INTRODUCTION TO INCOMPRESSIBLE FLUID DYNAMICS

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Abstract. We present a concise geometric introduction to incompressible fluid dynamics, emphasizing vorticity, nonlocality, and the evolution structure most relevant to the Navier–Stokes regularity problem. Classical kinematics is reformulated in terms of direction fields and dyadic geometry, providing a clear pathway from Eulerian dynamics to entropy-based rigidity and modern blow-up analysis.

Contents

1. Introduction

Fluid motion is one of the most familiar phenomena in nature, yet the equations governing it remain among the most difficult in mathematics. Whether water flowing through a pipe, smoke drifting upward, or atmospheric currents spanning continents, fluid motion follows universal principles: each parcel of fluid moves, twists, and interacts with every other. In many regimes, especially at moderate speeds or when density variations are small, these dynamics are well-approximated by the incompressible Navier–Stokes equations. Incompressibility here is not a literal physical claim—air does compress—but a mathematical idealization that captures a volume-preserving regime in which the geometry of the flow becomes especially transparent [1, 2].

A fluid is, at a microscopic level, an enormous collection of molecules undergoing constant collisions. Nothing at this scale suggests that velocity or pressure should be smooth functions of position. Yet in almost every physical regime outside extreme micro- and nano-scales, fluids behave as if they were continuous media: they bend smoothly, develop coherent vortices, transmit pressure, and influence nearby motion in ways that would be impossible if molecular discreteness dominated. The magnitude of this scale separation is extraordinary: a cubic millimeter of liquid water contains approximately

$$N = \frac{\rho\, V}{m_{\mathrm{H_2O}}} = \frac{(1000\,\mathrm{kg/m^3})\,(1\times 10^{-9}\,\mathrm{m^3})}{(2.99\times 10^{-26}\,\mathrm{kg})} \approx 3.34\times 10^{19}$$

molecules—hundreds of times more than the estimated number of stars in the observable universe. A typical water molecule has a mean free path of merely $\ell_{\mathrm{mfp}} \sim 0.1$–$0.3\,\mathrm{nm}$, while macroscopic flow structures may range from millimeters to thousands of kilometers [3, 4]. Such enormous disparity justifies describing a fluid by smoothly varying fields whose evolution is governed by partial differential equations.

Even more striking is the phenomenon of turbulence: when the dimensionless *Reynolds number*

$$\mathrm{Re} = \frac{UL}{\nu},$$

becomes large, fluid motion transitions from orderly to violently irregular. Here:

- U is a characteristic flow velocity (e.g. mean or peak velocity),
- L is a characteristic length scale of the flow (e.g. pipe diameter),
- ν is the kinematic viscosity of the fluid (for water, $\nu \approx 10^{-6}\,\mathrm{m^2/s}$).

Large Re indicates that inertial forces dominate viscous forces, enabling vortices to stretch, fold, and interact across a hierarchy of scales.

According to Kolmogorov's 1941 phenomenology, turbulent kinetic energy injected at large scales cascades to progressively smaller eddies until it reaches the *dissipation scale*

$$\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4},$$

where:

- ε is the mean kinetic energy dissipation rate per unit mass (units: $\mathrm{m^2/s^3}$),
- η is the smallest dynamically relevant length scale of the flow.

In water, with $\nu \approx 10^{-6}\,\mathrm{m^2/s}$ and typical laboratory values $\varepsilon \sim 10^{-3}$–$10^{-1}\,\mathrm{m^2/s^3}$, one obtains η on the order of tens of microns: far larger than molecular scales ($\sim 10^{-10}\,\mathrm{m}$), yet far smaller than observable macroscopic structures.

Between the large (integral) scale L at which energy is injected and the small dissipation scale η lies the turbulent *inertial range*, across which the energy flux remains

approximately constant. Within this range, empirical velocity increments obey the celebrated Kolmogorov 2/3-law for the second-order structure function,

$$S_2(r) = \mathbb{E}\big(|u(x+r) - u(x)|^2\big) \sim (\varepsilon\, r)^{2/3}, \qquad \eta \ll r \ll L,$$

a cornerstone of modern turbulence theory [5, 6].

A typical laboratory-scale turbulent flow may involve more than 10^{10} dynamically active degrees of freedom. Yet, at all scales and in every regime, the governing PDEs are the same: the incompressible Navier–Stokes equations.

