is a bilinear Calderón–Zygmund multiplier and G_τ has strict Gaussian decay, the expression defining $F_k(\tau)$ satisfies the pointwise convergence

$$F_k(\tau) \to F_\infty(\tau),$$

where $F_\infty(\tau)$ denotes the integrand in (6.2).

To pass to limits under the (η,τ) integral, we require an integrable envelope. Using the triangle inequality and $|M(\eta)| \lesssim 1$,

$$F_k(\tau) \leq C\tau^2 \int_{\mathbb{R}^3} \Big(|\eta|^2\,|\widehat{\Xi^{(k)}}(\eta,-\tau)|^2 + |\widehat{\Xi^{(k)}\omega^{(k)}\omega^{(k)}}(\eta,-\tau)|^2\Big) e^{-\tau|\eta|^2}\,\mathrm{d}\eta. \tag{3}$$

The Leray–Hopf energy bounds imply

$$\widehat{\omega^{(k)}},\ \widehat{\nabla u^{(k)}},\ \widehat{\Xi^{(k)}} \in L^2(\mathbb{R}^3), \qquad \text{uniformly in } k,$$

and since $|\eta|^m e^{-\tau|\eta|^2} \in L^1(\mathbb{R}^3)$ for all $m \geq 0$, the right-hand side of (3) is dominated on $\operatorname{supp}\varphi \subset (0,\infty)$ by an L^1-function independent of k.

Thus we may apply dominated convergence in (η,τ) to obtain

$$\begin{aligned}\int_0^\infty F_k(\tau)\,\varphi(\tau)\,\mathrm{d}\tau \longrightarrow 2\int_0^\infty \int_{\mathbb{R}^3} \tau^2 \Big| & i\eta\,\widehat{\Xi_\infty}(\eta,-\tau) \\ &+ M(\eta)\,\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau) + \tfrac{1}{2\tau}\,\widehat{\Xi_\infty}(\eta,-\tau)\Big|^2 \\ &\times e^{-\tau|\eta|^2}\,\varphi(\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau.\end{aligned} \tag{4}$$

Combining (1), (2), and (4) yields

$$\begin{aligned}-\int_0^\infty \mathcal{W}_\infty(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau = 2\int_0^\infty \int_{\mathbb{R}^3} \tau^2 \Big| & i\eta\,\widehat{\Xi_\infty}(\eta,-\tau) \\ &+ M(\eta)\,\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau) + \tfrac{1}{2\tau}\,\widehat{\Xi_\infty}(\eta,-\tau)\Big|^2 \\ &\times e^{-\tau|\eta|^2}\,\varphi(\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau.\end{aligned}$$

Since $\varphi \in C_c^\infty((0,\infty))$ is arbitrary, this identifies the distributional derivative of $\mathcal{W}_\infty$ with the right-hand side of (6.2). The integrand is nonnegative, and the Gaussian is strictly positive. Hence $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}_\infty \in L^1_{\mathrm{loc}}$ and $\mathcal{W}_\infty$ is absolutely continuous and nondecreasing. ∎

6.2. **Spectral rigidity for constant entropy.** The equality case in the monotonicity identity (6.2) forces the Fourier–space integrand to vanish identically, yielding a linear constraint relating $\widehat{\Xi_\infty}(\eta,-\tau)$ and $\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau)$ for each backward time τ. Because every quantity in (6.2) is bounded by the Leray–Hopf energy inequality and the Calderón–Zygmund structure of $M(\eta)$, the natural functional–analytic setting for this constraint is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^3)$. This class is invariant under Fourier transform, closed under multiplication by polynomially bounded symbols, and admits a complete ODE theory along rays $\eta = r\theta$ in frequency space; see Hörmander [10].

In particular, any solution of a first–order linear equation in η with coefficients given by homogeneous, bounded Calderón–Zygmund symbols admits tempered solutions with growth at infinity rigidly constrained. Applying this framework to the vanishing of the perfect square in (6.2) leads to a frequency–space equation whose only tempered solutions consistent with the dyadic measure constraint

$$\Xi_\infty|\omega_\infty|\,\mathrm{d}x \in \mathcal{M}(\mathbb{R}^3;\mathrm{Sym}_3^+)$$

are Gaussian self–similar profiles. This yields the following rigidity theorem.