Historically, the mathematical structure of fluid dynamics emerged gradually over more than two centuries. Euler's 1757 memoir established the equations of motion for an ideal (inviscid) fluid [7]. Navier, in 1822, introduced the first continuum-mechanical model of viscosity based on molecular interactions [8], and Stokes refined this framework in 1845 by formulating the modern stress–strain relation and completing the viscous term as it appears today [9] (1.6). A century later, Leray's 1934 work founded the modern theory of weak solutions, proving global existence of finite-energy solutions and introducing concepts—such as suitable weak limits and energy inequalities—that remain central to PDE analysis [10]. The Clay Mathematics Institute's Millennium Problem on existence and smoothness of Navier–Stokes solutions continues this development [11].

Fluid dynamics stands at the interface of physical modeling, empirical turbulence phenomena, and deep mathematical structure. To develop the analytical framework needed for incompressible flow, we begin by formalizing the objects of the theory: the measurable, vector- and tensor-valued fields defined on subsets of spacetime and the notation used to manipulate them. This requires a careful treatment of spacetime as a measure-theoretic domain, tensor indexing, and the weak-analytic tools on which all subsequent arguments rest.

1.1. **Preliminaries and notation.** A *subset of spacetime* is simply a Cartesian product $\Omega \times I$, where Ω is a spatial region and I is a time interval. Set-theoretically,

$$\Omega \times I = \{(x,t) : x \in \Omega,\ t \in I\},$$

and any field is a function whose domain is this product. Analytically, we regard Ω as equipped with its Borel σ-algebra $\mathcal{B}(\Omega)$ and the Lebesgue measure λ^3, so that $(\Omega, \mathcal{B}(\Omega), \lambda^3)$ is a measure space. Likewise, the time interval carries the one-dimensional Lebesgue measure λ^1. Their product

$$(\Omega \times I,\ \mathcal{B}(\Omega) \otimes \mathcal{B}(I),\ \lambda^3 \otimes \lambda^1)$$

is the canonical measure-theoretic model of spacetime in Euclidean fluid mechanics; integrals over Ω or $\Omega \times I$ are always understood with respect to these Lebesgue measures. Two functions that differ only on a set of measure zero are identified, since all integral identities and distributional formulations are insensitive to such differences.

Within this framework, a scalar field assigns to each point (x,t) a real number, while a vector field assigns a vector in $\mathbb{R}^3$. More generally, a *tensor field* is a measurable function

$$T\colon \Omega \times I \to \mathcal{T}^{(r,s)}(\mathbb{R}^3),$$

where $\mathcal{T}^{(r,s)}(\mathbb{R}^3)$ denotes the space of rank-(r,s) tensors. In practice we work in Euclidean space relative to the standard basis, so every tensor field is identified with its component functions, each of which is an element of an appropriate Lebesgue space on $\Omega \times I$. For instance, a vector field $u(x,t)$ is a rank-$(1,0)$ tensor with components $u_i(x,t)$, and a matrix field $A(x,t)$ is a rank-$(1,1)$ tensor with components $A_{ij}(x,t)$.

Higher-rank examples such as the Cauchy stress tensor or the vorticity direction dyadic $\Xi_{ij} = \xi \otimes \xi$ introduced later fit into the same componentwise description.

Whenever indices appear, they label components relative to the basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Thus

$$u(x,t) = u_i(x,t)e_i, \qquad (\sigma u)_i = \sigma_{ij}u_j,$$

and repeated indices are summed over $j = 1, 2, 3$ according to the Einstein summation convention. Component functions are measurable by default, and equalities such as $u_i = v_i$ are always interpreted almost everywhere with respect to the underlying Lebesgue measure. This convention is crucial: all weak derivatives, divergence identities, and integral formulations later in the paper rely on properties holding for almost every $(x,t) \in \Omega \times I$, rather than pointwise.

Because Lebesgue measure is complete and translation invariant, it provides the natural setting for the function spaces used throughout—$L^p(\Omega)$, $L^p(\Omega \times I)$, Sobolev spaces, and their Bochner counterparts. All tensorial identities, contractions, and differential operators introduced later are interpreted componentwise in this measure-theoretic sense.