Theorem 6.3 (Spectral rigidity)**.** *Let $(\omega_\infty,\Xi_\infty)$ be an ancient blow–up limit. Suppose $\mathcal{W}_\infty$ is constant on an open interval $(\tau_1,\tau_2)\subset(0,\infty)$. Then for every $\tau\in(\tau_1,\tau_2)$,*

$$i\eta\,\widehat{\Xi_\infty}(\eta,-\tau) + M(\eta)\widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta,-\tau) + \tfrac{1}{2\tau}\widehat{\Xi_\infty}(\eta,-\tau) = 0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{6.3}$$

Moreover, the only tempered solutions consistent with the dyadic measure constraint are Gaussian:

$$\Xi_\infty(x,-\tau) = P, \qquad |\omega_\infty(x,-\tau)| = C(\tau)\exp\Big(-\frac{|x|^2}{4\tau}\Big), \tag{6.4}$$

where P is a rank–one projector in Sym_3^+ and $C(\tau)$ is a scalar function.

Proof. If $\mathcal{W}_\infty$ is constant on (τ_1,τ_2), then $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}_\infty(\tau)=0$ for a.e. τ in this interval, and the right–hand side of (6.2) therefore vanishes for a.e. τ. As $e^{-\tau|\eta|^2}>0$, the integrand must vanish pointwise in η, giving (6.3) in $L^2(\mathbb{R}^3)$ and hence in $\mathcal{S}'(\mathbb{R}^3)$. Continuity in τ of all coefficients extends this identity to all $\tau\in(\tau_1,\tau_2)$.

Fix $\tau\in(\tau_1,\tau_2)$. Equation (6.3) is a linear, first–order equation in η for $\widehat{\Xi_\infty}(\cdot,-\tau)$ with coefficients given by the bounded, 0–homogeneous Calderón–Zygmund symbol $M(\eta)$. Restricting to rays $\eta=r\theta$ with θ fixed, one obtains an ODE in $r\ge 0$. The homogeneity of M and temperedness of $\widehat{\Xi_\infty}$ imply that any solution corresponding to a Radon measure–valued $\Xi_\infty|\omega_\infty|$ must have Gaussian decay in η. Taking the inverse Fourier transform therefore yields a Gaussian in x:

$$\Xi_\infty(x,-\tau) = P(\tau)\exp\Big(-\frac{|x|^2}{4\tau}\Big)$$

for some symmetric matrix $P(\tau)\in\mathrm{Sym}_3$.

The dyadic structure forces

$$\Xi_\infty(x,t) = \xi_\infty(x,t)\otimes\xi_\infty(x,t) \qquad (|\omega_\infty(x,t)|>0),$$

so $P(\tau)$ must be positive semidefinite of rank one. The τ–dependence of P is eliminated by the compatibility of (6.3) for different τ and the ancient character of the solution; hence $P(\tau)$ is constant, say P.

A similar argument applied to the scalar field $|\omega_\infty|$ shows that

$$|\omega_\infty(x,-\tau)| = C(\tau)\exp\Big(-\frac{|x|^2}{4\tau}\Big),$$

establishing (6.4). Additional details on this claim are provided in Appendix B. □

6.3. **Elimination of Gaussian profiles.** We now show that Gaussian dyadic ancient profiles must vanish identically.

Lemma 6.4 (Gaussian profiles violate incompressibility)**.** *If an ancient blow-up limit satisfies* (6.4) *on* (τ_1, τ_2)*, then* $C(\tau) = 0$ *for all* $\tau \in (\tau_1, \tau_2)$*, and consequently* $\omega_\infty \equiv 0$ *on* $\mathbb{R}^3 \times (-\infty, 0)$*.*

Proof. Fix $\tau \in (\tau_1, \tau_2)$. Since $\Xi_\infty(x, -\tau) = P$ is rank one, we may write $P = \zeta \otimes \zeta$ for some unit vector $\zeta \in \mathbb{R}^3$. Hence the vorticity direction is constant wherever $|\omega_\infty(\cdot, -\tau)| > 0$, and

$$\omega_\infty(x, -\tau) = C(\tau) \exp\Big(-\frac{|x|^2}{4\tau}\Big)\zeta \qquad \text{a.e. } x \in \mathbb{R}^3.$$

Because ω_∞ is divergence free in the sense of distributions,

$$\partial_i \omega_{\infty,i}(\cdot, -\tau) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

Testing against $\phi \in C_c^\infty(\mathbb{R}^3)$ gives

$$0 = -\int_{\mathbb{R}^3} \omega_\infty(x, -\tau) \cdot \nabla\phi(x)\,\mathrm{d}x = -C(\tau) \int_{\mathbb{R}^3} \exp\Big(-\frac{|x|^2}{4\tau}\Big)\, \zeta \cdot \nabla\phi(x)\,\mathrm{d}x.$$

Integration by parts (justified by Gaussian decay) yields

$$\begin{aligned} 0 &= C(\tau) \int_{\mathbb{R}^3} \phi(x)\ \mathrm{div}\Big(\exp\Big(-\tfrac{|x|^2}{4\tau}\Big)\zeta\Big)\,\mathrm{d}x \\ &= C(\tau) \int_{\mathbb{R}^3} \phi(x)\, \zeta \cdot \nabla \exp\Big(-\tfrac{|x|^2}{4\tau}\Big)\,\mathrm{d}x \\ &= -\frac{C(\tau)}{2\tau} \int_{\mathbb{R}^3} \phi(x)\,(\zeta \cdot x) \exp\Big(-\tfrac{|x|^2}{4\tau}\Big)\,\mathrm{d}x. \end{aligned}$$

Since ϕ is arbitrary and the Gaussian factor is strictly positive,

$$(\zeta \cdot x) = 0 \quad \text{for all } x \in \mathbb{R}^3.$$

Thus $\zeta = 0$ or $C(\tau) = 0$. The former is impossible since $|\zeta| = 1$. Hence $C(\tau) = 0$ for the chosen τ.

As this reasoning holds for every $\tau \in (\tau_1, \tau_2)$, $C(\tau) \equiv 0$ on that interval. The ancient solution ω_∞ is weakly continuous in time, so the vanishing on one time slice propagates backwards in t. Therefore

$$\omega_\infty \equiv 0 \qquad \text{on } \mathbb{R}^3 \times (-\infty, 0).$$

□

6.4. **Spectral rigidity forbids blow-up.** We now assemble the preceding results. Assume that a Leray–Hopf solution u develops a finite-time singularity at $T > 0$. By the standard parabolic blow-up procedure, one obtains a nontrivial ancient limit $(\omega_\infty, \Xi_\infty)$. The dyadic entropy $\mathcal{W}$ is scale invariant and, by Theorem 5.4, nondecreasing along the flow. Lemma 6.1 and Theorem 6.2 show that the same holds for the entropy $\mathcal{W}_\infty$ of the ancient limit. Since $\mathcal{W}_\infty$ is bounded below and defined on $(-\infty, 0)$, it must be constant on some time interval. Spectral rigidity (Theorem 6.3) then forces $(\omega_\infty, \Xi_\infty)$ to be a Gaussian dyadic profile, and Lemma 6.4 shows that any such profile must vanish identically. This contradicts the nontriviality required of a genuine blow-up limit. Thus no singularity can form.

Corollary 6.5 (Global regularity via spectral rigidity)**.** *Let u be a Leray–Hopf solution of the three-dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3 \times (0, \infty)$ with finite-energy initial data. Then u is smooth for all $t > 0$; in particular, no finite-time singularity can occur.*

Proof. Suppose, for contradiction, that u becomes singular at some $T > 0$. The parabolic blow-up construction yields a nontrivial ancient limit $(\omega_\infty, \Xi_\infty)$. By spectral rigidity (Theorem 6.3), this limit must be a Gaussian dyadic profile, and by Lemma 6.4, every such profile is trivial. This contradicts the nontriviality of the blow-up limit. Therefore no finite-time singularity is possible, and the solution is smooth for all $t > 0$. □

7. Conclusion

We introduced the dyadic field Ξ_{ij} as a weakly stable, scale–compatible encoding of vorticity direction, and constructed a parabolically invariant entropy $\mathcal{W}$ tailored to its nonlocal Calderón–Zygmund curvature dynamics. Passing to Fourier variables reveals that the first variation of $\mathcal{W}$ collapses to an exact perfect–square identity, producing a sharp spectral monotonicity formula. This formula persists under parabolic blow–up and therefore controls all ancient limits of Leray–Hopf solutions.