Vectors and matrices as tensors. A general vector $u \in \mathbb{R}^3$ has the component representation

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

and a general matrix (rank-$(1,1)$ tensor) has the form

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

The tensor (outer) product of two vectors u and v is the matrix $u \otimes v$ with components

$$(u \otimes v)_{ij} = u_i v_j, \qquad u \otimes v = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}.$$

Thus expressions such as $u_i u_j$ represent the i, j component of $u \otimes u$.

Contractions arise by summation over paired indices. The double contraction of two matrices A and B is the scalar

$$A : B = A_{ij}B_{ij},$$

the Frobenius inner product. In matrix notation,

$$A : B = A_{11}B_{11} + A_{12}B_{12} + \cdots + A_{33}B_{33}.$$

Likewise, the contraction of a matrix with a vector is

$$(Au)_i = A_{ij}u_j.$$

Covectors and differentials. In addition to vectors, we frequently use *covectors*, or rank-$(0,1)$ tensors, which act linearly on vectors. In $\mathbb{R}^3$ every covector can be written as

$$\alpha = \alpha_i \, \mathrm{d}x^i,$$

where $\{\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\}$ is the dual basis to $\{e_1, e_2, e_3\}$. The action on a vector $v = v_i e_i$ is

$$\alpha(v) = \alpha_i v_i.$$

Differentials arise naturally as covector fields. For a scalar field $\phi\colon \Omega \times I \to \mathbb{R}$, the differential is

$$\mathrm{d}\phi = \partial_i \phi \, \mathrm{d}x^i,$$

so that $\mathrm{d}\phi$ encodes the directional derivative of ϕ in every coordinate direction.

Volume forms and the wedge product. In differential geometry, integration over a region requires a *volume form*, often constructed from basic covectors using the *wedge product* $\wedge$. The wedge product is an antisymmetric multiplication satisfying

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\,\mathrm{d}x^j \wedge \mathrm{d}x^i, \qquad \mathrm{d}x^i \wedge \mathrm{d}x^i = 0.$$

In $\mathbb{R}^3$ the natural 3-form

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

encodes oriented infinitesimal volume, and in standard coordinates this reduces exactly to the usual Lebesgue volume element

$$\mathrm{d}x = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.$$

Thus, although the wedge product is a general geometric construction, in Euclidean coordinates it reproduces the familiar integration element of multivariable calculus.

Why spacetime uses $\mathrm{d}x\,\mathrm{d}t$ rather than a wedge form. When integrating over spacetime $\Omega \times I$, which is a subset of $\mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^4$, one could in principle use a 4-form

$$\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}t.$$

However, in PDE theory and functional analysis we do not treat time as a geometric covector direction but as an additional parameter for evolution. Instead of a geometric volume form, we use the *product Lebesgue measure*

$$\mathrm{d}x\,\mathrm{d}t,$$

the Cartesian product of the spatial Lebesgue measure $\mathrm{d}x$ and the one-dimensional Lebesgue measure $\mathrm{d}t$.

This is not merely a matter of notation: it reflects the analytic structure of parabolic PDEs, where time derivatives and space derivatives play fundamentally different roles. Using $\mathrm{d}x\,\mathrm{d}t$ highlights that integration is taken over a product measure space rather than a single geometric object with a unified orientation.

Lebesgue measure and why it matters. The Lebesgue measure on $\Omega \subseteq \mathbb{R}^3$ assigns a "volume" to each measurable set and provides the foundation for L^p spaces. Working with Lebesgue measure ensures:

- functions are considered equivalent if they differ only on sets of measure zero;
- weak derivatives, distributional identities, and Sobolev spaces are well-defined;
- mixed space–time integrability properties such as $L^2_t L^2_x$ or $L^\infty_t L^2_x$ behave correctly;
- Fubini's theorem applies, allowing spacetime integrals to be separated into time integrals of spatial integrals.

This analytic framework is essential for the Navier–Stokes equations, where solutions may be defined only almost everywhere, and where the time variable acts as an evolution parameter while space carries geometric and differential structure.