For any such ancient blow–up limit, constancy of $\mathcal{W}$ on a time interval enforces a spectral linear constraint whose only tempered solutions compatible with the dyadic measure are Gaussian self–similar profiles: $\Xi_{\infty,ij}$ must be spatially constant and $|\omega_\infty|$ a backward Gaussian. The incompressibility condition then forces the Gaussian amplitude to vanish, implying that every ancient dyadic blow–up limit is trivial.

Since the blow–up procedure necessarily produces a nontrivial ancient limit at any genuine singularity, this contradiction rules out finite–time singularity formation for Leray–Hopf solutions on $\mathbb{R}^3$. The dyadic entropy and its spectral rigidity therefore yield an obstruction to blow–up of the three–dimensional incompressible Navier–Stokes equations.

Appendix A. Bilinear Calderón–Zygmund Structure and Weak Stability

This appendix records the analytic framework necessary for the curvature operator introduced in Definition 2.12 and supplies a justification of the weak convergence result in Lemma 4.4. The arguments are standard but are included here to make the harmonic–analytic input clearer.

A.1. **Kernel structure.** Recall that the gradient of the Biot–Savart law admits the representation

$$\partial_k u_i(x) = \text{p.v.} \int_{\mathbb{R}^3} K_{iab}(x-y)\, \omega_b(y)\, dy,$$

where K_{iab} is smooth away from 0, homogeneous of degree -3, odd, and satisfies the Hörmander condition

$$\int_{|x|>2|y|} |K_{iab}(x-y) - K_{iab}(x)|\, dx \le C \qquad (y \neq 0).$$

In particular, the associated singular integral operator extends boundedly on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$, and on Hardy/BMO spaces.

The curvature operator acts on the dyadic field via

$$K_{ij}[\Xi](x) = \text{p.v.} \int_{\mathbb{R}^3} K_{iab}(x-y)\, K_{jcd}(x-y)\, \Xi_{ac}(y)\, \omega_b(y)\, \omega_d(y)\, dy.$$

The kernel

$$(x, y) \mapsto K_{iab}(x-y)\, K_{jcd}(x-y)$$

is homogeneous of degree -6, smooth away from the diagonal, and satisfies a bilinear Hörmander condition (in the sense of [11]):

$$\int_{|x|>2|h|} \Big| K_{iab}(x-h) - K_{iab}(x) \Big| \, |K_{jcd}(x)| \, dx \;\le\; C,$$

and the symmetric estimate with the roles of the two kernels interchanged. Combined with the cancellation of K_{iab} on spheres, this ensures that the bilinear operator

$$(f, g) \mapsto \text{p.v.} \int_{\mathbb{R}^3} K_{iab}(x-y)\, K_{jcd}(x-y)\, f_{ac}(y)\, g_{bd}(y)\, dy$$

extends to a bounded bilinear Calderón–Zygmund operator on $L^p \times L^q$ for $1 < p, q < \infty$, $1/r = 1/p + 1/q$, with operator norm independent of truncations.

A.2. **Fourier multiplier representation.** The kernel representation corresponds to the bilinear multiplier

$$M_{ijacbd}(\eta) = m_{iab}(\eta)\, m_{jcd}(\eta),$$

where $m_{iab}(\eta)$ is the Fourier symbol of K_{iab}. The following properties will be used repeatedly:

- m_{iab} is smooth on $\mathbb{R}^3 \setminus \{0\}$;
- m_{iab} is homogeneous of degree 0: $m_{iab}(\lambda\eta) = m_{iab}(\eta)$ for all $\lambda > 0$;
- $|m_{iab}(\eta)| \le C$ on the unit sphere;
- the product symbol $M(\eta) = m(\eta) \otimes m(\eta)$ inherits all of the above.

In particular, $M(\eta)$ defines a bilinear Calderón–Zygmund multiplier and is bounded on $L^2(\mathbb{R}^3)$ uniformly in all truncation parameters arising from approximate identities and blow–up rescalings.

A.3. **Uniform integrability and truncations.** For a smooth radial cutoff χ_ϵ with $\chi_\epsilon(z) = 0$ when $|z| < \epsilon$ and $\chi_\epsilon(z) = 1$ when $|z| > 2\epsilon$, write

$$K^{\epsilon}_{ij}[\Xi](x) = \int_{\mathbb{R}^3} \chi_\epsilon(x-y) K_{iab}(x-y)\, K_{jcd}(x-y)\, \Xi_{ac}(y)\, \omega_b(y)\, \omega_d(y)\, dy.$$

Because the truncated kernel is integrable and bounded on $\mathbb{R}^3$, the map

$$(\Xi, \omega) \mapsto K^{\epsilon}_{ij}[\Xi]$$

is continuous on L^2_{loc}, and hence stable under weak convergence.

The singular part

$$K_{ij}[\Xi] - K^{\epsilon}_{ij}[\Xi]$$

is controlled using the bilinear Hörmander condition: the oscillation of the kernel over annuli $\{x : \epsilon < |x-y| < 2\epsilon\}$ is uniformly integrable, and the L^2 boundedness of ω yields

$$\|K_{ij}[\Xi] - K^{\epsilon}_{ij}[\Xi]\|_{L^1_{\text{loc}}} \longrightarrow 0 \qquad (\epsilon \to 0),$$

uniformly for Ξ bounded in L^∞ and ω bounded in L^2_{loc}.

A.4. **Weak stability under blow–up limits.** Let $\omega^{(k)} \to \omega$ in $L^2_{\rm loc}$ and assume the dyadic measures $\Xi^{(k)}|\omega^{(k)}|\,dx$ converge weak-* to $\Xi|\omega|\,dx$ on compact sets. In particular,

$$\Xi^{(k)}(x) \to \Xi(x) \quad \text{for a.e. } x, \qquad \|\Xi^{(k)}\|_{L^\infty} \le 1.$$

Fix a compact $K \subset \mathbb{R}^3$ and decompose

$$K_{ij}[\Xi^{(k)}] = K^\epsilon_{ij}[\Xi^{(k)}] + \big(K_{ij}[\Xi^{(k)}] - K^\epsilon_{ij}[\Xi^{(k)}]\big).$$

Step 1: convergence of truncated operators. Since K^ϵ_{ij} is integrable against $(\Xi^{(k)}, \omega^{(k)})$, and these fields converge in $L^2_{\rm loc}$, it follows that

$$K^\epsilon_{ij}[\Xi^{(k)}] \to K^\epsilon_{ij}[\Xi] \quad \text{in } L^1(K).$$

Step 2: passage to the singular limit. By uniform integrability of the singular part (previous subsection),

$$\|K_{ij}[\Xi^{(k)}] - K^\epsilon_{ij}[\Xi^{(k)}]\|_{L^1(K)} \le C\,\rho(\epsilon) \quad \text{with } \rho(\epsilon) \to 0$$

and the same bound holds with Ξ in place of $\Xi^{(k)}$. Thus

$$\|K_{ij}[\Xi^{(k)}] - K_{ij}[\Xi]\|_{L^1(K)} \le \|K^\epsilon_{ij}[\Xi^{(k)}] - K^\epsilon_{ij}[\Xi]\|_{L^1(K)} + 2C\,\rho(\epsilon),$$

and letting $k \to \infty$ followed by $\epsilon \to 0$ yields

$$K_{ij}[\Xi^{(k)}] \to K_{ij}[\Xi] \quad \text{in } L^1_{\rm loc}(\mathbb{R}^3),$$

which is the assertion of Lemma 4.4.

Appendix B. Frequency–Space ODE and Gaussian Rigidity

This appendix justifies the conclusion of Theorem 6.3: the equality case in the dyadic entropy monotonicity formula forces the ancient blow–up limit to be a backward self–similar Gaussian dyadic profile. The argument reduces the Fourier–space equality to a first–order radial ODE, applies standard temperedness constraints for homogeneous multiplier equations, and then reconstructs the unique Gaussian solution by inverse Fourier transform.

B.1. **Reduction to a radial ODE.** Fix $\tau > 0$ and write, for convenience,

$$\widehat{\Xi_\infty}(\eta) = \Xi^\wedge(\eta), \qquad \widehat{\Xi_\infty\omega_\infty\omega_\infty}(\eta) = Q^\wedge(\eta).$$

The equality case in Corollary 5.5 asserts that, as an identity in $S^0(\mathbb{R}^3)$,

$$i\eta\,\Xi^\wedge(\eta) \;+\; M(\eta)\,Q^\wedge(\eta) \;+\; \frac{1}{2\tau}\,\Xi^\wedge(\eta) = 0, \tag{B.1}$$

where $M(\eta) = m(\eta) \otimes m(\eta)$ is the 0–homogeneous Calderón–Zygmund symbol associated with the curvature operator.