Meaning of "almost everywhere." Let $(X, \mathcal{M}, \mu)$ be a measure space; in our applications $X = \Omega \subseteq \mathbb{R}^3$ with Lebesgue measure $\mu = \lambda^3$, or $X = \Omega \times I$ with product Lebesgue measure $\lambda^3 \otimes \lambda^1$. A property $P(x)$ is said to hold *almost everywhere* (abbreviated a.e.) on X if

$$\mu\big(\{x \in X : P(x) \text{ fails}\}\big) = 0.$$

Equivalently, P holds at every point except on a set of measure zero. Two measurable functions $f, g\colon X \to \mathbb{R}^m$ are considered equal almost everywhere if

$$\mu\big(\{x \in X : f(x) \neq g(x)\}\big) = 0.$$

This convention is fundamental in L^p and Sobolev spaces: elements of $L^p(X)$ are equivalence classes of functions modulo sets of measure zero, and weak derivatives are defined through integral identities that ignore such negligible sets. Thus a vector field $u(x,t)$ representing a Navier–Stokes solution may fail to be defined or may be discontinuous on a null set without affecting any integral identity, energy estimate, or distributional formulation.

Lebesgue spaces $L^p(\Omega)$ and the role of measure. Let $(\Omega, \mathcal{L}^3)$ denote the measurable space consisting of the region $\Omega \subseteq \mathbb{R}^3$ equipped with the Lebesgue σ-algebra and Lebesgue measure $\mathcal{L}^3$. For $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined as

$$L^p(\Omega) = \Big\{ f\colon \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |f(x)|^p \,\mathrm{d}x < \infty \Big\}.$$

Two functions f and g are regarded as the same element of $L^p(\Omega)$ if they differ only on a set of Lebesgue measure zero. This identification is crucial for PDE theory, since weak derivatives and energy estimates rely on equivalence classes rather than pointwise values.

The space $L^p(\Omega)$ becomes a normed space under

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{1/p},$$

and $L^p(\Omega)$ is complete with respect to this norm, making it a Banach space. For $p = \infty$, we define

$$L^\infty(\Omega) = \{ f \in L^1_{\mathrm{loc}}(\Omega) : f \text{ is essentially bounded} \},$$
$$\|f\|_{L^\infty(\Omega)} = \inf\{ M > 0 : |f(x)| \leq M \text{ for a.e. } x \in \Omega \}.$$

Vector- and tensor-valued L^p spaces are defined componentwise: a vector field $u = (u_1, u_2, u_3)$ lies in $L^p(\Omega; \mathbb{R}^3)$ iff each component $u_i \in L^p(\Omega)$.

Mixed space–time spaces. On the product domain $\Omega \times I$ equipped with the product Lebesgue measure $\mathrm{d}x\,\mathrm{d}t$, the mixed spaces

$$L^q(I; L^p(\Omega)), \qquad p, q \in [1, \infty],$$

consist of measurable functions $u(x,t)$ such that

$$\|u\|_{L^q_t L^p_x} = \left(\int_I \left(\int_\Omega |u(x,t)|^p \,\mathrm{d}x \right)^{q/p} \mathrm{d}t \right)^{1/q} < \infty$$

(with the usual modifications for $p = \infty$ or $q = \infty$). These spaces are fundamental for parabolic PDEs: Leray–Hopf weak solutions satisfy

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

Thus, for any integrable scalar field ψ, we write

$$\int_\Omega \psi(x,t) \,\mathrm{d}x, \qquad \int_{\Omega \times I} \psi(x,t) \,\mathrm{d}x\,\mathrm{d}t,$$

the second being integration with respect to the product Lebesgue measure on $\Omega \times I$.

Scalar, vector, and tensor-valued integrals. Integration is not limited to scalar fields. If a field takes values in a finite-dimensional vector space, the integral is defined componentwise. Thus:

- A *vector-valued* integrand $u_i(x)$ produces a vector:
$$\int_\Omega u(x)\,\mathrm{d}x = \begin{pmatrix} \int_\Omega u_1\,\mathrm{d}x \\ \int_\Omega u_2\,\mathrm{d}x \\ \int_\Omega u_3\,\mathrm{d}x \end{pmatrix}.$$
- A *matrix-valued* integrand $A_{ij}(x)$ produces a matrix:
$$\left(\int_\Omega A\,\mathrm{d}x\right)_{ij} = \int_\Omega A_{ij}(x)\,\mathrm{d}x.$$
- A general rank-(r,s) tensor field $T^{j_1\cdots j_s}_{i_1\cdots i_r}(x)$ integrates to a tensor of the same rank:
$$\left(\int_\Omega T\,\mathrm{d}x\right)^{j_1\cdots j_s}_{i_1\cdots i_r} = \int_\Omega T^{j_1\cdots j_s}_{i_1\cdots i_r}(x)\,\mathrm{d}x.$$

Integration over spacetime proceeds analogously:
$$\int_{\Omega\times I} u_i(x,t)\,\mathrm{d}x\,\mathrm{d}t, \qquad \int_{\Omega\times I} A_{ij}(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$
each producing the corresponding tensor.

Such tensorial integrals are ubiquitous: total momentum $\int_\Omega \rho u_i$, kinetic energy $\int_\Omega |u|^2$, moment-of-inertia tensors, and stress-resultants all arise from componentwise integration.

Differential operators with indices. The gradient is the covector-valued operator
$$\nabla\phi = \partial_i\phi\,\mathrm{d}x^i,$$
while the array $\partial_j u_i$ forms a rank-$(1,1)$ tensor.

The divergence of a vector field is the contraction
$$\nabla\cdot u = \partial_i u_i.$$
The divergence of a matrix field σ is the vector field with components
$$(\nabla\cdot\sigma)_i = \partial_j\sigma_{ij}.$$

The Laplacian is the scalar second-order operator
$$\Delta = \partial_j\partial_j,$$
acting componentwise on vector fields:
$$(\Delta u)_i = \partial_j\partial_j u_i.$$

The advantage of tensor indexing is that it expresses the nonlinear structure of the equations without hiding coordinatewise interactions. For instance, the advective term
$$(u\cdot\nabla)u = u_j\partial_j u_i$$
makes clear that the ith component of the velocity changes along the direction of u through the directional derivative $u_j\partial_j$.

Strong and weak derivatives; distributions. Fluid mechanics naturally interacts with distribution theory, because the velocity and pressure fields in physically relevant solutions are rarely smooth. Instead, they belong to Lebesgue or Sobolev spaces where derivatives must be understood in a weak (distributional) sense.

To make this precise, let $\Omega \subseteq \mathbb{R}^3$ be a spatial domain equipped with Lebesgue measure $\mathrm{d}x$. A scalar field

$$\phi \colon \Omega \to \mathbb{R}$$

is said to have a *strong* partial derivative $\partial_i \phi$ at a point $x \in \Omega$ if the limit

$$\partial_i \phi(x) = \lim_{h \to 0} \frac{\phi(x + h e_i) - \phi(x)}{h}$$

exists. Here e_i denotes the ith standard basis vector of $\mathbb{R}^3$.

However, strong differentiability is too restrictive for weak solutions of the Navier–Stokes equations. We therefore enlarge our viewpoint and admit derivatives understood through their action on *test functions*. A test function is a smooth, compactly supported function

$$\varphi \in C_c^\infty(\Omega),$$

and the collection of all such functions is denoted $\mathcal{D}(\Omega)$.

A scalar field $\phi \in L^1_{\mathrm{loc}}(\Omega)$ is said to have a *weak* derivative $\partial_i \phi \in L^1_{\mathrm{loc}}(\Omega)$ if for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \phi \, \partial_i \varphi \, \mathrm{d}x = - \int_\Omega (\partial_i \phi) \, \varphi \, \mathrm{d}x.$$

This identity is interpreted as an integration-by-parts formula in which the boundary terms vanish because φ has compact support. The function $\partial_i \phi$ is then the derivative of ϕ in the sense of distributions.

Vector and tensor fields admit weak derivatives componentwise. A vector field

$$u \colon \Omega \to \mathbb{R}^3, \qquad u(x) = (u_1(x), u_2(x), u_3(x)),$$

belongs to the space of distributions $\mathcal{D}'(\Omega)$ if each component u_i defines a continuous linear functional on test functions. The weak derivative $\partial_j u_i$ is defined by

$$\int_\Omega u_i \, \partial_j \varphi \, \mathrm{d}x = - \int_\Omega (\partial_j u_i) \, \varphi \, \mathrm{d}x \qquad \forall \, \varphi \in C_c^\infty(\Omega).$$

Weak divergence of vectors and tensors. Using the same principle, we define the divergence of a vector field u_i distributionally by

$$\int_\Omega (\nabla \cdot u) \, \varphi \, \mathrm{d}x = - \int_\Omega u_i \, \partial_i \varphi \, \mathrm{d}x,$$

so that the weak divergence is the distribution $\nabla \cdot u \in \mathcal{D}'(\Omega)$ satisfying the above identity.