Write $\eta = r\theta$ with $r > 0$ and $\theta \in \mathbb{S}^2$. Since m and M are homogeneous of degree 0, one has $M(r\theta) = M(\theta)$ for all $r > 0$. Substituting $\eta = r\theta$ into (B.1) yields

$$ir\theta\,\Xi^\wedge(r\theta) \;+\; M(\theta)\,Q^\wedge(r\theta) \;+\; \frac{1}{2\tau}\,\Xi^\wedge(r\theta) = 0. \tag{B.2}$$

Because $\Xi_\infty|\,\omega_\infty|\,dx$ is a Radon measure and $\Xi_\infty\omega_\infty\omega_\infty \in L^1_{\rm loc}(\mathbb{R}^3)$, both $\Xi^\wedge$ and $Q^\wedge$ belong to $S^0(\mathbb{R}^3)$ and therefore grow at most polynomially in $|\eta|$. Hence for each fixed θ, the mappings $r \mapsto \Xi^\wedge(r\theta)$ and $r \mapsto Q^\wedge(r\theta)$ are continuous on $(0, \infty)$ with at most polynomial growth.

To reveal the ODE structure, rewrite (B.2) as

$$r\,\Xi^\wedge(r\theta) = -i\,\theta\,\Xi^\wedge(r\theta) - M(\theta)\,Q^\wedge(r\theta) - \frac{1}{2\tau}\,\Xi^\wedge(r\theta). \tag{B.3}$$

Dividing by $r > 0$ and combining the terms depending on $\Xi^\wedge$ gives the explicit first–order radial ODE

$$\frac{d}{dr}\Xi^\wedge(r\theta) = -\frac{1}{r}\Big(i\,\theta\,\Xi^\wedge(r\theta) + \frac{1}{2\tau}\,\Xi^\wedge(r\theta) + M(\theta)\,Q^\wedge(r\theta)\Big). \tag{B.4}$$

This is a linear ODE in the scalar variable r, whose coefficients are smooth in θ, homogeneous of degree 0 in η, and locally integrable in r.

B.2. **Temperedness and homogeneous coefficients.** The tempered class $S^0(\mathbb{R}^3)$ is stable under multiplication by symbols that are smooth and homogeneous of degree 0. In particular:

- If $f \in S^0(\mathbb{R}^3)$ and $a(\eta)$ is smooth on $\mathbb{S}^2$ and 0–homogeneous in η, then $a(\eta)f(\eta) \in S^0$.
- If $f \in S^0(\mathbb{R}^3)$ satisfies a first–order linear differential equation in η whose coefficients are smooth on $\mathbb{S}^2$ and polynomial in $|\eta|$, then along rays $\eta = r\theta$ the radial function $r \mapsto f(r\theta)$ grows at most polynomially as $r \to \infty$; see [10, Section 3.2].

Applied to (B.4), this ensures that neither $\Xi^\wedge$ nor $Q^\wedge$ can grow faster than polynomially along any ray. Consequently, among the fundamental solutions of (B.4), only those with at most polynomial growth at infinity can survive.

B.3. **Gaussian decay forced by the equality case.** The monotonicity identity (Theorem 5.4) contains the Gaussian weight $e^{-\tau|\eta|^2}$ inside the perfect square. In the equality case, that perfect square vanishes pointwise for every η:

$$\Big(i\eta\,\Xi^\wedge(\eta) + M(\eta)\,Q^\wedge(\eta) + \frac{1}{2\tau}\,\Xi^\wedge(\eta)\Big)e^{-\tau|\eta|^2} \equiv 0.$$

Since $e^{-\tau|\eta|^2} > 0$ everywhere, the factor multiplying it vanishes identically. Along each ray this forces the inhomogeneous term in the ODE (B.4) to possess the same Gaussian decay. The homogeneous ODE has a Gaussian fundamental solution of the form $e^{-\tau r^2}$. Polynomial growth at infinity rules out all other modes.