If σ_{ij} is a matrix field (rank-$(1,1)$ tensor), its divergence is the vector field with components

$$(\nabla \cdot \sigma)_i = \partial_j \sigma_{ij}$$

in the distributional sense:

$$\int_\Omega (\nabla \cdot \sigma)_i \, \varphi_i \, \mathrm{d}x = - \int_\Omega \sigma_{ij} \, \partial_j \varphi_i \, \mathrm{d}x, \qquad \varphi_i \in C_c^\infty(\Omega).$$

Weak Laplacian. The Laplacian of a scalar field ϕ is defined distributionally by

$$\int_\Omega (\Delta \phi) \, \varphi \, \mathrm{d}x = \int_\Omega \partial_i \phi \, \partial_i \varphi \, \mathrm{d}x, \qquad \forall \, \varphi \in C_c^\infty(\Omega),$$

where the right-hand side is well-defined whenever $\phi \in H^1_{\mathrm{loc}}(\Omega)$.

The weak Laplacian of a vector field is computed componentwise:

$$(\Delta u)_i = \partial_j \partial_j u_i \in \mathcal{D}'(\Omega).$$

These constructions are indispensable in fluid analysis because Leray–Hopf weak solutions have finite kinetic energy but may fail to be pointwise differentiable. Weak derivatives allow the Navier–Stokes equations to make sense despite this low regularity.

Function spaces and regularity assumptions. Weak formulations of the Navier–Stokes equations rely on a functional–analytic framework which is often intimidating to students; we will endeavor here to make it clear. The basic building blocks are the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $H^k(\Omega)$ (or the more general $W^{k,p}(\Omega)$). These encode the minimal integrability and differentiability required for weak derivatives, energy estimates, and compactness arguments.

Lebesgue spaces. For a measurable set $\Omega \subseteq \mathbb{R}^3$ with Lebesgue measure $\mathrm{d}x$, the space $L^p(\Omega)$, $1 \le p < \infty$, is defined by

$$L^p(\Omega) = \left\{ u \colon \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^p \, \mathrm{d}x < \infty \right\}.$$

Two functions are identified if they agree almost everywhere. Endowed with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, \mathrm{d}x \right)^{1/p},$$

$L^p(\Omega)$ is a Banach space. For $p = \infty$, one defines the essential supremum norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|.$$

Vector- and tensor-valued L^p spaces are defined by requiring each component to lie in $L^p(\Omega)$.

Sobolev spaces. Weak derivatives naturally lead to the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \partial_i u \in L^p(\Omega) \right\}.$$

The most important case for incompressible flow is $p = 2$, giving the Hilbert space

$$H^1(\Omega) = W^{1,2}(\Omega), \qquad \|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^{3} \|\partial_i u\|_{L^2(\Omega)}^2.$$

Weak derivatives are understood in the sense of distributions, so that $\partial_i u$ satisfies

$$\int_\Omega u \, \partial_i \varphi \, \mathrm{d}x = - \int_\Omega (\partial_i u) \, \varphi \, \mathrm{d}x \qquad \forall \, \varphi \in C_c^\infty(\Omega).$$

The Sobolev space $H^1(\Omega)$ is the canonical energy space for incompressible flow, since the kinetic energy

$$\frac{1}{2} \int_\Omega |u(x,t)|^2 \, \mathrm{d}x$$

controls the L^2 norm, and viscosity controls the H^1 norm through $\|\nabla u\|_{L^2}$.

Divergence-free spaces. Define

$$L^2_\sigma(\Omega) = \{ u \in L^2(\Omega; \mathbb{R}^3) : \partial_i u_i = 0 \text{ in } \mathcal{D}'(\Omega), \ u \cdot n|_{\partial\Omega} = 0 \},$$

the space of solenoidal vector fields. Similarly define

$$H^1_\sigma(\Omega) = H^1(\Omega) \cap L^2_\sigma(\Omega).$$

These are closed subspaces of $L^2(\Omega)$ and $H^1(\Omega)$, respectively, and capture the incompressibility constraint.