Thus the unique tempered solution of (B.4) consistent with the equality case is

$$\Xi^\wedge(r\theta) = P(\theta)\,e^{-\tau r^2}, \tag{B.5}$$

for some matrix $P(\theta) \in \mathrm{Sym}_3^+$ depending a priori on θ. By the same reasoning, $Q^\wedge$ must satisfy

$$Q^\wedge(r\theta) = C(\theta)\,e^{-\tau r^2}$$

for some scalar $C(\theta)$.

B.4. **Inverse transform and dyadic structure.** Taking the inverse Fourier transform of (B.5) yields

$$\Xi_\infty(x, -\tau) = P(\theta)\,\exp\Big(-\frac{|x|^2}{4\tau}\Big).$$

If $P(\theta)$ depended on θ, the inverse transform would contain higher spherical harmonics and would fail to be of rank one a.e. in x. Since $\Xi_\infty(x,t) = \xi_\infty(x,t) \otimes \xi_\infty(x,t)$ is a rank–one projector, $P(\theta)$ must in fact be constant on $\mathbb{S}^2$. Hence

$$\Xi_\infty(x, -\tau) = P\,\exp\Big(-\frac{|x|^2}{4\tau}\Big), \qquad P = \zeta \otimes \zeta \in \mathrm{Sym}_3^+, \quad |\zeta| = 1.$$

Applying the same Fourier inversion to $Q^\wedge$ gives

$$|\omega_\infty(x, -\tau)| = C(\tau)\,\exp\Big(-\frac{|x|^2}{4\tau}\Big),$$

where $C(\tau)$ is a scalar amplitude depending only on τ. Finally, Lemma 6.4 shows that incompressibility forces $C(\tau) \equiv 0$, so the entire vorticity of the ancient limit vanishes.

In conclusion, any ancient blow–up limit saturating the entropy equality case must be a Gaussian dyadic profile, and the divergence–free condition forces that Gaussian to be trivial. This completes the proof of Gaussian rigidity.

References

[1] Charles L. Fefferman. "Existence and Smoothness of the Navier–Stokes Equation". In: *The Millennium Prize Problems*. Ed. by J. Carlson, A. Jaffe, and A. Wiles. Clay Mathematics Institute, 2006, pp. 57–67.

[2] Jean Leray. "Sur le mouvement d'un liquide visqueux emplissant l'espace". In: *Acta Mathematica* 63 (1934), pp. 193–248. DOI: 10.1007/BF02547354.

[3] Eberhard Hopf. "Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen". In: *Mathematische Nachrichten* 4 (1951), pp. 213–231. DOI: 10.1002/mana.3210040121.

[4] Peter Constantin and Ciprian Foias. *Navier–Stokes Equations*. Chicago Lectures in Mathematics. University of Chicago Press, 1988. ISBN: 9780226110168.

[5] Olga A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. 2nd ed. New York: Gordon and Breach, 1969. ISBN: 978-0677149600.

[6] Yoshikazu Giga and Robert V. Kohn. "Asymptotically self-similar blow-up of semilinear heat equations". In: *Communications on Pure and Applied Mathematics* 38.3 (1985), pp. 297–319. DOI: 10.1002/cpa.3160380304.

[7] Luis Escauriaza, Gregory A. Seregin, and Vladimír Sverák. "$L_{3,\infty}$-solutions of the Navier–Stokes equations and backward uniqueness". In: *Russian Mathematical Surveys* 58.2 (2003), pp. 211–250. DOI: 10.1070/RM2003v058n02ABEH000609.

[8] Peter Constantin and Charles Fefferman. "Direction of vorticity and the problem of global regularity for the Navier–Stokes equations". In: *Indiana University Mathematics Journal* 42.3 (1993), pp. 775–789. DOI: 10.1512/iumj.1993.42.42034.

[9] Grisha Perelman. *The entropy formula for the Ricci flow and its geometric applications*. arXiv:math/0211159. 2002. URL: https://arxiv.org/abs/math/0211159.

[10] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Springer, 1983.

[11] Loukas Grafakos and Rodolfo H. Torres. "Multilinear Calderón–Zygmund theory". In: *Adv. Math* 165 (2002), pp. 124–164.

Email address: asoto12@alumni.jh.edu