Bochner spaces and mixed space–time norms. Parabolic PDEs require function spaces on $\Omega \times I$ where the time variable is treated separately from space. For a Banach space X (e.g. $X = L^2(\Omega)$ or $X = H^1(\Omega)$), the Bochner space

$$L^q(I;X) = \{u\colon I \to X \text{ Bochner-measurable} : \|u(t)\|_X \in L^q(I)\}$$

consists of time-indexed functions with values in X. The norm is

$$\|u\|_{L^q_t X_x} = \left(\int_I \|u(t)\|_X^q \,\mathrm{d}t\right)^{1/q},$$

with the obvious modification when $q = \infty$.

For fluid dynamics, the fundamental spaces are

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \qquad \text{and} \qquad u \in L^2(0,T;H^1_\sigma(\Omega)),$$

meaning:

$$\text{finite kinetic energy: } \sup_{t\in[0,T]} \|u(t)\|_{L^2} < \infty,$$

$$\text{finite dissipation: } \int_0^T \|\nabla u(t)\|_{L^2}^2 \,\mathrm{d}t < \infty.$$

The Leray–Hopf framework. A Leray–Hopf weak solution of the incompressible Navier–Stokes equations satisfies:

- $u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega))$,
- $\partial_t u \in L^{4/3}(0,T;H^{-1}(\Omega))$ (distributional time derivative),
- the weak form of the momentum equation

  $$\int_0^T \int_\Omega \big(u_i \partial_t \varphi_i + u_j u_i \,\partial_j \varphi_i + \partial_i u_j \,\partial_i \varphi_j\big) \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega f_i \,\varphi_i \,\mathrm{d}x \,\mathrm{d}t,$$

  for all $\varphi \in C_c^\infty(\Omega \times (0,T);\mathbb{R}^3)$ with $\partial_i \varphi_i = 0$,
- the *energy inequality*

  $$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u(s)\|_{L^2}^2 + \int_s^t \int_\Omega f \cdot u \,\mathrm{d}x \,\mathrm{d}\tau$$

  for almost every $0 \le s < t \le T$.

The spaces $L^2(0,T;H^1_\sigma(\Omega))$ and $L^\infty(0,T;L^2_\sigma(\Omega))$ are precisely those in which this energy inequality is valid and in which the nonlinear term $u_j \partial_j u_i$ makes distributional sense. The dual space $H^{-1}(\Omega)$ is required to interpret $\partial_t u$ as a distribution acting on divergence-free test functions.

The weak formulation relies on:

- L^2–control of u for kinetic energy;
- L^2–control of ∇u for dissipation;
- Bochner measurability to ensure $u(\cdot,t)$ is defined for a.e. t;
- Fubini's theorem to interchange space–time integrals;
- weak compactness theorems (Banach–Alaoglu, Aubin–Lions) for existence.

This is the analytic bedrock on which all modern Navier–Stokes theory is built, and it will remain the default functional setting throughout this work.

Parabolic weak forms. Weak formulations on spacetime require test functions with compact support on $\Omega \times I$. We denote these by

$$\varphi \in C_c^\infty(\Omega \times I).$$

For a vector field $u_i(x,t)$, the weak time derivative satisfies

$$\int_{\Omega\times I} u_i\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega\times I} (\partial_t u_i)\,\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

meaning that $\partial_t u_i$ is a distribution on $\Omega\times I$.

In the parabolic setting, nonlinear terms naturally produce tensorial integrands. For example, the transport term $(u\cdot\nabla)u$ appears weakly as

$$\int_{\Omega\times I} u_j\,\partial_j u_i\,\varphi_i\,\mathrm{d}x\,\mathrm{d}t,$$

and the stress divergence as

$$\int_{\Omega\times I} \sigma_{ij}\,\partial_j\varphi_i\,\mathrm{d}x\,\mathrm{d}t.$$

Each quantity is interpreted componentwise using the conventions above. These weak identities allow the Navier–Stokes system to be interpreted in $\mathcal{D}'(\Omega\times I)$ even when u is far from smooth.

This framework will reappear when we pass to the vorticity formulation, the Biot–Savart representation, and the tensorial dyadic structures central to the analysis of nonlocal stretching mechanisms.

Importance of tensors in fluid dynamics. The power of index notation is that it keeps the structure of the PDE explicit. The nonlinear advection term is

$$(u\cdot\nabla)u = u_j\partial_j u_i,$$

making clear that the ith component of the velocity changes along the direction of the velocity itself.

This notation will be essential later when we pass to vorticity, introduce the dyadic Ξ_{ij}, and express curvature and entropy quantities in terms of indexed derivatives, contractions, and tensor products.

1.2. **Fluids from first principles.** With this notation established, we now return to the governing principles. We regard the fluid as occupying a region $\Omega\subseteq\mathbb{R}^3$ and evolving over a time interval $I\subseteq\mathbb{R}$. The macroscopic state is described by functions

$$\rho\colon \Omega\times I\to(0,\infty), \qquad u\colon \Omega\times I\to\mathbb{R}^3, \quad u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t)).$$

These are coarse-grained averages of microscopic quantities, but at the continuum scale they are treated as smooth functions of space and time.

The first structural constraint is conservation of mass. For any measurable region $E\subseteq\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_E \rho(x,t)\,\mathrm{d}x = -\int_{\partial E}\rho\,u\cdot n\,\mathrm{d}S.$$

Applying the divergence theorem and passing to a local description yields the continuity equation

$$\partial_t\rho + \partial_j(\rho\,u_j) = 0 \qquad \text{in } \Omega\times I. \tag{1.1}$$

In many physical regimes one assumes $\rho(x,t)\equiv\rho_0$ for some constant $\rho_0>0$. Substituting this into (1.1) and dividing by ρ_0 gives the divergence-free constraint

$$\partial_j u_j = 0 \qquad \text{in } \Omega\times I, \tag{1.2}$$

expressing the volume-preserving nature of incompressible flow.

Momentum balance supplies the second principle. A fluid parcel follows a trajectory $X(t)$ solving the ODE

$$\dot X(t) = u(X(t),t),$$

and its acceleration equals the total (material) derivative of the velocity:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}(X(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(u_i(X(t),t)\big).$$

Applying the chain rule gives the Eulerian expression

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \partial_t u_i + u_j\,\partial_j u_i \qquad \text{in } \Omega \times I. \tag{1.3}$$

Internal forces on a fluid parcel arise from stresses within the medium and are encoded by the Cauchy stress tensor

$$\sigma\colon \Omega \times I \to \mathbb{R}^{3\times 3}, \qquad \sigma = (\sigma_{ij})_{1\le i,j\le 3}.$$

Its divergence produces the internal force density:

$$(\nabla \cdot \sigma)_i = \partial_j \sigma_{ij}.$$

If $f_i(x,t)$ denotes external force density, Newton's second law becomes

$$\rho\,\frac{\mathrm{d}u_i}{\mathrm{d}t} = \partial_j \sigma_{ij} + f_i \qquad \text{in } \Omega \times I. \tag{1.4}$$

A constitutive law specifies σ_{ij} in terms of u. For a Newtonian viscous fluid, the stress depends linearly and isotropically on the rate of strain. Writing $p(x,t)$ for pressure, the symmetric gradient

$$D_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i),$$

and using viscosity coefficients $\mu > 0$ (shear viscosity) and $\lambda \in \mathbb{R}$ (bulk viscosity), the constitutive relation is

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\,D_{ij} + \lambda\,(\partial_k u_k)\,\delta_{ij}. \tag{1.5}$$

Under the incompressibility constraint (1.2), the bulk term vanishes. Moreover, one computes

$$\partial_j(2\mu D_{ij}) = \mu\,\partial_j\partial_j u_i = \mu\,\Delta u_i,$$

where $\Delta = \partial_j\partial_j$ is the Laplacian. Substituting (1.5) into (1.4) therefore yields

$$\rho\big(\partial_t u_i + u_j \partial_j u_i\big) = -\partial_i p + \mu\,\Delta u_i + f_i \qquad \text{in } \Omega \times I. \tag{1.6}$$

Together with (1.2), this is the incompressible Navier–Stokes system in physical units, expressed entirely in indexed tensor form.

For analytical purposes it is convenient to nondimensionalize so that $\rho = \mu = 1$. Under this normalization, (1.6) becomes

$$\partial_t u_i + u_j \partial_j u_i = \Delta u_i - \partial_i p + f_i, \qquad \partial_j u_j = 0, \tag{1.7}$$

a system of nonlinear PDEs on the product domain $\Omega \times I$ for the unknowns (u_i, p).

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